A SURFACE PHASE FIELD MODEL FOR TWO-PHASE BIOLOGICAL MEMBRANES [*]

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Abstract. We study vesicles formed by lipid bilayers that are governed by an elastic bending energy and on which the lipids laterally separate forming two different phases. The energy laden phase interfaces may be modeled as curves on the hyper-surface representing the membrane (sharp interface model). The phase field methodology is another powerful tool to model such phase separation phenomena where thin layers describe the interfaces (diffuse interface model). For both approaches we characterize equilibrium shapes in terms of the Euler-Lagrange equations of the total membrane energy subject to constraints on the area of the two phases and the volume. We further show by matching appropriate formal asymptotic expansions that the sharp interface model is obtained from the diffuse interface model as the thickness of the phase interface tends to zero. The essential challenge lies in the fact that also the geometry of the membrane is unknown and depends on a small parameter representing the interface thickness.

Key words. biomembrane, phase field method, matched asymptotic expansions

AMS subject classifications. 92C10, 82B26, 35C20, 49Q10

1. Introduction. Biomembranes are bilayers of lipid molecules and a basic component of the boundaries of cells and cell organelles [27]. Observations of vesicles formed by biomembranes exhibit an interesting variety of shapes and shape transitions [29]. Established models treat biomembranes as deformable inextensible fluid surfaces of infinitesimal thickness, unable to sustain shear stress, and governed by bending energy functionals with the membrane strain energy depending on the curvature of the surface. A classical model for the elastic bending energy is the Canham-Helfrich-Evans energy functional [8, 16, 20]

$$\mathcal{F}_{CEH}(\Gamma) = \int_{\Gamma} \frac{k_{\kappa}}{2}(\kappa - \kappa_s)^2 + \int_{\Gamma} k_g g \qquad (1.1)$$

where the membrane is modeled as a closed hyper-surface Γ in $\mathbb{R}^3$ enclosing a bounded domain Ω. The mean curvature of the membrane is denoted by κ and the Gaussian curvature by g. Note that κ is the sum of the principle curvatures rather than the arithmetic mean and hence differs from the common definition by a factor 2. The positive real numbers k_{κ} (bending rigidity) and k_g (Gaussian bending rigidity) are material dependent elasticity parameters whilst κ_s is called spontaneous curvature.

Shape transition phenomena such as bud formation, pearling and vesicle fission have recently been observed in two-component giant unilamellar vesicles involving a separation into two phases [6, 5]. Line tension is observed at the phase interface, and in [23, 24] an energy functional of the form

$$\mathcal{F}_{SI}(\Gamma) = \mathcal{F}_B + \mathcal{F}_L = \sum_{i=1}^{2} \Big(\int_{\Gamma_i} \frac{k_{\kappa}^{(i)}}{2}(\kappa - \kappa_s^{(i)})^2 + \int_{\Gamma_i} k_g^{(i)} g \Big) + \int_{\gamma} \bar{\sigma} \qquad (1.2)$$

has been proposed for the two-phase membrane which is composed of two smooth surfaces Γ_i with a common boundary γ. Then $\bar{\sigma}$ denotes the (constant) energy density

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of the excess free energy of the phase transition located on γ. It is assumed that the lipid bilayer structure of the membrane remains intact across the phase interface so that the whole surface $\Gamma = \Gamma_1 \cup \gamma \cup \Gamma_2$ is of the class C^1 across γ and fulfills some more regularity properties that will be specified in Section 3. We emphasize that these assumptions are consistent with the assumptions that usually are made for axisymmetric shapes, see [11] for instance.

In this paper, we investigate the idea of replacing the line energy $\mathcal{F}_L$ by a Ginzburg-Landau energy of the form

$$\mathcal{F}_{GL} = \int_\Gamma \sigma \Big(\frac{\varepsilon}{2} |\nabla_\Gamma c|^2 + \frac{1}{\varepsilon} \psi(c) \Big), \tag{1.3}$$

where c is an order parameter to distinguish the two phases, ∇_Γ stands for the surface gradient, $\psi(c) = \frac{1}{2}(1 - c^2)^2$ is a double-well potential and ε a small length scale. The coefficient σ is related to the line energy coefficient $\bar{\sigma}$ by

$$\bar{\sigma} = \frac{4}{3}\sigma \tag{1.4}$$

which will be motivated by an asymptotic analysis. The double-well potential w has two minima in the points $c = \pm 1$ so that $c \approx 1$ and $c \approx -1$ in the two phases, whilst the phase interface γ is replaced by a thin layer of a thickness scaling with ε across which c changes its value smoothly but quickly. The total membrane energy in this diffuse interface model reads

$$\mathcal{F}_{DI}(\Gamma, c) = \mathcal{F}_{MC}(\Gamma, c) + \mathcal{F}_{GC}(\Gamma, c) + \mathcal{F}_{GL}(\Gamma, c)$$
$$= \int_\Gamma \frac{1}{2} k_\kappa(c) \big| \kappa - \kappa_s(c) \big|^2 + \int_\Gamma k_g(c) g + \int_\Gamma \Big(\frac{\sigma \varepsilon}{2} |\nabla_\Gamma c|^2 + \frac{\sigma}{\varepsilon} \psi(c) \Big). \tag{1.5}$$

With respect to the bending rigidities and the spontaneous curvature we set

$$k_\kappa(c) = \begin{cases} k_\kappa^{(1)} & \text{if } 1 \le c, \\ \frac{k_\kappa^{(1)} + k_\kappa^{(2)}}{2} + \frac{k_\kappa^{(1)} - k_\kappa^{(2)}}{4} c(3 - c^2) & \text{if } -1 < c < 1, \\ k_\kappa^{(2)} & \text{if } c \le -1, \end{cases}$$

and similarly for $\kappa_s(c)$ and $k_g(c)$ but other interpolations of the same smoothness are sufficient to establish the results presented in the study.

The idea of replacing the line energy by (1.3) has already been proposed earlier, see for example [3, 28, 25, 31, 30] yet a formal asymptotic analysis seems to be lacking which motivates this study. Furthermore, the phase field method provides a convenient way to compute equilibrium membrane shapes using surface finite elements [14]. We remark that it has also been used already to describe the membrane [7, 35, 26].

We first introduce some concepts of a surface calculus, including surface gradients, an integration by parts formula and a transport identity (Leibniz formula) for evolving surfaces. This calculus is easily accessible to the discretization using surface finite elements as they have been used for solving partial differential equations on evolving surfaces [13] and geometric evolution equations as Willmore flow [12]. Based on this calculus we carefully derive the Euler-Lagrange equations of the sharp interface membrane energy (1.2), the *sharp interface model*. Appropriate two-phase membranes and admissible deformations that respect the required regularity properties are defined for this purpose.

Important works in this context are [22, 15] where deformed sheets with discontinuities along certain curves are considered and the standard notation of differential geometry is used. That notation is also used in [1] on two-phase biomembranes. There, conditions not only for critical points but for minimizers are derived, yet explicit statements of the interface conditions are lacking for an energy density as in (1.1) since, later on, the dependence on the Gaussian curvature is dropped. Furthermore, there are approaches for open membranes containing numerous computations using the usual notation in differential geometry [9, 36, 4] or differential forms [32, 33] that may be used to treat the present case of two open membranes glued together. The essential novelty of our approach lies in using the mentioned surface calculus instead. The two-component case is also considered in [32, 34] but due to different smoothness assumptions on the membrane across the phase interface the resulting equilibrium equations slightly differ from our equations. We will come back to this issue in Appendix B.

Using the surface calculus again we will proceed with the membrane energy (1.5) and derive the *diffuse interface model*. Varying the line energy $\mathcal{F}_L$ requires tangential membrane deformations but now we restrict it to the normal direction because energy changes due to deformations of the phase interface in lateral direction correspond to variations of the order parameter. In both models we take hard constraints on the surface areas of the two phases and the volume of the enclosed domain into account. We refer to [29] for the physical regime where this is of relevance.

Finally, an asymptotic analysis is performed by matching suitable asymptotic ε-expansions in the bulk domain with others in the interfacial layers. This way we recover the equations of the sharp interface model in the limit as the interface thickness tends to zero ($\varepsilon \to 0$). The essential difficulty is that the membrane surface itself depends on ε so that standard techniques have to be extended. In Appendix A we relate the equations of the sharp interface model to those in [10, 11] for axisymmetric surfaces.

2. Notation, material derivative and Leibniz formula.

2.1. Calculus on surfaces. In this section we consider smooth oriented two-dimensional hypersurfaces $\tilde{\Gamma} \subset \mathbb{R}^3$ which, if not closed, have smooth boundaries $\partial\tilde{\Gamma}$. To fix the orientation let $\boldsymbol{\nu} = (\nu_i)_{i=1}^3$ denote the a unit normal field on $\tilde{\Gamma}$. Further, let $\boldsymbol{\mu} = (\mu_i)_{i=1}^3$ denote the outer co-normal of $\tilde{\Gamma}$ on $\partial\tilde{\Gamma}$, i.e., $\boldsymbol{\mu}$ is tangential to $\tilde{\Gamma}$ and normal to $\partial\tilde{\Gamma}$.

For any function η defined on a neighborhood $\mathcal{N} \subset \mathbb{R}^3$ of $\tilde{\Gamma}$ we define its tangential gradient on $\tilde{\Gamma}$ by $\nabla_{\tilde{\Gamma}}\eta := \nabla\eta - \nabla\eta \cdot \boldsymbol{\nu}\,\boldsymbol{\nu}$ where $\cdot$ denotes the usual scalar product and $\nabla\eta$ denotes the usual gradient on $\mathbb{R}^3$. The tangential gradient $\nabla_{\tilde{\Gamma}}\eta$ only depends on the values of η restricted to $\tilde{\Gamma}$, and $\nabla_{\tilde{\Gamma}}\eta \cdot \boldsymbol{\nu} = 0$. The components of the tangential gradient will be denoted by $\nabla_{\tilde{\Gamma}}\eta = (\underline{D}_i\eta)_{i=1}^3$.

If $\boldsymbol{w} = (w_i)_{i=1}^3, \boldsymbol{z} = (z_i)_{i=1}^3 : \tilde{\Gamma} \to \mathbb{R}^3$ are smooth vector fields then $\nabla_{\tilde{\Gamma}}\boldsymbol{w}$ is the matrix with components $(\nabla_{\tilde{\Gamma}}\boldsymbol{w})_{ij} = \underline{D}_j w_i$, and we write $(\nabla_{\tilde{\Gamma}}\boldsymbol{w})^\perp = (\underline{D}_i w_j)_{i,j}$ for its transpose and use the scalar product $\nabla_{\tilde{\Gamma}}\boldsymbol{w} : \nabla_{\tilde{\Gamma}}\boldsymbol{z} = \sum_{i,j} \underline{D}_j w_i \underline{D}_j z_i$. We will furthermore use the notation $\boldsymbol{w} \otimes \boldsymbol{z}$ for the matrix with entries $w_i z_j$. The surface divergence is defined by $\nabla_{\tilde{\Gamma}} \cdot \boldsymbol{w} = \text{tr}(\nabla_{\tilde{\Gamma}}\boldsymbol{w})$. The Laplace-Beltrami operator on $\tilde{\Gamma}$ is defined as the tangential divergence of the tangential gradient, $\Delta_{\tilde{\Gamma}}\eta = \nabla_{\tilde{\Gamma}} \cdot \nabla_{\tilde{\Gamma}}\eta$.

At any point $\boldsymbol{x} \in \tilde{\Gamma}$ we define the projection $\boldsymbol{P}(\boldsymbol{x}) := \boldsymbol{I} - \boldsymbol{\nu}(\boldsymbol{x}) \otimes \boldsymbol{\nu}(\boldsymbol{x}) \in \mathbb{R}^{3\times3}$ (where $\boldsymbol{I}$ is the identity matrix) to the tangent space $T_{\boldsymbol{x}}\tilde{\Gamma}$. With the help of $\boldsymbol{P}$ we can write $\nabla_{\tilde{\Gamma}}\eta = \boldsymbol{P}\nabla\eta$, $\nabla_{\tilde{\Gamma}}\boldsymbol{w} = \nabla\boldsymbol{w}\boldsymbol{P}$, $\nabla_{\tilde{\Gamma}} \cdot \boldsymbol{w} = \boldsymbol{P} : \nabla_{\tilde{\Gamma}}\boldsymbol{w}$.

