NAPDE Sheet 4

February 23, 2010

1. Consider the finite element space

$$V_h := \{v_h \in C(\bar{I}) : V(0) = v(1) = 0, v|_{x_{i-1},x_i} \text{ is linear } i = 1, N\}$$

where $x_i := ih$ and $Nh = 1$. Show that

$$||v_h||^2_{L^2(I)} = \frac{h}{6}\sum_{i=1}^{N}(V_iV_{i-1} + 4V_i^2 + V_iV_{i+1}) \geq \frac{h}{3}\sum_{i=1}^{N-1}|V_i|^2$$

and

$$||(v_h)'||^2_{L^2(I)} \leq \frac{4}{h}\sum_{i=1}^{N-1}|V_i|^2.$$

Hence obtain the *inverse inequality*

$$||(v_h)'||_{L^2(I)} \leq \frac{\sqrt{12}}{h}||v_h||_{L^2(I)}\ , \quad \forall v_h \in V_h.$$

2. Consider the heat equation with Neumann boundary conditions:

$$\begin{array}{rcll} \frac{\partial u}{\partial t} & = & \frac{\partial^2 u}{\partial x^2} & (x,t) \in (0,1) \times (0,\infty), \\ \frac{\partial u}{\partial x} & = & 0 & (x,t) \in \{0,1\} \times (0,\infty), \\ u & = & g & (x,t) \in [0,1] \times \{0\}. \end{array} \tag{1}$$

Write down the forward and backward Euler finite element methods using the piecewise linear finite element space on a uniform grid, $x_j = jh, h = 1/J$. Derive the algebraic formulae satisfied by the nodal values of the finite element solution.

3. Consider the heat equation with Dirichlet boundary conditions:

$$\begin{array}{rcll} \frac{\partial u}{\partial t} & = & \frac{\partial^2 u}{\partial x^2} & (x,t) \in (0,1) \times (0,\infty), \\ u & = & 0 & (x,t) \in \{0,1\} \times (0,\infty), \\ u & = & g & (x,t) \in [0,1] \times \{0\}. \end{array} \tag{2}$$

Write down the forward and backward Euler finite element methods using the piecewise linear finite element space on a uniform grid, $x_j = jh, h = 1/J$. Derive the algebraic formulae satisfied by the nodal values of the finite element solution. Find stability bounds; in the case of the forward Euler scheme use the inverse inequality of Question 1.

4. Consider the heat equation with Neumann boundary conditions:

$$\begin{aligned} \frac{\partial u}{\partial t} &= -u + \Delta u + f && (x,t) \in \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} &= 0 && (x,t) \in \partial\Omega \times (0,\infty), \\ u(\cdot,0) &= g(\cdot) && . \end{aligned} \tag{3}$$

Write down the continuous in time finite element approximation using piecewise linear elements on a triangulation of a polygonal domain Ω where $g, f \in L^2(\Omega)$. Prove stability of the backward Euler discretization in time.