NAPDE Sheet 3

February 25, 2010

1. Consider the elliptic equation

$$-\Delta u = f \quad \text{for}(x,y) \in \Omega = (0,1)^2$$

subject to the homogeneous Dirichlet boundary condition $u = 0$ on Γ_D and the non-homogeneous Neumann boundary condition $\frac{\partial u}{\partial x} = 1$ on Γ_N where $\Gamma_N = \{(x,y) \in \partial\Omega; x = 1\}$ and $\Gamma_D = \partial\Omega \setminus \Gamma_N$. State the weak formulation of the problem.

Consider a triangulation of Ω which has been obtained from a square mesh of spacing $h = 1/N, N \geq 2$ in both coordinate directions by subdividing each square into two triangles with the diagonal of negative slope. Using continuous piecewise linear basis functions on this triangulation, obtain the finite element approximation to the boundary value problem. Rewrite the finite element method as a system of linear algebraic equations and comment on the structure of the matrix.

2. Given that Ω is a bounded open set in $\mathbb{R}^n$, consider the elliptic boundary value problem (labelled (P)):

$$-\Delta u + cu = f \quad \text{for } x \in \Omega, \quad u = 0 \; x \in \partial\Omega$$

with $C_s \geq c(x) \geq 0$ and $f \in L^2(\Omega)$.

a) Formulate the problem as a minimization problem for a quadratic functional $J : H_0^1(\Omega) \to \mathbb{R}$ which has at most only one solution.

b)Show that if u solves the minimization problem then u is a weak solution of (P).

c)Assume that $n = 1, \Omega = (0,1)$ and $c(x) = 1$. Suppose that V_h is the finite element space consisting of piece-wise linear continuous functions on a uniform mesh with $h = 1/N$. Formulate the minimization problem on V_h. Show that it is equivalent to solving a system of linear algebraic equations with a tridiagnal matrix A whose entries you should calculate.

3. Consider the two point boundary value problem

$$-(p(x)u')' + q(x)u = f(x), \quad x \in (0,1), \; u(0) = 0, \; u(1) = 0,$$

where $p(x) \geq \bar{c} > 0, q(x) \geq 0$ for $x \in [0,1], p \in C^1[0,1], q \in C[0,1]$ and $f \in L^2(0,1)$. Given that u_h denotes the continuous piecewise linear finite element approximation to u on a uniform subdivision of $[0,1]$ into subintervals of length $h = 1/N, N \geq 2$, show that

$$||u - u_h||_{H^1(0,1)} \leq C_1 h ||u''||_{L^2(0,1)}$$

where C_1 is a positive constant that you should specify. Show further that there is a positive constant C_2 such that

$$||u - u_h||_{H^1(0,1)} \leq C_2 h ||f||_{L^2(0,1)}.$$

Calculate the right hand side of these inequalities in the case when $p(x) = 1, q(x) = 0, f(x) = 1$ and $h = 1/1000$.

Derive an error bound in the L^2 norm using the Aubin-Nitsche duality argument.

4. Let $\Omega := (0,1)^2$. Consider the finite element space $V_h \subset H^1(\Omega)$ consisting of all continuous piecewise linear functions on a triangulation of Ω obtained from a uniform square mesh of size $h = 1/N, N \geq 2$, by subdividing each square into two triangles with the diagonal of negative slope. Given that $u \in H^2(\Omega)$ let $\mathcal{I}_h u$ denote its continuous piecwise linear interpolant from V_h. You may take it for granted that

$$||u - \mathcal{I}_h u||_{H^1(\Omega)} \leq K_1 h |u|_{H^2(\Omega)},$$

where K_1 is a positive constant, independent of u, u_h and h.

Now consider the elliptic boundary value problem

$$-\Delta u + u = f(x,y) \text{ in } \Omega,\ \ u = 0 \text{ on } \partial\Omega,$$

where $f \in L^2(\Omega)$.

Show that the finite element approximation of this boundary value problem satisfies the error bound

$$||u - u_h||_{H^1(\Omega)} \leq K_2 h |u|_{H^2(\Omega)}$$

where K_2 is a positive constant independent of u, u_h and h.

Use the Aubin-Nitsche duality argument to derive an error bound in the L^2 norm. You may assume elliptic regularity

$$||u||_{H^2(\Omega)} \leq C||f||_{L^2(\Omega)}$$

for the solution of the boundary value problem.

Show that the finite element method can be restated as a seven point difference scheme for the vector U of nodal values of the finite element function u_h. Comment on the sparsity structure of the matrix of the associated system of linear algebraic equations.

5. Suppose that Ω is a two dimensional bounded polygonal domain which is the union of right-angled isosceles triangles which intersect only along an edge or a vertex. Let K be a triangle with vertices P_1, P_2 and P_3 such that the edges P_1P_2 and P_1P_3 have length h. Denote by $\phi_i(x,y)$, for $i = 1,2,3$, the linear function on K such that

$$\phi_i(P_j) = \delta_{ij} \quad j = 1,2,3.$$

(a) Using local coordinates (ξ, η) so that P_1 and P_2 are on the η axis and P_1 and P_3 are on the ξ axis, show that :

(i)

$$\nabla\phi_1 \cdot \nabla\phi_j = -\frac{1}{h^2} \quad j = 2,3 \quad \text{and} \quad \nabla\phi_2 \cdot \nabla\phi_3 = 0$$

$$\nabla\phi_1 \cdot \nabla\phi_1 = \frac{2}{h^2} \quad \text{and} \quad \nabla\phi_i \cdot \nabla\phi_i = \frac{1}{h^2} \quad i = 2,3.$$

(ii)

$$\int_K \phi_i dxdy = \frac{h^2}{6} \quad \forall i.$$

(b) Let v_h be a linear function on K and set $V_i := v_h(P_i),\ i = 1, 2, 3$, to be the values of v_h at the vertices. Show that

$$\int_K |\nabla v_h|^2 dxdy = \mathbf{V}^T A^K \mathbf{V}$$

where A^K is the element stiffness matrix and $\mathbf{V}^T = (V_1, V_2, V_3)$. Use (a)(i) to calculate the numerical values of the entries of the element stiffness matrix A^K.