The mean curvature of $\tilde{\Gamma}$ with respect to $\boldsymbol{\nu}$ is defined by

$$\kappa = -\nabla_{\tilde{\Gamma}} \cdot \boldsymbol{\nu}. \tag{2.1}$$

The symmetric matrix $\nabla_{\tilde{\Gamma}} \boldsymbol{\nu}$ of the tangential derivatives of the normal field is known as the Weingarten map or shape operator. It satisfies $|\nabla_{\tilde{\Gamma}} \boldsymbol{\nu}|^2 = \kappa_1^2 + \kappa_2^2 = \kappa^2 - 2g$ where κ_i, $i = 1, 2$, are the principle curvatures. Further we have $\kappa = \kappa_1 + \kappa_2$ and $g = \kappa_1 \kappa_2$. We infer from this the formula

$$g = \frac{1}{2}\big(\kappa^2 - |\nabla_{\tilde{\Gamma}} \boldsymbol{\nu}|^2\big). \tag{2.2}$$

Furthermore, we recall the following identity:

$$\Delta_{\tilde{\Gamma}} \boldsymbol{\nu} = -|\nabla_{\tilde{\Gamma}} \boldsymbol{\nu}|^2 \boldsymbol{\nu} - \nabla_{\tilde{\Gamma}} \kappa. \tag{2.3}$$

The formula for partial integration on surfaces reads

$$\int_{\tilde{\Gamma}} \xi \underline{D}_i \eta = -\int_{\tilde{\Gamma}} \eta \underline{D}_i \xi - \int_{\tilde{\Gamma}} \xi \eta \kappa \boldsymbol{\nu}_i + \int_{\partial \tilde{\Gamma}} \xi \eta \boldsymbol{\mu}_i. \tag{2.4}$$

2.2. Boundary identities. Let us write $\tilde{\gamma} = \partial \tilde{\Gamma}$ for the boundary curve of a surface $\tilde{\Gamma}$ as in the previous subsection and let $\boldsymbol{\tau}$ denote the unit tangential field along $\tilde{\gamma}$ such that $(\boldsymbol{\tau}, \boldsymbol{\mu}, \boldsymbol{\nu})$ constitutes a positively oriented orthonormal basis in every point on $\tilde{\gamma}$. The notation $\nabla_{\tilde{\gamma}} f$ stands for the derivative of a field $f : \tilde{\gamma} \to \mathbb{R}$ along $\tilde{\gamma}$: Using a parameterization $\boldsymbol{r}(s)$ for the curve $\tilde{\gamma}$ we have that

$$\nabla_{\tilde{\gamma}} f = \frac{1}{|\partial_s \boldsymbol{r}(s)|} \partial_s (f \circ \boldsymbol{r})(s)\, \boldsymbol{\tau}.$$

The curvature vector of $\tilde{\gamma}$ is denoted by $\boldsymbol{h}$ and fulfills

$$\boldsymbol{h} = \nabla_{\tilde{\gamma}} \boldsymbol{\tau} \boldsymbol{\tau} = \frac{1}{|\partial_s \boldsymbol{r}(s)|} \partial_s \Big(\frac{\partial_s \boldsymbol{r}(s)}{|\partial_s \boldsymbol{r}(s)|} \Big). \tag{2.5}$$

It is normal to the curve whence we may write $\boldsymbol{h} = h_g \boldsymbol{\mu} + h_{\boldsymbol{\nu}} \boldsymbol{\nu}$. The quantity $h_g = \boldsymbol{h} \cdot \boldsymbol{\mu}$ is the *geodesic curvature* of $\tilde{\gamma}$ and $h_{\boldsymbol{\nu}} = \boldsymbol{h} \cdot \boldsymbol{\nu}$ is known as its *normal curvature* (with respect to $\tilde{\Gamma}$).

In analogy to (2.4) we have that

$$\int_{\tilde{\gamma}} \zeta \nabla_{\tilde{\gamma}} f = -\int_{\tilde{\gamma}} f \nabla_{\tilde{\gamma}} \zeta - \int_{\tilde{\gamma}} f \boldsymbol{h} \tag{2.6}$$

where no boundary term occurs since $\partial \tilde{\gamma} = \emptyset$.

Close to $\tilde{\gamma}$ we may extend the fields $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ to $\tilde{\Gamma}$. For this purpose, we consider the distance of a point $\boldsymbol{x} \in \tilde{\Gamma}$ to $\tilde{\gamma}$ and define the function

$$d(\boldsymbol{x}) := \mathrm{dist}_{\tilde{\Gamma}}(\boldsymbol{x}, \tilde{\gamma}) = \inf \Big\{ \int_0^1 \|\boldsymbol{g}'(y)\| dy \,\Big|\, \boldsymbol{g} \in C^1([0,1], \tilde{\Gamma}),\, \boldsymbol{g}(0) = \boldsymbol{x},\, \boldsymbol{g}(1) \in \tilde{\gamma} \Big\}. \tag{2.7}$$

By the smoothness of $\tilde{\Gamma}$ there is a thin tube around $\tilde{\gamma}$ such that for each point $\boldsymbol{x}$ in that tube there is a unique geodesic realizing the distance. We define $\boldsymbol{\mu}(\boldsymbol{x}) := -\nabla_{\tilde{\Gamma}} d(\boldsymbol{x})$ and choose (the unique) $\boldsymbol{\tau}(\boldsymbol{x})$ such that $(\boldsymbol{\tau}, \boldsymbol{\mu}, \boldsymbol{\nu})$ is a positively oriented orthonormal basis again.

By the smoothness of $\tilde{\Gamma}$ the differentials $\nabla_{\tilde{\Gamma}}\boldsymbol{\tau}$, $\nabla_{\tilde{\Gamma}}\boldsymbol{\mu}$, and $\nabla_{\tilde{\Gamma}}\boldsymbol{\nu}$ then have limits when approaching $\tilde{\gamma}$. From $0 = \boldsymbol{\nu} \cdot \boldsymbol{\tau}$ we conclude that

$$\boldsymbol{\tau} \cdot \nabla_{\tilde{\Gamma}}\boldsymbol{\nu}\boldsymbol{\tau} = \boldsymbol{\tau} \cdot \nabla_{\tilde{\gamma}}\boldsymbol{\nu}\boldsymbol{\tau} = -\boldsymbol{\nu} \cdot \nabla_{\tilde{\gamma}}\boldsymbol{\tau}\boldsymbol{\tau} = -\boldsymbol{\nu} \cdot \boldsymbol{h} = -h_{\boldsymbol{\nu}}$$

which is an extension of the normal curvature of $\tilde{\gamma}$ to the surrounding tube. We also define the quantities

$$h_p := -\boldsymbol{\mu}\nabla_{\tilde{\Gamma}}\boldsymbol{\nu}\boldsymbol{\mu}, \quad h_d := -\boldsymbol{\tau} \cdot \nabla_{\tilde{\Gamma}}\boldsymbol{\nu}\boldsymbol{\mu} \, (= -\boldsymbol{\mu} \cdot \nabla_{\tilde{\Gamma}}\boldsymbol{\nu}\boldsymbol{\tau}).$$

It can easily shown that

$$\kappa = h_{\boldsymbol{\nu}} + h_p, \quad |\nabla_{\tilde{\Gamma}}\boldsymbol{\nu}|^2 = h_{\boldsymbol{\nu}}^2 + h_p^2 + 2h_d^2, \quad g = h_{\boldsymbol{\nu}} h_p - h_d^2. \tag{2.8}$$

2.3. Leibniz formulae. Deforming a surface leads to the notion of an evolving surface $\{\tilde{\Gamma}(\tau)\}_{\tau}$ depending smoothly on a *time* parameter $\tau \in (-\tau_0, \tau_0)$. We will usually omit the dependence of fields and surfaces on τ since it is clear from the context whether we deal with the evolving surface or a surface at a specific time. In particular, we just write $\nabla_{\tilde{\Gamma}}$ for $\nabla_{\tilde{\Gamma}(\tau)}$ whence this operator contains only spatial derivatives but no time derivatives.

With each mass point $\boldsymbol{x} \in \tilde{\Gamma}$ we can associate a vector field $\boldsymbol{v}$ which is its material velocity. Given the normal, one can decompose the velocity in the form $\boldsymbol{v} = v_{\boldsymbol{\nu}}\boldsymbol{\nu} + \boldsymbol{v}_T$ into a scalar normal component $v_{\boldsymbol{\nu}} := \boldsymbol{v} \cdot \boldsymbol{\nu}$ and a tangential vector field $\boldsymbol{v}_T := \boldsymbol{v} - v_{\boldsymbol{\nu}}\boldsymbol{\nu}$.

By $\partial_{\tau}^{\bullet}$ we denote the *material derivative* of a scalar function η defined on the evolving surface $\{\tilde{\Gamma}(\tau)\}_{\tau}$. Occasionally we will also use the *normal time derivative* which is the material derivative where only the normal contribution of the velocity is taken into account: $\partial_{\tau}^{\circ}\eta = \eta_t + v_{\boldsymbol{\nu}}\frac{\partial \eta}{\partial \boldsymbol{\nu}}$. As a consequence of the splitting of $\boldsymbol{v}$ into a normal an a tangential part we have the relation $\partial_{\tau}^{\bullet}\eta = \partial_{\tau}^{\circ}\eta + \boldsymbol{v}_T \cdot \nabla_{\tilde{\Gamma}}\eta$.

The following formulae for the differentiation of a parameter dependent surface integral will play a decisive role.

LEMMA 2.1 (Leibniz Formula). *Let* $\{\tilde{\Gamma}(\tau)\}_{\tau}$ *be an evolving surface and* η, ψ *be smooth scalar fields on* $\tilde{\Gamma}$ *such that all the following integrals exist. Then*

$$\frac{d}{d\tau} \int_{\tilde{\Gamma}} \eta = \int_{\tilde{\Gamma}} (\partial_{\tau}^{\bullet}\eta + \eta \, \nabla_{\tilde{\Gamma}} \cdot \boldsymbol{v}) \tag{2.9}$$

$$= \int_{\tilde{\Gamma}} (\partial_{\tau}^{\circ}\eta - \eta v_{\boldsymbol{\nu}}\kappa + \nabla_{\tilde{\Gamma}} \cdot (\eta \boldsymbol{v}_T)) = \int_{\tilde{\Gamma}} (\partial_t^{\circ}\eta - \eta v_{\boldsymbol{\nu}}\kappa) + \int_{\partial\tilde{\Gamma}} \eta \boldsymbol{v}_T \cdot \boldsymbol{\mu} \tag{2.10}$$

where we used (2.4) *for the last identity.*
With the rate of deformation tensor $D(\boldsymbol{v})_{ij} = \frac{1}{2}\left(\underline{D}_i\boldsymbol{v}_j + \underline{D}_j\boldsymbol{v}_i\right)$ $(i, j = 1, \ldots, n)$,

$$\frac{d}{d\tau} \int_{\tilde{\Gamma}} \nabla_{\tilde{\Gamma}}\eta \cdot \nabla_{\tilde{\Gamma}}\psi = \int_{\tilde{\Gamma}} \nabla_{\tilde{\Gamma}}\psi \cdot \nabla_{\tilde{\Gamma}}\partial_{\tau}^{\bullet}\eta + \int_{\tilde{\Gamma}} \nabla_{\tilde{\Gamma}}\partial_{\tau}^{\bullet}\psi \cdot \nabla_{\tilde{\Gamma}}\eta$$

$$+ \int_{\tilde{\Gamma}} \nabla_{\tilde{\Gamma}}\eta \cdot (\nabla_{\tilde{\Gamma}} \cdot \boldsymbol{v} - 2D(\boldsymbol{v}))\nabla_{\tilde{\Gamma}}\psi \tag{2.11}$$

A proof of this Lemma is given in [13].

