NAPDE Sheet 1

January 2010

1. Draw the graph of the function ϕ defined by
$$\phi(x) := (1 - |x|)_+, \quad x \in [-2, 2].$$
Is it true that $\phi \in C[-2, 2] \cup C^1(-2, 2)$?
Calculate the first (weak) derivative $\phi' = D\phi$ of ϕ on the interval [-2,2]. Verify that $\phi, \phi' \in L^p(-2, 2)$ for all $p \in [1, \infty]$. Hence deduce that $\phi \in W^{1,p}(-2, 2)$ for $p \in [1, \infty]$.

2. Suppose that $u(x) = x^\alpha, x \in [0, 1]$, where α is a fixed real number, $0 < \alpha < 1$. Show that $u \in C^\infty(0, 1)$ but that $u \notin W^{1,p}(0, 1)$ for $p \geq (1 - \alpha)^{-1}$.

3. Given that (a, b) is an open interval of the real line,
let $H^1_{E_0}(a, b) = \{v \in H^1(a, b) : v(a) = 0\}$.
   - a)By writing
   $$v(x) = \int_a^x v'(\xi) d\xi, \quad a \leq x \leq b,$$
   for $v \in H^1_{E_0}(a, b)$, show that the following (Poincare-Friedrichs) inequality holds for each $v \in H^1_{E_0}(a, b)$:
   $$||v||^2_{L^2(a,b)} \leq \frac{1}{2}(b - a)^2 |v|^2_{H^1(a,b)}.$$
   - b) By writing
   $$(v(x))^2 = \int_a^x \frac{d}{d\xi}(v(\xi))^2 d\xi = 2\int_a^x v(\xi)v'(\xi) d\xi, \quad a \leq x \leq b,$$
   for $v \in H^1_{E_0}(a, b)$, show that the following (Agmon's) inequality holds for each $v \in H^1_{E_0}(a, b)$:
   $$\max_{x \in [a,b]} v(x)^2 \leq 2||v||_{L^2(a,b)} |v|_{H^1(a,b)}.$$

4. Let $u(x) \in H^1((0, 1))$. Prove that
$$\sup_{x \in (0,1)} |u(x) - \int_0^1 u(y) dy| \leq \|u'\|_2.$$
Hence prove that
$$\sup_{x \in (0,1)} |u(x)| \leq \|u'\|_2 + \|u\|_2.$$
Finally deduce that
$$\|u\|_\infty \leq \sqrt{2}\|u\|_{H^1}.$$

5. Let $v(x) = x^{-\beta}$ with $0 < \beta < \frac{1}{2}$ and define $I = (0, 1)$ so that $\bar{I} = [0, 1]$. Show that $v(x) \in L^2(I)$ but that $v(x) \notin H^1(I)$ and $v(x) \notin C(\bar{I})$. Can you find a function which is in $L^2(I)$ and $C(\bar{I})$ but not in $H^1(I)$?

6. Show that

$$AB \leq \frac{\epsilon^2}{2}A^2 + \frac{1}{2\epsilon^2}B^2, \quad \forall A, B \in R \text{ and } \epsilon > 0. \tag{1}$$

7. Let $I = (a, b)$ and solve the eigenvalue problem

$$\begin{aligned} \phi_k'' + \lambda_k \phi_k &= 0, \quad x \in (a, b) \\ \phi(a) = \phi(b) &= 0. \end{aligned}$$

Note that $\{\phi_k\}_{k=1}^{\infty}$ forms an orthogonal basis for $L^2(I)$ so that any $v(x) \in L^2(I)$ may be uniquely represented as

$$v(x) = \sum_{k=1}^{\infty} a_k \phi_k(x).$$

Show that

$$\|v(x)\|_{L^2}^2 = \sum_{k=1}^{\infty} a_k^2$$

if the ϕ_k are appropriately normalized. Show also that, if $v(a) = v(b) = 0$, then

$$\|v'\|_{L_2}^2 = -\int_a^b v(x)v''(x)\, dx,$$

and hence prove that

$$\|v\|_{L^2} \leq C_p \|v'\|_{L_2}$$

for $v \in C^2(I, R) \bigcap H_0^1(I)$. [The result can be extended to arbitrary functions $v \in H_0^1(I)$ by approximation and, also, to functions in $H_0^1(\Omega)$, with $\Omega \subset \mathbb{R}^d$ bounded and open.].

8. (a) If

$$g(x, y) = (x^2 + y^2)^{-p/2}$$

then calculate $\nabla g(x, y)$. Let Ω be the unit disk $x^2 + y^2 \leq 1$ and evaluate $\|\nabla g\|_{L^2}^2$.

(b) Now do the same for the function

$$g(x, y) = \log[\log(r)], \quad r^2 = x^2 + y^2$$

on the disk Ω with $r \leq \frac{1}{2}$.

(c) Does the final result of Exercise 4 have a direct analog in two dimensions?

(d) Show that $g \in H^1(\Omega)$ but $g \notin C(\Omega)$.

[*Hint:* recall the change of variable formula:

$$\int_\Omega f(x, y) dx dy = \int_\Omega f(r\cos\theta, r\sin\theta) r dr d\theta.]$$

9. Consider the linear equations

$$A\xi = b \tag{2}$$

where A is an $m \times m$ positive definite, symmetric matrix. Let $(\cdot, \cdot)$ denote the standard real inner-product on $\mathbb{R}^m$ and let $\|\cdot\|$ denote the induced (Euclidean) norm.

(a) Show that solving (2) is equivalent to minimizing

$$g(\eta) := \frac{1}{2}(\eta, A\eta) - (b, \eta)$$

and that

$$g(\eta) \geq -\frac{1}{2}(b, A^{-1}b) \quad \forall \eta \in \mathbb{R}^m.$$

(b) Consider the system of ordinary differential equations

$$\frac{du}{dt} = b - Au. \tag{3}$$

Show that

$$\frac{d}{dt}g(u(t)) = -\|\frac{du}{dt}(t)\|^2.$$

(c) Apply the forward Euler method to (3) to obtain a method in the form

$$U_{n+1} = U_n - \Delta t_n r_n$$

where r_n should be specified and Δt_n is the (adaptively chosen) time-step. Find the choice of Δt_n which maximizes the decrease in $g(u_m)$ from $m = n$ to $m = n + 1$. Show that with this choice of time-step

$$\frac{g(u_{n+1}) - g(u_n)}{\Delta t_n} = -\frac{1}{2}\|r_n\|^2.$$

What iterative method for (2) have you just derived?