(c) Consider the finite element approximation of

$$-\nabla(q\nabla u) = 1 \text{ on } \Omega,$$

where q is a given positive function, using continuous piece-wise linear finite element functions on a union of isosceles right-angled triangles. Consider the triangles whose vertices are $(O, N, E), (O, N, W), (O, E, S), (O, W, S)$ with corresponding coordinates $O = (0, 0), N = (0, h), W = (-h, 0), S = (0, -h)$ and $E = (h, 0)$.

Find the algebraic version of the finite element equation

$$\int_\Omega q\nabla u_h \cdot \nabla\phi_O \mathrm{d}x\mathrm{d}y = \int_K \phi_O \mathrm{d}x\mathrm{d}y$$

involving the nodal values U_N, U_E, U_W, U_S and U_O of the finite element approximation u_h, where ϕ_O is the basis function for vertex O.

6. (a)Let $\hat{K}$ be the square in the $X - Y$ plane with vertices (-1,-1), (1,-1), (-1,1) and (1,1) labelled Q_1, Q_2, Q_3 and Q_4 respectively.

(i) Find the bilinear basis functions $\Phi_i(X, Y)$, for $i = 1, 2, 3, 4$ which satisfy

$$\Phi_i(Q_j) = \delta_{ij} \quad j = 1, 2, 3, 4.$$

(ii) Show that for $i, j = 1, 2, 3, 4$ (use symmetry in order to reduce the number of calculations!)

$$\int_{\hat{K}} \frac{\partial\Phi_i}{\partial X}(X, Y)\frac{\partial\Phi_j}{\partial X}(X, Y)dXdY = \begin{cases} 1/3 & \text{if } i = j, \\ 1/6 & \text{if } |i - j| = 2, \\ -1/6 & \text{if } i + j = 5, \\ -1/3 & \text{otherwise.} \end{cases}$$

and

$$\int_{\hat{K}} \frac{\partial\Phi_i}{\partial Y}(X, Y)\frac{\partial\Phi_j}{\partial Y}(X, Y)dXdY = \begin{cases} 1/3 & \text{if } i = j, \\ -1/3 & \text{if } |i - j| = 2, \\ -1/6 & \text{if } i + j = 5, \\ 1/6 & \text{otherwise.} \end{cases}$$

Calculate

$$\int_{\hat{K}} \nabla\Phi_i(X, Y)\nabla\Phi_j(X, Y)dXdY\ , \quad i, j = 1, 2, 3, 4$$

where

$$\nabla\Phi_i = (\frac{\partial\Phi_i}{\partial X}, \frac{\partial\Phi_i}{\partial Y}).$$

(b)Let K be a square in the $x-y$ plane with vertices P_1, P_2, P_3 and P_4 whose coordinates are, respectively, $((x^*, y^*), (x^*+h, y^*), (x^*, y^*+h)$ and (x^*+h, y^*+h). Denote by $\phi_i(x,y)$, for $i = 1,2,3,4$, the bilinear function such that

$$\phi_i(P_j) = \delta_{ij} \quad j = 1,2,3,4.$$

Let v_h be a bilinear function on K and set $V_i := v_h(P_i),\ i = 1,2,3,4$, to be the values of v_h at the vertices.

Use part (a) to write down the entries of the element stiffness matrix A^K where

$$\int_K \nabla v_h \cdot \nabla w_h dxdy = V^T A^K W$$

and $V^T = (V_1, V_2, V_3, V_4)$.

(c) Derive a generic nine point formula for approximating

$$-\Delta u = 0$$

in a square domain with non-zero Dirichlet boundary conditions using a square mesh of size h and bilinear finite elements.

7. Consider the following fourth order differential equation:

$$\begin{aligned}
\frac{d^2}{dx^2}\left(p(x)\frac{d^2u(x)}{dx^2}\right) - \frac{d^2u}{dx^2} + q(x)u &= f(x), \quad x \in (0,1), \\
u(0) = u(1) &= 0, \\
\frac{d^2u}{dx^2}(0) = \frac{d^2u}{dx^2}(1) &= 0.
\end{aligned}$$

Derive an appropriate variational form of this problem over a space $V \subset H^2(0,1)$ which you should specify. Use the Lax-Milgram theorem to prove existence and uniqueness.

8. Consider the **biharmonic equation**

$$\begin{aligned}
\Delta^2 u &= f, \quad x \in \Omega, \\
u = \Delta u &= 0, \quad x \in \partial\Omega.
\end{aligned}$$

Write this as a pair of coupled second-order problems then derive a weak form of the problem. By introducing an appropriate piecewise linear approximation space, derive a finite element method and prove an error bound for the method.