Further useful formulae are

$$\partial_{\tau}^{\circ}\boldsymbol{\nu} = -\nabla_{\tilde{\Gamma}}(\boldsymbol{v} \cdot \boldsymbol{\nu}) = -\nabla_{\tilde{\Gamma}}v_{\boldsymbol{\nu}}, \quad \partial_{\tau}^{\bullet}\boldsymbol{\nu} = -(\nabla_{\tilde{\Gamma}}\boldsymbol{v})^{\perp}\boldsymbol{\nu} \tag{2.12}$$

$$\partial_{\tau}^{\circ}\kappa = \Delta_{\tilde{\Gamma}}(\boldsymbol{\nu} \cdot \boldsymbol{v}) + |\nabla_{\tilde{\Gamma}}\boldsymbol{\nu}|^2 \boldsymbol{\nu} \cdot \boldsymbol{v} \tag{2.13}$$
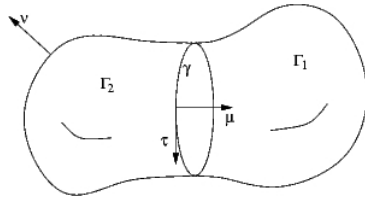
FIG. 3.1. *Sketch of an admissible two-phase membrane with vectors $\boldsymbol{\tau}$, $\boldsymbol{\mu}$, and $\boldsymbol{\nu}$.*

as proved in [19]. We conclude this section with some formulae on the boundary curves $\tilde{\gamma}(\tau) = \partial\tilde{\Gamma}(\tau)$ where, as for $\tilde{\Gamma}$, the dependence on τ usually is omitted. Clearly, $\partial_\tau^\bullet \boldsymbol{\tau}$ is orthogonal to $\boldsymbol{\tau}$ since the vector field has unit length. A short calculation yields that

$$\partial_\tau^\bullet \boldsymbol{\tau} = \boldsymbol{P}_{\tilde{\gamma}^\perp} \nabla_{\tilde{\gamma}} \boldsymbol{v}\boldsymbol{\tau} \quad \text{where } \boldsymbol{P}_{\tilde{\gamma}^\perp} = \boldsymbol{I} - \boldsymbol{\tau} \otimes \boldsymbol{\tau} \tag{2.14}$$

is the projection to the plane normal to $\tilde{\gamma}$. The material derivative of $\boldsymbol{h}$ is obtained by computing the derivative with respect to τ of the right hand side of (2.5) where $\boldsymbol{r}(s)$ is replaced by $\boldsymbol{r}(\tau, s)$ with $\partial_\tau \boldsymbol{r} = \boldsymbol{v}$:

$$\partial_\tau^\bullet \boldsymbol{h} = -\nabla_{\tilde{\gamma}} \cdot \boldsymbol{v}\,\boldsymbol{h} + \nabla_{\tilde{\gamma}}\big(\boldsymbol{P}_{\tilde{\gamma}^\perp} \nabla_{\tilde{\gamma}} \boldsymbol{v}\boldsymbol{\tau}\big)\boldsymbol{\tau} \tag{2.15}$$

3. Sharp Interface Equilibrium Equations. Using the calculus presented in the previous section we will now derive the Euler-Lagrange equations of the membrane energy $\mathcal{F}_{SI}$ defined in (1.2) under constraints on enclosed volume and areas of the two membrane phases.

DEFINITION 3.1. *An* **admissible two-phase surface** *for $\mathcal{F}_{SI}$ is the boundary Γ of a bounded, simply connected open domain $\Omega \subset \mathbb{R}^3$ and can be decomposed in the form $\Gamma = \Gamma_1 \cup \gamma \cup \Gamma_2$ where*
- *Γ_1 and Γ_2 are two-dimensional smooth oriented not necessarily connected hypersurfaces with smooth boundaries that coincide and correspond to γ which consists of a finite number of smooth curves, $\partial\Gamma_1 = \partial\Gamma_2 = \gamma$,*
- *locally around γ the surface Γ can be parametrized by a C^1 map.*

Limits of quantities on γ that may be discontinuous will get an upper index of the form $^{(1)}$ or $^{(2)}$ depending on whether γ is approached from Γ_1 or Γ_2, and by $[\cdot]_{(1)}^{(2)} = (\cdot)^{(2)} - (\cdot)^{(1)}$ we denote the jump across γ. We denote by $\boldsymbol{\mu}$ the outer co-normal of Γ_2. We also recall the notation of $\boldsymbol{\tau}$ for the unit tangential vector field along γ such that $(\boldsymbol{\tau}, \boldsymbol{\mu}, \boldsymbol{\nu})$ is positively oriented (cf. Figure 3.1). After extension as described around (2.7) these vector fields are continuous across γ since Γ is C^1. Differentiating the jump $[\boldsymbol{\nu}]_{(1)}^{(2)} = 0$ along γ, i.e., in the direction $\boldsymbol{\tau}$, we see that $\nabla_\Gamma \boldsymbol{\nu}\boldsymbol{\tau}$ is continuous across γ (also see [22]) so that $h_{\boldsymbol{\nu}} = -\boldsymbol{\tau} \cdot \nabla_\Gamma \boldsymbol{\nu}\boldsymbol{\tau}$ and $h_d = -\boldsymbol{\mu} \cdot \nabla_\Gamma \boldsymbol{\nu}\boldsymbol{\tau}$ are continuous, too. Observe that $h_p = -\boldsymbol{\mu} \cdot \nabla_\Gamma \boldsymbol{\nu}\boldsymbol{\mu}$ may be discontinuous across γ.

REMARK 3.2. *The regularity assumptions on $h_{\boldsymbol{\nu}}$ and h_p are consistent with the approaches in [23, 24, 5, 11] where surfaces of revolution are considered (implying that $h_d = 0$). See also Appendix A. We refer to [21] for a study of the axisymmetric case where the assumption of Γ being C^1 across γ is dropped.*

DEFINITION 3.3. *Assume that positive numbers V, A_1, and A_2 are given such that*

$$A_1 + A_2 \geq 4\pi(3V/4\pi)^{2/3}. \tag{3.1}$$

For admissible two-phase membrane $\Gamma = \Gamma_1 \cup \gamma \cup \Gamma_2$ *enclosing the domain* Ω *the* **constraint functionals** *are*

$$\mathcal{C}_V(\Gamma) := |\Omega| - V = \frac{1}{3} \int_\Gamma \boldsymbol{x} \cdot \boldsymbol{\nu} - V, \qquad \mathcal{C}_{A_i}(\Gamma) := |\Gamma_i| - A_i, \quad i = 1, 2.$$

We remark that (3.1) is a natural requirement due to the fact that the sphere minimizes the area enclosing a given volume. The constraints then read

$$\mathcal{C}_V(\Gamma) = 0, \qquad \mathcal{C}_{A_i}(\Gamma) = 0, \quad i = 1, 2, . \tag{3.2}$$

DEFINITION 3.4. *Suppose that an admissible two-phase surface* $\Gamma = \Gamma_1 \cup \gamma \cup \Gamma_2$ *is given as well as a map* $\boldsymbol{w} \in C^1(\Gamma; \mathbb{R}^3)$ *which is smooth on* Γ_1, Γ_2, *and on* γ. *The deformed surface* $\Gamma(\tau) = \Gamma_1(\tau) \cup \gamma(\tau) \cup \Gamma_2(\tau)$ *in the direction* $\boldsymbol{w}$ *is defined by*

$$\Gamma(\tau) := \{\boldsymbol{x}(\tau) := \boldsymbol{x} + \tau \boldsymbol{w}(\boldsymbol{x}) \,|\, \boldsymbol{x} \in \Gamma\}.$$

Such a map $\boldsymbol{w}$ *is called* **admissible deformation field** *for the admissible two-phase surface* Γ *if there is a small* $\bar{\tau}$ *such that for all* $\tau \in (-\bar{\tau}, \bar{\tau})$ *the set* $\Gamma(\tau)$ *is an admissible two-phase surface.*

In the context of the above definition, the fields $\boldsymbol{w}$, $\boldsymbol{P}\boldsymbol{w}$, $\nabla_\Gamma \boldsymbol{w}$ are continuous across γ yet the field $\nabla_\Gamma(\boldsymbol{w} \cdot \boldsymbol{\nu})$ may not be because $\nabla_\Gamma \boldsymbol{\nu}$ may be discontinuous. But the field $\nabla_\gamma(\boldsymbol{w} \cdot \boldsymbol{\nu})\boldsymbol{\tau}$ defined along γ is continuous.

DEFINITION 3.5. *Given a function* $\mathcal{E}$ *defined on admissible two-phase surfaces and given an admissible two-phase surface* Γ *and an admissible deformation field* $\boldsymbol{w}$ *the* **variation** *of* $\mathcal{E}$ *in direction* $\boldsymbol{w}$ *is defined by*

$$\langle \delta \mathcal{E}(\Gamma), \boldsymbol{w} \rangle := \frac{d}{d\tau} \mathcal{E}(\Gamma(\cdot)) \Big|_{\tau=0}.$$

LEMMA 3.6. **Variation of the bending energy.** *Assume that* $\Gamma = \Gamma_1 \cup \gamma \cup \Gamma_2$ *is an admissible two-phase membrane and* $\boldsymbol{w}$ *an admissible deformation field. Then*

$$\langle \delta \mathcal{F}_B(\Gamma), \boldsymbol{w} \rangle = \sum_{i=1,2} \int_{\Gamma_i} k_\kappa^{(i)} \Big(\Delta_\Gamma \kappa + |\nabla_\Gamma \boldsymbol{\nu}|^2 (\kappa - \kappa_s^{(i)}) - \frac{1}{2}(\kappa - \kappa_s^{(1)})^2 \kappa \Big) \boldsymbol{\nu} \cdot \boldsymbol{w}$$

$$+ \int_\gamma \big[k_\kappa (h_{\boldsymbol{\nu}} + h_p - \kappa_s) \big]_{(1)}^{(2)} \boldsymbol{\nu} \cdot \nabla_\Gamma \boldsymbol{w} \boldsymbol{\mu} - \big[k_\kappa \nabla_\Gamma (h_{\boldsymbol{\nu}} + h_p) \cdot \boldsymbol{\mu} \big]_{(1)}^{(2)} \boldsymbol{\nu} \cdot \boldsymbol{w}$$

$$+ \int_\gamma \big[\tfrac{1}{2} k_\kappa (h_{\boldsymbol{\nu}} + h_p - \kappa_s)^2 \boldsymbol{\mu} + k_\kappa (h_{\boldsymbol{\nu}} + h_p - \kappa_s) \nabla_\Gamma \boldsymbol{\nu} \boldsymbol{\mu} \big]_{(1)}^{(2)} \cdot \boldsymbol{P} \boldsymbol{w}$$

$$+ \int_\gamma \big[k_g h_{\boldsymbol{\nu}} \big]_{(1)}^{(2)} \boldsymbol{\nu} \cdot \nabla_\Gamma \boldsymbol{w} \boldsymbol{\mu} + \big[k_g \big]_{(1)}^{(2)} (\nabla_\gamma h_d \cdot \boldsymbol{\tau}) (\boldsymbol{\nu} \cdot \boldsymbol{w})$$

$$+ \int_\gamma \big[k_g (h_{\boldsymbol{\nu}} h_p - h_d^2) \boldsymbol{\mu} + k_g h_{\boldsymbol{\nu}} \nabla_\Gamma \boldsymbol{\nu} \boldsymbol{\mu} \big]_{(1)}^{(2)} \cdot \boldsymbol{P} \boldsymbol{w}. \tag{3.3}$$

Proof. We start with the mean curvature bending terms in (1.2). The Leibniz

formula (2.10) together with identity (2.13) yields

$$\frac{d}{d\tau}\int_{\Gamma_1(\tau)}\frac{k_\kappa^{(1)}}{2}\big|\kappa(\cdot)-\kappa_s^{(1)}\big|^2\Big|_{\tau=0}$$

$$=\int_{\Gamma_1}k_\kappa^{(1)}\Big(\partial_t^\circ(\kappa-\kappa_s^{(1)})\Big)(\kappa-\kappa_s^{(1)})-\frac{1}{2}k_\kappa^{(1)}(\kappa-\kappa_s^{(1)})^2\boldsymbol{\kappa}\cdot\boldsymbol{w}$$

$$+\int_\gamma\frac{1}{2}k_\kappa^{(1)}(\kappa^{(1)}-\kappa_s^{(1)})^2(-\boldsymbol{\mu})\cdot\boldsymbol{w}$$

$$=\int_{\Gamma_1}k_\kappa^{(1)}\big(\Delta_\Gamma(\boldsymbol{\nu}\cdot\boldsymbol{w})+|\nabla_{\Gamma_1}\boldsymbol{\nu}|^2\boldsymbol{\nu}\cdot\boldsymbol{w}\big)(\kappa-\kappa_s^{(1)})-\frac{1}{2}k_\kappa^{(1)}(\kappa-\kappa_s^{(1)})^2\kappa\boldsymbol{\nu}\cdot\boldsymbol{w}$$

$$+\int_\gamma-\frac{1}{2}k_\kappa^{(1)}(\kappa^{(1)}-\kappa_s^{(1)})^2\boldsymbol{\mu}\cdot\boldsymbol{P}\boldsymbol{w},$$

and applying (2.4) twice to the first and once to the last term gives

$$=\int_{\Gamma_1}\Big(k_\kappa^{(1)}\Delta_\Gamma\kappa+k_\kappa^{(1)}|\nabla_{\Gamma_1}\boldsymbol{\nu}|^2(\kappa-\kappa_s^{(1)})-\frac{1}{2}k_\kappa^{(1)}(\kappa-\kappa_s^{(1)})^2\kappa\Big)(\boldsymbol{\nu}\cdot\boldsymbol{w})$$

$$+\int_\gamma k_\kappa^{(1)}(\nabla_\Gamma\kappa)^{(1)}\cdot\boldsymbol{\mu}(\boldsymbol{\nu}\cdot\boldsymbol{w})-k_\kappa^{(1)}(\kappa^{(1)}-\kappa_s^{(1)})\big(\nabla_\Gamma(\boldsymbol{\nu}\cdot\boldsymbol{w})\big)^{(1)}\cdot\boldsymbol{\mu}$$

$$+\int_\gamma-\frac{1}{2}k_\kappa^{(1)}(\kappa^{(1)}-\kappa_s^{(1)})^2\boldsymbol{\mu}\cdot\boldsymbol{P}\boldsymbol{w}$$

$$=\int_{\Gamma_1}\Big(k_\kappa^{(1)}\Delta_{\Gamma_1}\kappa+k_\kappa^{(1)}|\nabla_{\Gamma_1}\boldsymbol{\nu}|^2(\kappa-\kappa_s^{(1)})-\frac{1}{2}k_\kappa^{(1)}(\kappa-\kappa_s^{(1)})^2\kappa\Big)(\boldsymbol{\nu}\cdot\boldsymbol{w}) \qquad (3.4)$$

$$+\int_\gamma k_\kappa^{(1)}\big(\nabla_\Gamma(h_{\boldsymbol{\nu}}+h_p)\big)^{(1)}\cdot\boldsymbol{\mu}(\boldsymbol{\nu}\cdot\boldsymbol{w})-k_\kappa^{(1)}(h_{\boldsymbol{\nu}}+h_p^{(1)}-\kappa_s^{(1)})\boldsymbol{\nu}\cdot\nabla_\Gamma\boldsymbol{w}\boldsymbol{\mu}$$

$$+\int_\gamma-\frac{1}{2}k_\kappa^{(1)}(h_{\boldsymbol{\nu}}+h_p^{(1)}-\kappa_s^{(1)})^2\boldsymbol{\mu}\cdot\boldsymbol{P}\boldsymbol{w}-k_\kappa^{(1)}(h_{\boldsymbol{\nu}}+h_p^{(1)}-\kappa_s^{(1)})\big(\nabla_\Gamma\boldsymbol{\nu}\boldsymbol{\mu}\big)^{(1)}\cdot\boldsymbol{w}.$$

Since $\nabla_\Gamma\boldsymbol{\nu}$ is a tangential tensor we may replace $\boldsymbol{w}$ by $\boldsymbol{P}\boldsymbol{w}$ in the last term. The computation of the variation of the mean curvature bending energy of Γ_2 is similar. Adding the terms together we recover the first three lines on the right hand side of the asserted identity (3.3).

With respect to the Gaussian curvature terms in (1.2) we observe that thanks to the Gauss-Bonnet formula $\int_{\Gamma_1}k_g^{(1)}g=2\pi k_g^{(1)}-\int_\gamma k_g^{(1)}h_g$ and $\int_{\Gamma_2}k_g^{(2)}g=2\pi k_g^{(2)}+\int_\gamma k_g^{(2)}h_g$. As $\{\gamma(\tau)\}_\tau$ is just a one-dimensional evolving surface with velocity field $\boldsymbol{w}$ we can apply (2.9) to see that

$$\frac{d}{d\tau}\int_{\gamma(\cdot)}k_g^{(2)}h_g(\cdot)\Big|_{\tau=0}=k_g^{(2)}\int_\gamma\partial_\tau^\bullet\boldsymbol{h}\cdot\boldsymbol{\mu}+\boldsymbol{h}\cdot\partial_\tau^\bullet\boldsymbol{\mu}+h_g\nabla_\gamma\cdot\boldsymbol{w}. \qquad (3.5)$$

Using the orthonormality of $\boldsymbol{\mu}$, $\boldsymbol{\tau}$, and $\boldsymbol{\nu}$ and the identities (2.14) and (2.12) we obtain

$$\partial_\tau^\bullet\boldsymbol{\mu}=\partial_\tau^\bullet\boldsymbol{\mu}\cdot\boldsymbol{\tau}\,\boldsymbol{\tau}+\partial_\tau^\bullet\boldsymbol{\mu}\cdot\boldsymbol{\nu}\,\boldsymbol{\nu}$$

$$=-\boldsymbol{\mu}\cdot\partial_\tau^\bullet\boldsymbol{\tau}\,\boldsymbol{\tau}-\boldsymbol{\mu}\cdot\partial_\tau^\bullet\boldsymbol{\nu}\,\boldsymbol{\nu}=-(\boldsymbol{\mu}\cdot\nabla_\gamma\boldsymbol{w}\,\boldsymbol{\tau})\boldsymbol{\tau}+(\boldsymbol{\nu}\cdot\nabla_\Gamma\boldsymbol{w}\,\boldsymbol{\mu})\boldsymbol{\nu}.$$

Thanks to (2.15) and since $\boldsymbol{h}$ is orthogonal to $\boldsymbol{\tau}$ we obtain from (3.5)

$$\frac{d}{d\tau}\int_{\gamma(\cdot)}k_g^{(2)}h_g(\cdot)\Big|_{\tau=0}=k_g^{(2)}\int_\gamma\big(\nabla_\gamma(\boldsymbol{P}_{\gamma^\perp}\nabla_\gamma\boldsymbol{w}\,\boldsymbol{\tau})\boldsymbol{\tau}\big)\cdot\boldsymbol{\mu}+k_g^{(2)}\int_\gamma h_{\boldsymbol{\nu}}(\boldsymbol{\nu}\cdot\nabla_\Gamma\boldsymbol{w}\,\boldsymbol{\mu}).$$

When integrating by parts with respect to the first term this is

$$= k_g^{(2)} \int_\gamma -(\nabla_\gamma \boldsymbol{\mu}\boldsymbol{\tau}) \cdot \boldsymbol{P}_{\gamma^\perp} \nabla_\gamma \boldsymbol{w}\boldsymbol{\tau} + k_g^{(2)} \int_\gamma h_{\boldsymbol{\nu}}(\boldsymbol{\nu} \cdot \nabla_\Gamma \boldsymbol{w}\boldsymbol{\mu})$$
$$=: I + II. \tag{3.6}$$

The vector field $\nabla_\gamma \boldsymbol{\mu}\boldsymbol{\tau}$ is orthogonal to $\boldsymbol{\mu}$, and since $\boldsymbol{P}_{\gamma^\perp}$ maps vectors to the space $\text{span}\{\boldsymbol{\mu}, \boldsymbol{\nu}\}$ in each point on γ we have that

$$- (\nabla_\gamma \boldsymbol{\mu}\boldsymbol{\tau}) \cdot \boldsymbol{P}_{\gamma^\perp} \nabla_\gamma \boldsymbol{w}\boldsymbol{\tau} = -(\boldsymbol{\nu} \cdot \nabla_\gamma \boldsymbol{\mu}\boldsymbol{\tau})(\nabla_\gamma \boldsymbol{w}\boldsymbol{\tau} \cdot \boldsymbol{\nu})$$
$$= (\boldsymbol{\mu} \cdot \nabla_\gamma \boldsymbol{\nu}\boldsymbol{\tau})(\boldsymbol{\nu} \cdot \nabla_\gamma \boldsymbol{w}\boldsymbol{\tau}) = -h_d(\boldsymbol{\nu} \cdot \nabla_\gamma \boldsymbol{w}\boldsymbol{\tau}).$$

We now split the deformation field in the form

$$\boldsymbol{w} = (\boldsymbol{w} \cdot \boldsymbol{\nu})\boldsymbol{\nu} + (\boldsymbol{w} \cdot \boldsymbol{\mu})\boldsymbol{\mu} + (\boldsymbol{w} \cdot \boldsymbol{\tau})\boldsymbol{\tau} =: w_{\boldsymbol{\nu}}\boldsymbol{\nu} + w_{\boldsymbol{\mu}}\boldsymbol{\mu} + w_{\boldsymbol{\gamma}}\boldsymbol{\tau}. \tag{3.7}$$

We assume that $w_\gamma = 0$ because deformations in direction $\boldsymbol{\tau}$ neither change the membrane shape nor the position of the phase interface and, hence, don't result in any energy change. In fact, if one kept the terms with w_γ in the following one would see that the contributions to I and II cancel each other. We then have that

$$\nabla_\gamma \boldsymbol{w} = \boldsymbol{\nu} \otimes \nabla_\gamma w_{\boldsymbol{\nu}} + \nabla_\gamma \boldsymbol{\nu} \, w_{\boldsymbol{\nu}} + \boldsymbol{\mu} \otimes \nabla_\gamma w_{\boldsymbol{\mu}} + \nabla_\gamma \boldsymbol{\mu} \, w_{\boldsymbol{\mu}}$$

and then

$$\boldsymbol{\nu} \cdot \nabla_\gamma \boldsymbol{w}\boldsymbol{\tau} = \nabla_\gamma w_{\boldsymbol{\nu}} \cdot \boldsymbol{\tau} + \boldsymbol{\nu} \cdot \nabla_\gamma \boldsymbol{\mu}\boldsymbol{\tau} \, w_{\boldsymbol{\mu}}$$
$$= \nabla_\gamma w_{\boldsymbol{\nu}} \cdot \boldsymbol{\tau} - \boldsymbol{\mu} \cdot \nabla_\gamma \boldsymbol{\nu}\boldsymbol{\tau} \, w_{\boldsymbol{\mu}} = \nabla_\gamma w_{\boldsymbol{\nu}} \cdot \boldsymbol{\tau} + h_d w_{\boldsymbol{\mu}}.$$

Altogether we end up with

$$I = k_g^{(2)} \int_\gamma -h_d \nabla_\gamma w_{\boldsymbol{\nu}} \cdot \boldsymbol{\tau} - h_d^2 w_{\boldsymbol{\mu}} = k_g^{(2)} \int_\gamma \nabla_\gamma h_d \cdot \boldsymbol{\tau} \, w_{\boldsymbol{\nu}} - h_d^2 w_{\boldsymbol{\mu}}. \tag{3.8}$$

We also want to employ the splitting (3.7) to deal with II but we have to deal with the fact that $\nabla_\Gamma w_{\boldsymbol{\nu}} = \nabla_\Gamma(\boldsymbol{w} \cdot \boldsymbol{\nu})$ may not be continuous across γ. Since we are computing the variation of $\int_{\Gamma_2} k_g^{(2)} g$ we consider limits of the fields on γ when approaching it from Γ_2 and indicate this by the usual upper index wherever necessary. From the splitting (3.7) we get

$$\nabla_\Gamma \boldsymbol{w} = \boldsymbol{\nu} \otimes (\nabla_\Gamma w_{\boldsymbol{\nu}})^{(2)} + (\nabla_\Gamma \boldsymbol{\nu})^{(2)} \, w_{\boldsymbol{\nu}} + \boldsymbol{\mu} \otimes \nabla_\Gamma w_{\boldsymbol{\mu}} + (\nabla_\Gamma \boldsymbol{\mu})^{(2)} \, w_{\boldsymbol{\mu}}$$

which yields

$$\boldsymbol{\nu} \cdot \nabla_\Gamma \boldsymbol{w}\boldsymbol{\mu} = (\nabla_\Gamma w_{\boldsymbol{\nu}})^{(2)} \cdot \boldsymbol{\mu} + \boldsymbol{\nu} \cdot (\nabla_\Gamma \boldsymbol{\mu})^{(2)} \boldsymbol{\mu} \, w_{\boldsymbol{\mu}}$$
$$= \nabla_\Gamma \boldsymbol{w}\boldsymbol{\mu} \cdot \boldsymbol{\nu} + (\nabla_\Gamma \boldsymbol{\nu}\boldsymbol{\mu})^{(2)} \cdot \boldsymbol{w} - \boldsymbol{\mu} \cdot (\nabla_\Gamma \boldsymbol{\nu})^{(2)} \boldsymbol{\mu} \, w_{\boldsymbol{\mu}}$$
$$= \boldsymbol{\nu} \cdot \nabla_\Gamma \boldsymbol{w}\boldsymbol{\mu} + (\nabla_\Gamma \boldsymbol{\nu}\boldsymbol{\mu})^{(2)} \cdot \boldsymbol{P}\boldsymbol{w} + h_p^{(2)} w_{\boldsymbol{\mu}}.$$

We finally end up with

$$II = k_g^{(2)} \int_\gamma h_{\boldsymbol{\nu}} \big(\boldsymbol{\nu} \cdot \nabla_\Gamma \boldsymbol{w}\boldsymbol{\mu} + (\nabla_\Gamma \boldsymbol{\nu}\boldsymbol{\mu})^{(2)} \cdot \boldsymbol{P}\boldsymbol{w} + h_p^{(2)} w_{\boldsymbol{\mu}} \big). \tag{3.9}$$

Writing $w_{\boldsymbol{\mu}} = \boldsymbol{\mu} \cdot \boldsymbol{w} = \boldsymbol{\mu} \cdot \boldsymbol{P}\boldsymbol{w}$ we get from (3.6), (3.8), (3.9)

$$
\frac{d}{d\tau} \int_{\gamma(\cdot)} k_g^{(2)} h_g(\cdot) \Big|_{\tau=0} = k_g^{(2)} \int_\gamma \Big((\nabla_\gamma h_d \cdot \boldsymbol{\tau})(\boldsymbol{\nu} \cdot \boldsymbol{w}) + (h_{\boldsymbol{\nu}} h_p^{(2)} - h_d^2)\boldsymbol{\mu} \cdot \boldsymbol{P}\boldsymbol{w}
$$
$$
+ h_{\boldsymbol{\nu}} \boldsymbol{\nu} \cdot \nabla_\Gamma \boldsymbol{w}\boldsymbol{\mu} + h_{\boldsymbol{\nu}}(\nabla_\Gamma \boldsymbol{\nu}\boldsymbol{\mu})^{(2)} \cdot \boldsymbol{P}\boldsymbol{w} \Big). \quad (3.10)
$$

Similarly for the variation of $\int_\gamma k_g^{(1)} h_g$, and we obtain the last two lines of the asserted identity (3.3) which completes the proof. $\square$

REMARK 3.7. *The formula for the variation of the Gaussian curvature term (3.10) is the same as the one obtained in the appendix of [4] where a different calculus has been used. To see this one has to observe that $\boldsymbol{\mu}$ is the inner co-normal in our notation and that the shape operator is denoted by $\boldsymbol{k} = -\nabla_\Gamma \boldsymbol{\nu}$.*

LEMMA 3.8. **Variation of line energy and constraint functionals.** *Assume that $\Gamma = \Gamma_1 \cup \gamma \cup \Gamma_2$ is an admissible two-phase membrane and that $\boldsymbol{w}$ is an admissible deformation field. Then*

$$
\langle \delta \mathcal{F}_L(\Gamma), \boldsymbol{w} \rangle = \int_\gamma -\bar{\sigma}\boldsymbol{h} \cdot \boldsymbol{w} = \int_\gamma -\bar{\sigma} h_g \boldsymbol{\mu} \cdot \boldsymbol{P}\boldsymbol{w} - \bar{\sigma} h_{\boldsymbol{\nu}} \boldsymbol{\nu} \cdot \boldsymbol{w}, \quad (3.11)
$$

$$
\langle \delta \mathcal{C}_V(\Gamma), \boldsymbol{w} \rangle = \int_\Gamma \boldsymbol{\nu} \cdot \boldsymbol{w}, \quad (3.12)
$$

$$
\langle \delta \mathcal{C}_{A_1}(\Gamma), \boldsymbol{w} \rangle = \int_{\Gamma_1} -\kappa \boldsymbol{\nu} \cdot \boldsymbol{w} + \int_\gamma (-\boldsymbol{\mu}) \cdot \boldsymbol{P}\boldsymbol{w}, \quad (3.13)
$$

$$
\langle \delta \mathcal{C}_{A_2}(\Gamma), \boldsymbol{w} \rangle = \int_{\Gamma_2} -\kappa \boldsymbol{\nu} \cdot \boldsymbol{w} + \int_\gamma \boldsymbol{\mu} \cdot \boldsymbol{P}\boldsymbol{w}. \quad (3.14)
$$

Proof. The first identity is consequence of the fact that $-\boldsymbol{h}$ is the variation of the length of γ. The subsequent identities follow from (2.10) applied to Ω and the Γ_i, respectively. $\square$

DEFINITION 3.9. *An admissible two-phase surface $\Gamma = \Gamma_1 \cup \gamma \cup \gamma_2$ is a* **critical point** *of* (1.2) *subject to the constraints* (3.2) *if for all admissible deformations $\boldsymbol{w}$*

$$
0 = \big\langle (\delta \mathcal{F}_B + \delta \mathcal{F}_L + \lambda_V \delta \mathcal{C}_V + \lambda_A^{(1)} \delta \mathcal{C}_{A_1} + \lambda_A^{(2)} \delta \mathcal{C}_{A_2})(\Gamma), \boldsymbol{w} \big\rangle
$$

where λ_V, $\lambda_A^{(1)}$, and $\lambda_A^{(2)}$ are appropriate Lagrange multipliers.

To formulate the Euler-Lagrange equations we use the identities (3.3), (3.11), (3.12), (3.13), and (3.14) and the fact that the fields $\boldsymbol{\nu} \cdot \boldsymbol{w}$, $\boldsymbol{P}\boldsymbol{w}$, and $\nabla_\Gamma \boldsymbol{w}\boldsymbol{\mu}$ are independent on γ. From the terms in duality with $\boldsymbol{P}\boldsymbol{w}$ we obtain

$$
0 = \big[\tfrac{k_\kappa}{2}(h_{\boldsymbol{\nu}} + h_p - \kappa_s)^2 \boldsymbol{\mu} + k_g(h_{\boldsymbol{\nu}} h_p - h_d^2)\boldsymbol{\mu} \big]_{(1)}^{(2)}
$$
$$
+ \big[(k_\kappa(h_{\boldsymbol{\nu}} + h_p - \kappa_s) + k_g h_{\boldsymbol{\nu}})\nabla_\Gamma \boldsymbol{\nu}\boldsymbol{\mu} \big]_{(1)}^{(2)} - \bar{\sigma} h_g \boldsymbol{\mu} + (\lambda_A^{(2)} - \lambda_A^{(1)})\boldsymbol{\mu}
$$

on γ. On may multiply with $\boldsymbol{\mu}$ this yields a scalar equation where we may replace $\nabla_\Gamma \boldsymbol{\nu}\boldsymbol{\mu} \cdot \boldsymbol{\mu} = -h_p$ which yields equation (3.18) below. We observe that multiplying with $\boldsymbol{\tau}$ does not give further equations since $\nabla_\Gamma \boldsymbol{\nu}\boldsymbol{\mu} \cdot \boldsymbol{\tau} = -h_d$ is continuous by assumption and $k_\kappa(h_{\boldsymbol{\nu}} + h_p - \kappa_s) + k_g h_{\boldsymbol{\nu}}$ is continuous, too, by condition (3.16) below.

PROBLEM 3.10. **Sharp interface equilibrium equations.** *For given real values V, A_1, A_2 fulfilling* (3.1) *find an admissible two-phase membrane $\Gamma = \Gamma_1 \cup \gamma \cup \Gamma_2$*

and find Lagrange multipliers λ_V, $\lambda_A^{(1)}$, and $\lambda_A^{(2)}$ such that

$$0 = k_\kappa^{(i)} \Delta_{\Gamma_i} \kappa + k_\kappa^{(i)} |\nabla_{\Gamma_i} \boldsymbol{\nu}|^2 (\kappa - \kappa_s^{(i)})$$
$$- \tfrac{1}{2} k_\kappa^{(i)} (\kappa - \kappa_s^{(i)})^2 \kappa + \lambda_V - \lambda_A^{(i)} \kappa \qquad \qquad on\ \Gamma_i,\ i = 1, 2, \quad (3.15)$$

$$0 = \left[k_\kappa (h_{\boldsymbol{\nu}} + h_p - \kappa_s) + k_g h_{\boldsymbol{\nu}} \right]_{(1)}^{(2)} \qquad \qquad on\ \gamma, \qquad \qquad (3.16)$$

$$0 = -\left[k_\kappa \nabla_\Gamma (h_{\boldsymbol{\nu}} + h_p) \right]_{(1)}^{(2)} \cdot \boldsymbol{\mu} + \left[k_g \right]_{(1)}^{(2)} \nabla_\gamma h_d \cdot \boldsymbol{\tau} - \bar{\sigma} h_{\boldsymbol{\nu}} \qquad on\ \gamma, \qquad \qquad (3.17)$$

$$0 = \left[\tfrac{k_\kappa}{2} (h_{\boldsymbol{\nu}} + h_p - \kappa_s)^2 + k_g (h_{\boldsymbol{\nu}} h_p - h_d^2) \right]_{(1)}^{(2)}$$
$$- \left[(k_\kappa (h_{\boldsymbol{\nu}} + h_p - \kappa_s) + k_g h_{\boldsymbol{\nu}}) h_p \right]_{(1)}^{(2)} - \bar{\sigma} h_g + \left[\lambda_A \right]_{(1)}^{(2)} \qquad on\ \gamma, \qquad \qquad (3.18)$$

$$0 = |\Omega| - V, \qquad \qquad (3.19)$$

$$0 = |\Gamma_i| - A_i,\ i = 1, 2. \qquad \qquad (3.20)$$

REMARK 3.11. *Equation (3.15) is the (normal) force balance for the membrane and is independent of the phase separation. On the phase interface, (3.16) is a continuity condition, and the conditions (3.17) and (3.18) can be considered as force balances in the normal and co-normal direction, respectively. We show in Appendix A that for axisymmetric shapes the conditions coincide with (1), (3)–(5) in [11].*

4. Diffuse Interface Equilibrium Equations. We now consider the phase field version (1.5) for the membrane energy.

DEFINITION 4.1. *An **admissible phase field surface** for the membrane energy (1.5) is the smooth boundary Γ of a bounded, simply connected open set $\Omega \subset \mathbb{R}^3$ together with a smooth field $c : \Gamma \to \mathbb{R}$ which is called **order parameter** or **phase field variable**.*

As specified in the introduction we are interested in critical points (Γ, c) of $\mathcal{F}(\cdot, \cdot)$ defined by (1.5) subject to side conditions concerning the areas of the two phases and the volume of the enclosed domain. To take the area constraints into account in the phase field model we consider the function $h(c) = \tfrac{1}{2} c(3 - c^2)$ if $-1 < c < 1$, $h(c) = 1$ if $c \geq 1$, and $h(c) = -1$ if $c \leq -1$ and impose a constraint on $\int_\Gamma h(c)$ and on $|\Gamma|$. In fact, in the limit as $\varepsilon \to 0$ one expects that $\int_\Gamma h(c) \to |\Gamma_1| - |\Gamma_2|$. Recalling that we want to preserve the areas of Γ_1 and Γ_2 in this limit motivates to preserve $\int_\Gamma h(c)$ and $|\Gamma| = |\Gamma_1| + |\Gamma_2|$ instead. The constraints on the total area and on the phase area difference read

$$\mathcal{C}_A(\Gamma, c) = 0, \quad \mathcal{C}_c(\Gamma, c) = 0 \qquad \qquad (4.1)$$

in terms of the functionals

$$\mathcal{C}_A(\Gamma, c) := |\Gamma| - (A_1 + A_2), \quad \mathcal{C}_c(\Gamma, c) := \int_\Gamma h(c) - (A_1 - A_2).$$

The constraint $\mathcal{C}_c$ will be called *mass constraint* in the following with the notion behind that $\int_\Gamma h(c)$ could correspond to some kind of mass. The volume constraint

$$\mathcal{C}_V(\Gamma, c) := |\Omega| - V = 0 \qquad \qquad (4.2)$$

is kept in the diffuse interface setting where we use the same notation $\mathcal{C}_V$ for the functional for convenience.

DEFINITION 4.2. *Given an admissible phase field surface* (Γ, c), *a smooth normal vector field* $\boldsymbol{w} = w\boldsymbol{\nu} : \Gamma \to \mathbb{R}^3$ *and a smooth function* $\eta : \Gamma \to \mathbb{R}$, *the deformed admissible phase field surface* $(\Gamma(\tau), c(\tau))$ *in direction* (w, η) *for a small* $\tau \in \mathbb{R}$ *is defined by*

$$\Gamma(\tau) := \{\boldsymbol{x}(\tau) := \boldsymbol{x} + \tau w(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) \,|\, \boldsymbol{x} \in \Gamma\},$$
$$c(\tau) \,:\, \Gamma(\tau) \to \mathbb{R}, \quad c(\tau, \boldsymbol{x}(\tau)) := c(\boldsymbol{x}) + \tau\eta(\boldsymbol{x}). \qquad (4.3)$$

Such a pair (w, η) *is called* **admissible deformation (field)** *for an admissible phase field surface.*

By the regularity assumptions on admissible phase field surfaces there is a small $\tau_0 > 0$ so that $(\Gamma(\tau), c(\tau))$ is indeed is admissible for all $\tau \in (-\tau_0, \tau_0)$. In particular, for each point $\boldsymbol{x}(\tau)$ on $\Gamma(\tau)$ there is a unique point $\boldsymbol{x} \in \Gamma$ with $\boldsymbol{x}(\tau) = \boldsymbol{x} + \tau w(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x})$. Therefore

$$\frac{d}{d\tau} c(\cdot, \boldsymbol{x}(\cdot))\big|_{\tau=0} = \partial_\tau c(0, \boldsymbol{x}(0)) + \partial_\tau \boldsymbol{x}(0) \cdot \nabla c(0, \boldsymbol{x}(0))$$
$$= \partial_\tau c(0, \boldsymbol{x}) + w(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) \cdot \nabla c(\boldsymbol{x}) = \partial_\tau^\circ c(\cdot, \boldsymbol{x}(\cdot))|_{\tau=0}.$$

On the other hand, from (4.3) we see that $\frac{d}{d\tau} c(\cdot, \boldsymbol{x}(\cdot))|_{\tau=0} = \eta(\boldsymbol{x})$, whence

$$\partial_\tau^\bullet c(\cdot, \boldsymbol{x}(\cdot))\big|_{\tau=0} = \partial_\tau^\circ c(\cdot, \boldsymbol{x}(\cdot))\big|_{\tau=0} = \eta(\boldsymbol{x}). \qquad (4.4)$$

In the case $\eta = 0$ this means that we extend the phase field constantly in the normal direction away from Γ in order to define it on the deformed surface $\Gamma(\tau)$.

DEFINITION 4.3. *Let* $\mathcal{E} = \mathcal{E}(\Gamma, c)$ *be a functional defined on admissible phase field surfaces, let* (Γ, c) *be an admissible surface and let* (w, η) *be an admissible deformation field. The* **variation** *of* $\mathcal{E}$ *in* (Γ, c) *in direction* (w, η) *is defined by*

$$\big\langle \delta\mathcal{E}(\Gamma, c), (w, \eta) \big\rangle = \frac{d}{d\tau} \mathcal{E}\big(\Gamma(\cdot), c(\cdot)\big)\Big|_{\tau=0}.$$

LEMMA 4.4. **Variation of the mean curvature bending energy.** *For an admissible phase field surface* (Γ, c) *with admissible deformation* (w, η) *we have that*

$$\big\langle \delta\mathcal{F}_{MC}(\Gamma, c), (w, \eta) \big\rangle = \int_\Gamma \Big(\frac{1}{2}(\kappa - \kappa_s(c))^2 k_\kappa'(c) - k_\kappa(c)(\kappa - \kappa_s(c))\kappa_s'(c) \Big)\eta$$
$$+ \int_\Gamma \Big(\Delta_\Gamma\big(k_\kappa(c)(\kappa - \kappa_s(c))\big) + |\nabla_\Gamma\boldsymbol{\nu}|^2 k_\kappa(c)(\kappa - \kappa_s(c)) - \frac{1}{2}k_\kappa(c)(\kappa - \kappa_s(c))^2\kappa \Big)w. \qquad (4.5)$$

Proof. Using (2.10) and then (2.13) and (4.4) we obtain

$$\frac{d}{d\tau} \mathcal{F}_{MC}\big(\Gamma(\cdot), c(\cdot)\big)\Big|_{\tau=0}$$
$$= \int_\Gamma \frac{1}{2}k_\kappa'(c)(\kappa - \kappa_s(c))^2\eta - k_\kappa(c)(\kappa - \kappa_s(c))\kappa_s'(c)\eta$$
$$+ \int_\Gamma k_\kappa(c)(\kappa - \kappa_s(c))(\Delta_\Gamma w + |\nabla_\Gamma\boldsymbol{\nu}|^2 w) - \frac{1}{2}k_\kappa(c)\big|\kappa - \kappa_s(c)\big|^2\kappa w.$$

Twice integrating by parts in the term with $\Delta_\Gamma w$ yields the assertion. □

LEMMA 4.5. **Variation of the Gaussian curvature bending energy.** *For an admissible phase field surface* (Γ, c) *with admissible deformation field* (w, η) *we have that*

$$\langle \delta \mathcal{F}_{GC}(\Gamma, c), (w, \eta) \rangle = \int_\Gamma \Big(g k_g'(c) \Big) \eta + \int_\Gamma \Big(\nabla_\Gamma \cdot \big(k_g'(c)(\kappa \boldsymbol{I} + \nabla_\Gamma \boldsymbol{\nu}) \nabla_\Gamma c \big) \Big) w \quad (4.6)$$

Proof. We use formula (2.2) for the Gaussian curvature. With (2.10), (4.4), (2.12)

$$\frac{d}{d\tau} \int_{\Gamma(\cdot)} \frac{1}{2} k_g(c(\cdot)) |\nabla_{\Gamma(\cdot)} \boldsymbol{\nu}(\cdot)|^2 \Big|_{\tau=0}$$

$$= \sum_{i=1}^3 \int_\Gamma \frac{1}{2} \partial_\tau^\circ \big(k_g(c) |\nabla_\Gamma \boldsymbol{\nu}_i|^2 \big) - \frac{1}{2} k_g(c) |\nabla_\Gamma \boldsymbol{\nu}_i|^2 \kappa w$$

$$= \sum_i \int_\Gamma \frac{1}{2} k_g'(c) \partial_\tau^\circ c |\nabla_\Gamma \boldsymbol{\nu}_i|^2 + k_g(c) \nabla_\Gamma \boldsymbol{\nu}_i \cdot \nabla_\Gamma \partial_\tau^\circ \boldsymbol{\nu}_i$$

$$\quad + \sum_i \int_\Gamma -k_g(c) \nabla_\Gamma \boldsymbol{\nu}_i \cdot D(w\boldsymbol{\nu}) \nabla_\Gamma \boldsymbol{\nu}_i - \frac{1}{2} k_g(c) |\nabla_\Gamma \boldsymbol{\nu}_i|^2 \kappa w$$

$$= \int_\Gamma \frac{1}{2} k_g'(c) |\nabla_\Gamma \boldsymbol{\nu}|^2 \eta + \int_\Gamma k_g(c) \sum_i \nabla_\Gamma \boldsymbol{\nu}_i \cdot \nabla_\Gamma (-\underline{D}_i w)$$

$$\quad - \int_\Gamma k_g(c) \Big(\sum_i \nabla_\Gamma \boldsymbol{\nu}_i \otimes \nabla_\Gamma \boldsymbol{\nu}_i : \boldsymbol{\nu} \Big) w - \int_\Gamma \frac{1}{2} k_g(c) |\nabla_\Gamma \boldsymbol{\nu}|^2 \kappa w.$$

Using (2.3) we obtain that

$$\int_\Gamma k_g(c) \sum_i \nabla_\Gamma \boldsymbol{\nu}_i \cdot \nabla_\Gamma (-\underline{D}_i w) = \int_\Gamma \sum_i \nabla_\Gamma \big(k_g(c) \nabla_\Gamma \boldsymbol{\nu}_i \big) \cdot \underline{D}_i w$$

$$= \int_\Gamma \sum_i k_g'(c) \nabla_\Gamma c \cdot \nabla_\Gamma \boldsymbol{\nu}_i \underline{D}_i w + k_g(c) \Delta_\Gamma \boldsymbol{\nu}_i \underline{D}_i w$$

$$= \int_\Gamma k_g'(c) \nabla_\Gamma \boldsymbol{\nu} \nabla_\Gamma c \cdot \nabla_\Gamma w - k_g(c) \nabla_\Gamma \kappa \cdot \nabla_\Gamma w$$

$$= \int_\Gamma \nabla_\Gamma \cdot \big(-k_g(c) \nabla_\Gamma \boldsymbol{\nu} \nabla_\Gamma c + k_g(c) \nabla_\Gamma \kappa \big) w.$$

Applying (4.5) with $k_\kappa = k_g$ and $\kappa_s = 0$ we get

$$\frac{d}{d\tau} \int_{\Gamma(\cdot)} \frac{1}{2} k_g(c(\cdot)) \kappa(\cdot)^2 \Big|_{\tau=0} = \int_\Gamma \Big(\frac{1}{2} k_g'(c) \kappa^2 \Big) \eta$$

$$\quad + \int_\Gamma \Big(\Delta_\Gamma \big(k_g(c) \kappa \big) + k_g(c) |\nabla_\Gamma \boldsymbol{\nu}|^2 \kappa - \frac{1}{2} k_g(c) \kappa^2 \kappa \Big) w.$$

Altogether

$$
\frac{d}{d\tau} \int_{\Gamma(\cdot)} \frac{1}{2} k_g(c(\cdot)) \big(\kappa^2 - |\nabla_{\Gamma(\cdot)} \boldsymbol{\nu}(\cdot)|^2 \big) \Big|_{\tau=0}
$$

$$
= \int_\Gamma \frac{1}{2} \kappa^2 k_g'(c) \eta + \int_\Gamma \Big(\Delta_\Gamma \big(k_g(c)\kappa \big) + |\nabla_\Gamma \boldsymbol{\nu}|^2 k_g(c)\kappa - \frac{1}{2} k_g(c)\kappa^2 \kappa \Big) w
$$

$$
- \int_\Gamma \frac{1}{2} |\nabla_\Gamma \boldsymbol{\nu}|^2 k_g'(c) \eta - \int_\Gamma \nabla_\Gamma \cdot \big(- k_g'(c) \nabla_\Gamma \boldsymbol{\nu} \nabla_\Gamma c + k_g(c) \nabla_\Gamma \kappa \big) w
$$

$$
- \int_\Gamma -k_g(c) \Big(\sum_i \nabla_\Gamma \boldsymbol{\nu}_i \otimes \nabla_\Gamma \boldsymbol{\nu}_i : \nabla_\Gamma \boldsymbol{\nu} \Big) w + \int_\Gamma \frac{1}{2} k_g(c) |\nabla_\Gamma \boldsymbol{\nu}|^2 \kappa w
$$

$$
= \int_\Gamma g k_g'(c) \eta + \nabla_\Gamma \cdot \big(\nabla_\Gamma (k_g(c)\kappa) - k_g(c) \nabla_\Gamma \kappa + k_g'(c) \nabla_\Gamma \boldsymbol{\nu} \nabla_\Gamma \big) w
$$

$$
+ \int_\Gamma \Big(|\nabla_\Gamma \boldsymbol{\nu}|^2 \kappa + \sum_i \nabla_\Gamma \boldsymbol{\nu}_i \otimes \nabla_\Gamma \boldsymbol{\nu}_i : \nabla_\Gamma \boldsymbol{\nu} - g\kappa \Big) k_g(c) w.
$$

Observe that $\nabla_\Gamma (k_g(c)\kappa) - k_g(c)\nabla_\Gamma \kappa = \kappa k_g'(c)\nabla_\Gamma c$. Furthermore,

$$
|\nabla_\Gamma \boldsymbol{\nu}|^2 \kappa + \sum_i \nabla_\Gamma \boldsymbol{\nu}_i \otimes \nabla_\Gamma \boldsymbol{\nu}_i : \nabla_\Gamma \boldsymbol{\nu} - g\kappa = 0 \tag{4.7}
$$

so that we end up with the claimed formula (4.6) for the variation of the Gaussian curvature bending energy. To show (4.7) one can employ an orthogonal matrix $Q \in \mathbb{R}^{3\times3}$ such that $Q^{-1} \nabla_\Gamma \boldsymbol{\nu} Q = \mathrm{diag}(-\kappa_1, -\kappa_2, 0)$ where κ_1, κ_2 are the principal curvatures and proceed using that each of the summands is invariant under such a similarity transformation. □

REMARK 4.6. *If $k_g(c) = k_g$ is a constant independent of c then the energy $\int_\Gamma k_g g$ is a topological invariant by the Gauss-Bonnet theorem so that its variation must vanish. Formula (4.6) indeed then yields $\langle \mathcal{F}_{GC}(\Gamma, c), (w, \eta) \rangle = 0$ since $k_g'(c) = 0$.*

LEMMA 4.7. **Variation of line energy and constraint functionals.** *For an admissible phase field surface (Γ, c) with admissible deformation field (w, η) we have that*

$$
\langle \delta \mathcal{F}_{GL}(\Gamma, c), (w, \eta) \rangle = \int_\Gamma \Big(- \sigma \varepsilon \Delta_\Gamma c + \frac{\sigma}{\varepsilon} \psi'(c) \Big) \eta \tag{4.8}
$$

$$
- \int_\Gamma \sigma \Big(\varepsilon \nabla_\Gamma c \otimes \nabla_\Gamma c : \nabla_\Gamma \boldsymbol{\nu} + \big(\frac{\varepsilon}{2} |\nabla_\Gamma c|^2 + \frac{1}{\varepsilon} \psi(c) \big) \kappa \Big) w, \tag{4.9}
$$

$$
\langle \delta \mathcal{C}_c(\Gamma, c), (w, \eta) \rangle = \int_\Gamma h'(c)\eta - h(c)\kappa w, \tag{4.10}
$$

$$
\langle \delta \mathcal{C}_V(\Gamma, c), (w, \eta) \rangle = \int_\Gamma w, \quad \langle \delta \mathcal{C}_A(\Gamma, c), (w, \eta) \rangle = - \int_\Gamma \kappa w. \tag{4.11}
$$

Proof. For the first assertion, we use (2.11) for the term involving $\nabla_\Gamma c$, (2.9) for the term with the double well potential, (2.4) for partial integration (recall that Γ is

closed whence no boundary term appears), (2.1) and (4.4) to obtain (4.9):

$$\frac{d}{d\tau}\mathcal{F}_{GL}\big(\Gamma(\cdot),c(\cdot)\big)\Big|_{\tau=0} = \sigma\int_\Gamma \varepsilon\nabla_\Gamma c\cdot\nabla_\Gamma\partial_\tau^\bullet c - \varepsilon\nabla_\Gamma c\otimes\nabla_\Gamma c : 2D(\boldsymbol{w}) + \frac{1}{\varepsilon}\psi'(c)\partial_\tau^\bullet c$$

$$+ \sigma\int_\Gamma \Big(\frac{\varepsilon}{2}|\nabla_\Gamma c|^2 + \frac{1}{\varepsilon}\psi(c)\Big)\nabla_\Gamma\cdot\boldsymbol{w}$$

$$= \sigma\int_\Gamma -\varepsilon\Delta_\Gamma c\,\eta + \frac{1}{\varepsilon}\psi'(c)\eta - \varepsilon\nabla_\Gamma c\otimes\nabla_\Gamma c : \nabla_\Gamma\boldsymbol{\nu}\,w$$

$$+ \sigma\int_\Gamma -\Big(\frac{\varepsilon}{2}|\nabla_\Gamma c|^2 + \frac{1}{\varepsilon}\psi(c)\Big)\kappa w.$$

The other identities can be proved similarly. □

DEFINITION 4.8. *For given values V, A_1, A_2 fulfilling (3.1), an admissible phase field surface (Γ,c) is a* **critical point** *of the diffuse interface membrane energy (1.5) subject to constraints (4.1) and (4.2) if*

$$0 = \big(\delta\mathcal{F}_{MC} + \delta\mathcal{F}_{GC} + \delta\mathcal{F}_{GL} + \delta\mathcal{F}_M + \lambda_V\delta\mathcal{C}_V + \lambda_A\delta\mathcal{C}_A + \lambda_c\delta\mathcal{C}_c\big)(\Gamma,c)$$

where λ_V, λ_A, and λ_c are appropriate Lagrange multipliers.

Using (4.5), (4.6), (4.9), (4.11), and (4.10) critical points fulfill

PROBLEM 4.9. **Diffuse interface equilibrium equations.** *For given values V, A_1, A_2 fulfilling (3.1) find an admissible phase field surface (Γ,c) and Lagrange multipliers λ_V, λ_A, and λ_c such that*

$$0 = \Delta_\Gamma\big(k_\kappa(c)(\kappa - \kappa_s(c))\big) + |\nabla_\Gamma\boldsymbol{\nu}|^2 k_\kappa(c)(\kappa - \kappa_s(c)) - \tfrac{1}{2}k_\kappa(c)(\kappa - \kappa_s(c))^2\kappa$$

$$+ \nabla_\Gamma\cdot\big(k_g'(c)(\kappa\boldsymbol{I} + \nabla_\Gamma\boldsymbol{\nu})\nabla_\Gamma c\big)$$

$$- \sigma\varepsilon\nabla_\Gamma c\otimes\nabla_\Gamma c : \nabla_\Gamma\boldsymbol{\nu} - \sigma\big(\tfrac{\varepsilon}{2}|\nabla_\Gamma c|^2 + \tfrac{1}{\varepsilon}\psi(c)\big)\kappa$$

$$+ \lambda_V - \big(\lambda_A + \lambda_c h(c)\big)\kappa, \tag{4.12}$$

$$0 = \tfrac{1}{2}(\kappa - \kappa_s(c))^2 k_\kappa'(c) - k_\kappa(c)(\kappa - \kappa_s(c))\kappa_s'(c) + g k_g'(c)$$

$$- \varepsilon\sigma\Delta_\Gamma c + \tfrac{\sigma}{\varepsilon}\psi'(c) + \lambda_c h'(c), \tag{4.13}$$

$$0 = |\Omega| - V, \quad 0 = |\Gamma| - (A_1 + A_2), \quad 0 = \int_\Gamma h(c) - (A_1 - A_2). \tag{4.14}$$

We may consider the level set $\gamma_\varepsilon := \{\boldsymbol{x}\in\Gamma\,|\,c(\boldsymbol{x}) = 0\}$ as an approximation to the phase interface γ in the sharp interface model. For performing an asymptotic analysis of the above diffuse interface model it is convenient to write the governing equations close to γ_ε in terms of the curvatures $h_{\boldsymbol{\nu}}$, h_p, and h_d which are defined close to γ_ε as described in Section 2.2. Recalling the relations (2.8) we obtain

$$0 = \Delta_\Gamma\big(k_\kappa(c)(h_{\boldsymbol{\nu}} + h_p - \kappa_s(c))\big) + (h_{\boldsymbol{\nu}}^2 + h_p^2 + 2h_d^2)k_\kappa(c)(h_{\boldsymbol{\nu}} + h_p - \kappa_s(c))$$

$$- \tfrac{1}{2}k_\kappa(c)(h_{\boldsymbol{\nu}} + h_p - \kappa_s(c))^2(h_{\boldsymbol{\nu}} + h_p)$$

$$+ \nabla_\Gamma\cdot\big(k_g'(c)((h_{\boldsymbol{\nu}} + h_p)\boldsymbol{I} + \nabla_\Gamma\boldsymbol{\nu})\nabla_\Gamma c\big)$$

$$- \sigma\varepsilon\nabla_\Gamma c\otimes\nabla_\Gamma c : \nabla_\Gamma\boldsymbol{\nu} - \sigma\big(\tfrac{\varepsilon}{2}|\nabla_\Gamma c|^2 + \tfrac{1}{\varepsilon}\psi(c)\big)(h_{\boldsymbol{\nu}} + h_p)$$

$$+ \lambda_V - \big(\lambda_A + \lambda_c h(c)\big)(h_{\boldsymbol{\nu}} + h_p), \tag{4.15}$$

$$0 = \tfrac{1}{2}(h_{\boldsymbol{\nu}} + h_p - \kappa_s(c))^2 k_\kappa'(c) - k_\kappa(c)(h_{\boldsymbol{\nu}} + h_p - \kappa_s(c))\kappa_s'(c)$$

$$+ (h_{\boldsymbol{\nu}}h_p - h_d^2)k_g'(c) - \varepsilon\sigma\Delta_\Gamma c + \tfrac{\sigma}{\varepsilon}\psi'(c) + \lambda_c h'(c) \tag{4.16}$$

where

$$\nabla_\Gamma \boldsymbol{\nu} = -\, h_{\boldsymbol{\nu}} \boldsymbol{\tau} \otimes \boldsymbol{\tau} - h_p \boldsymbol{\mu} \otimes \boldsymbol{\mu} - h_d \boldsymbol{\tau} \otimes \boldsymbol{\mu} - h_d \boldsymbol{\mu} \otimes \boldsymbol{\tau} \qquad (4.17)$$

which we get from $\nabla_\Gamma \boldsymbol{\nu} = S^T S \nabla_\Gamma \boldsymbol{\nu} S^T S$ with the orthogonal matrix $S = (\boldsymbol{\tau}, \boldsymbol{\mu}, \boldsymbol{\nu})$.

5. Asymptotic Analysis. The goal is now to relate the diffuse interface problem to the sharp interface problem by matching appropriate asymptotic ε-expansions. The technique is carefully explained in [17] yet we have to extend it since the membrane shape, i.e., the underlying space, depends on ε.

5.1. Assumptions. Let $\{(\Gamma_\varepsilon, c_\varepsilon), \lambda_{V,\varepsilon}, \lambda_{A,\varepsilon}, \lambda_{c,\varepsilon}\}_\varepsilon$ be a family of solutions to Problem 4.9. Denoting the curvatures associated with the solutions appearing in (4.12) and (4.13) by κ_ε and g_ε, respectively, the fields occurring in (4.15), (4.16), and (4.17) are denoted by $h_{\boldsymbol{\nu},\varepsilon}$, $h_{p,\varepsilon}$, $h_{d,\varepsilon}$, $\boldsymbol{\mu}_\varepsilon$, and $\boldsymbol{\tau}_\varepsilon$. We assume that there is a limiting surface Γ_0 to which the surfaces Γ_ε converge as $\varepsilon \to 0$ and that this limit is an admissible two-phase surface. In fact, in numerical simulations [14] we have observed that κ_ε remains bounded in L^∞ as $\varepsilon \to 0$. The compactness and regularity properties of integral varifolds [2] to which the Γ_ε belong suggest that any limiting surface indeed is a C^1 surface.

More precisely, we assume that we can parametrize Γ_ε over Γ_0 in the form $\Gamma_\varepsilon = \{\boldsymbol{p}_\varepsilon(\boldsymbol{x}) \,|\, \boldsymbol{x} \in \Gamma_0\}$ where the functions $\boldsymbol{p}_\varepsilon$ are C^1 across γ_0 and smooth in $\Gamma_{0,1}$ and $\Gamma_{0,2}$ and can be expanded as

$$\boldsymbol{p}_\varepsilon(\boldsymbol{x}) = \boldsymbol{x} + \varepsilon \boldsymbol{p}_1(\boldsymbol{x}) + O(\varepsilon^2), \qquad (5.1)$$

with $\boldsymbol{p}_1 \in C^1(\Gamma_0)$. Further, we assume that the level sets

$$\gamma_\varepsilon := \{\boldsymbol{x}_\varepsilon \in \Gamma_\varepsilon \,|\, c_\varepsilon(\boldsymbol{x}_\varepsilon) = 0\} \qquad (5.2)$$

converge to a finite number of smooth curves γ_0 on Γ_0, and additionally we assume that the parameterization is such that

$$\gamma_\varepsilon = \{\boldsymbol{p}_\varepsilon(\boldsymbol{x}) \,|\, \boldsymbol{x} \in \gamma_0\}. \qquad (5.3)$$

Writing $\Gamma_0 = \Gamma_{0,1} \cup \gamma_0 \cup \Gamma_{0,2}$ in the sense of Definition 3.1 the domains $\Gamma_{0,i}$, $i = 1, 2$, are the limits of the sets $\Gamma_{\varepsilon,1} := \{c_\varepsilon > 0\}$, $\Gamma_{\varepsilon,2} := \{c_\varepsilon < 0\}$.

We note that (5.1) implies linear convergence of Γ_ε in ε which is what we have observed in our numerical simulations, see [14] (a paper on the more general model presented in this study is in preparation). We also observed linear convergence of γ_ε in ε which motivates assumption (5.3). Intuitively, one may think of $\boldsymbol{p}_\varepsilon(\boldsymbol{x}) - \boldsymbol{x}$ pointing in the normal direction with respect to Γ_0 but the additional assumption (5.3) means that, in general, there will be tangential contributions. We stress that for the following asymptotic analysis up to first order in ε this is not of relevance.

The unit normal on Γ_ε can be expanded in the form

$$\boldsymbol{\nu}_\varepsilon(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = \boldsymbol{\nu}_0(\boldsymbol{x}) + \varepsilon \boldsymbol{\nu}_1(\boldsymbol{x}) + O(\varepsilon^2) \qquad (5.4)$$

where $\boldsymbol{\nu}_1$ is a vector field tangential to Γ_0. Furthermore, one can show that the expansion of the surface gradient on Γ_ε for any field $f_\varepsilon : \Gamma_\varepsilon \to \mathbb{R}$ is

$$\nabla_{\Gamma_\varepsilon} f_\varepsilon(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = \nabla_{\Gamma_0}(f_\varepsilon \circ \boldsymbol{p}_\varepsilon) - \varepsilon(\nabla_{\Gamma_0} \boldsymbol{p}_1)^\perp \nabla_{\Gamma_0}(f_\varepsilon \circ \boldsymbol{p}_\varepsilon) + O(\varepsilon^2). \qquad (5.5)$$

If we further assume that the field f_ε can be expanded in the form

$$f_\varepsilon(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = f_0(\boldsymbol{x}) + \varepsilon f_1(\boldsymbol{x}) + O(\varepsilon^2) \tag{5.6}$$

then (5.5) results in

$$\nabla_{\Gamma_\varepsilon} f_\varepsilon(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = \nabla_{\Gamma_0} f_0 + \varepsilon\big(- (\nabla_{\Gamma_0}\boldsymbol{p}_1)^\perp \nabla_{\Gamma_0} f_0 + \nabla_{\Gamma_0} f_1\big) + O(\varepsilon^2).$$

5.2. Outer expansions. Away from the interfacial layer around the curve γ_ε we assume that we may expand curvature, order parameter, and Lagrange multipliers in the form

$$\kappa_\varepsilon(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = \kappa_0(\boldsymbol{x}) + \varepsilon\kappa_1(\boldsymbol{x}) + O(\varepsilon^2),$$
$$c_\varepsilon(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = c_0(\boldsymbol{x}) + \varepsilon c_1(\boldsymbol{x}) + O(\varepsilon^2),$$
$$\lambda_{i,\varepsilon} = \lambda_{i,0} + \varepsilon\lambda_{i,1} + O(\varepsilon^2), \quad i = V, A, c \tag{5.7}$$

with smooth functions κ_i, c_i on each domain $\Gamma_{0,j}$, $j = 1, 2$, that are bounded when approaching γ_0. From the expansion (5.4) and since Γ_0 is smooth away from γ_0 we conclude that $\nabla_{\Gamma_\varepsilon}\boldsymbol{\nu}_\varepsilon = \nabla_{\Gamma_0}\boldsymbol{\nu}_0 + O(\varepsilon)$. All these expansions are plugged into the equations of Problem 4.9 which are expanded in ε-series again.

5.3. Outer solutions. To order ε^{-1} the equation (4.13) yields $0 = \psi'(c_0)$. The only stable solutions are the minima of w, whence $c_0 = \pm 1$. Recalling the definition of $\Gamma_{\varepsilon,i}$, $i = 1, 2$, we see that we will have $c_0 = 1$ in $\Gamma_{0,1}$ and $c_0 = -1$ in $\Gamma_{0,2}$. The equation to the next order ε^0 reads $0 = -\sigma\psi''(c_0)c_1 - \lambda_{c,0}h'(c_0)$. Since $\psi''(\pm 1) > 0$ and $h'(\pm 1) = 0$ we obtain that $c_1 = 0$.

Since $k_G'(c_0) = 0$, $\nabla_{\Gamma_0}c_0 = 0$, $\psi'(c_0) = 0$, and $h(c_0) = c_0 = -(-1)^i$ on $\Gamma_{0,i}$, $i = 1, 2$, the equation determining the membrane shape (4.12) yields to leading order

$$0 = \Delta_{\Gamma_0}\big(k_\kappa(c_0)(\kappa_0 - \kappa_s(c_0))\big) + |\nabla_{\Gamma_0}\boldsymbol{\nu}_0|^2\big(k_\kappa(c_0)(\kappa_0 - \kappa_s(c_0))\big)$$
$$\quad - \frac{1}{2}k_\kappa(c_0)(\kappa_0 - \kappa_s(c_0))^2\kappa_0 + \lambda_{V,0} - \big(\lambda_{A,0} + h(c_0)\lambda_{c,0}\big)\kappa_0$$
$$= \Delta_{\Gamma_0}\big(k_\kappa^{(i)}(\kappa_0 - \kappa_s^{(i)})\big) + |\nabla_{\Gamma_0}\boldsymbol{\nu}_0|^2\big(k_\kappa^{(i)}(\kappa_0 - \kappa_s^{(i)})\big)$$
$$\quad - \frac{1}{2}k_\kappa^{(i)}(\kappa_0 - \kappa_s^{(i)})^2\kappa_0 + \lambda_{V,0} - \big(\lambda_{A,0} - (-1)^i\lambda_{c,0}\big)\kappa_0. \tag{5.8}$$

To check that this equation becomes (3.15) we have to identify the Lagrange multipliers.

For the first constraint in (4.14) we obtain to leading order that

$$0 = |\Omega_0| - V = \frac{1}{3}\int_{\Gamma_0} \boldsymbol{x}_0 \cdot \boldsymbol{\nu}_0 - V = \mathcal{C}_V(\Gamma_0)$$

which is (3.19). We see that $\lambda_{V,0}$ in (5.8) does the job of λ_V in (3.15). Since

$$\int_{\Gamma_\varepsilon} h(c_\varepsilon) = \int_{\Gamma_0} h(c_0) + O(\varepsilon) = |\Gamma_{0,1}| - |\Gamma_{0,2}| + O(\varepsilon)$$

the second and third constraint in (4.14) yield $0 = |\Gamma_{0,1}| + |\Gamma_{0,2}| - (A_1 + A_2)$ and $0 = |\Gamma_{0,1}| - |\Gamma_{0,2}| - (A_1 - A_2)$ to leading order, respectively, which is equivalent to (3.20). We identify

$$\lambda_A^{(1)} = \lambda_{A,0} + \lambda_{c,0}, \quad \lambda_A^{(2)} = \lambda_{A,0} - \lambda_{c,0} \tag{5.9}$$

but we will need to check that the recurrence of the $\lambda_A^{(i)}$ in (3.18) is correctly recovered.

5.4. Interface coordinates. Since the interfacial layer has a thickness scaling with ε we want to use a coordinate across the interface scaling in the same way. In other words, in this new coordinate the interfacial layer is blown up to a thickness of order one which enables to investigate the limits of fields such as c_ε and κ_ε in a sensible way. Again, expansions of the fields will be inserted into the governing equations. In order to obtain boundary values for the resulting problems these inner expansions are matched with the outer expansions employed in the phases.

Let us denote by $\boldsymbol{\tau}_0$ a unit tangent along γ_0 and by $\boldsymbol{\mu}_0$ the outer unit co-normal of $\Gamma_{0,2}$ (= inner unit co-normal of $\Gamma_{0,1}$) on γ_0 where we may assume that $(\boldsymbol{\tau}_0, \boldsymbol{\mu}_0, \boldsymbol{\nu}_0)$ is positively oriented. The fields $\boldsymbol{\tau}_0$ and $\boldsymbol{\mu}_0$ are extended away from γ_0 as explained in Section 2.2. We also recall the definition of $d(\boldsymbol{x})$ in (2.7) which we use for points $\boldsymbol{x} \in \Gamma_{0,1}$ so that $\boldsymbol{\mu}_0(\boldsymbol{x}) = \nabla_{\Gamma_0} d(\boldsymbol{x})$. The latter identity is extended to $\Gamma_{0,2}$ by considering $-d(\boldsymbol{x})$ in points $\boldsymbol{x}$, i.e., d is the signed distance to γ_0 with positive sign in $\Gamma_{0,1}$ and negative sign in $\Gamma_{0,2}$.

Let $\tilde{\boldsymbol{x}}(s)$ be a parameterization of γ_0 by arc-length s with $\partial_s \tilde{\boldsymbol{x}}(s) = \boldsymbol{\tau}_0(\tilde{\boldsymbol{x}}(s))$. We may extend it to a parameterization $\boldsymbol{x}(s, r)$ of Γ_0 locally around γ_0 by requiring that $d(\boldsymbol{x}(s,r), \gamma_0) = d(\boldsymbol{x}(s,r), \tilde{\boldsymbol{x}}(s)) = r$. Furthermore, $\partial_d \boldsymbol{x}(s,r) = \boldsymbol{\mu}_0(\boldsymbol{x}(s,r))$, and for fixed s the curve $r \mapsto \boldsymbol{x}(s,r)$ is a geodesic. We infer that $\nabla_{\Gamma_0}(\nabla_{\Gamma_0} d)\boldsymbol{\mu}_0$ points in the direction of $\boldsymbol{\nu}_0$ so that

$$\boldsymbol{\tau}_0 \cdot \nabla_{\Gamma_0}(\nabla_{\Gamma_0} d)\boldsymbol{\mu}_0 = 0, \quad \boldsymbol{\mu}_0 \cdot \nabla_{\Gamma_0}(\nabla_{\Gamma_0} d)\boldsymbol{\mu}_0 = 0. \tag{5.10}$$

With $z(\boldsymbol{x}) := d(\boldsymbol{x})/\varepsilon$ we now introduce a **scaled distance function** as a new coordinate. In a point $\boldsymbol{x}(s, \varepsilon z)$ we then have

$$\nabla_{\Gamma_0} d(\boldsymbol{x}) = \boldsymbol{\mu}_0(\tilde{\boldsymbol{x}}) + \varepsilon \nabla_{\Gamma_0}(\nabla_{\Gamma_0} d(\tilde{\boldsymbol{x}}))\boldsymbol{\mu}_0(\tilde{\boldsymbol{x}})z + O(\varepsilon^2). \tag{5.11}$$

The arc-length parameter s may also be considered as a function of $\boldsymbol{x}$, and then $\nabla_{\Gamma_0} s(\boldsymbol{x}) = \boldsymbol{\tau}_0(\boldsymbol{x})$ for all $\boldsymbol{x} \in \gamma_0$. An expansion of $\nabla_{\Gamma_0} s$ is obtained in a similar fashion to that for $\nabla_{\Gamma_0} d$:

$$\nabla_{\Gamma_0} s(\boldsymbol{x}) = \boldsymbol{\tau}_0(\tilde{\boldsymbol{x}}) + \varepsilon \nabla_{\Gamma_0}(\nabla_{\Gamma_0} s(\tilde{\boldsymbol{x}}))\boldsymbol{\mu}_0(\tilde{\boldsymbol{x}})z + O(\varepsilon^2). \tag{5.12}$$

Given any field f_ε on Γ_ε, writing $\tilde{f}_\varepsilon(s(\boldsymbol{x}), z(\boldsymbol{x})) := f_\varepsilon(\boldsymbol{p}_\varepsilon(\boldsymbol{x}))$, and recalling (5.5) we see that

$$\nabla_{\Gamma_\varepsilon} f_\varepsilon(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = \big(\boldsymbol{I} - \varepsilon(\nabla_{\Gamma_0}\boldsymbol{p}_1)^\perp\big)\big(\boldsymbol{\tau}_0 + \varepsilon\nabla_{\Gamma_0}(\nabla_{\Gamma_0} s)\boldsymbol{\mu}_0 z\big)\partial_s \tilde{f}_\varepsilon(s(\boldsymbol{x}), z(\boldsymbol{x}))$$
$$+ \frac{1}{\varepsilon}\big(\boldsymbol{I} - \varepsilon(\nabla_{\Gamma_0}\boldsymbol{p}_1)^\perp\big)\big(\boldsymbol{\mu}_0 + \varepsilon\nabla_{\Gamma_0}(\nabla_{\Gamma_0} d)\boldsymbol{\mu}_0 z\big)\partial_z \tilde{f}_\varepsilon(s(\boldsymbol{x}), z(\boldsymbol{x})) + O(\varepsilon) \tag{5.13}$$

where $\nabla_{\Gamma_0}\boldsymbol{p}_1$, $\boldsymbol{\tau}_0$, $\boldsymbol{\mu}_0$, and the derivatives of s and d are evaluated at $\tilde{\boldsymbol{x}}(s(\boldsymbol{x}))$. Assume now that the field f_ε can be expanded in the form

$$f_\varepsilon(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = F_0(s, z) + \varepsilon F_1(s, z) + O(\varepsilon^2) \tag{5.14}$$

close to the interface γ_ε in terms of the new coordinates (s, z). Then we end up with an ε-expansion of the form

$$\nabla_{\Gamma_\varepsilon} f_\varepsilon(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = \tfrac{1}{\varepsilon}\boldsymbol{\mu}_0\partial_z F_0$$
$$+ \big(\boldsymbol{\tau}_0\partial_s F_0 - (\nabla_{\Gamma_0}\boldsymbol{p}_1)^\perp \boldsymbol{\mu}_0\partial_z F_0 + \nabla_{\Gamma_0}(\nabla_{\Gamma_0} d)\boldsymbol{\mu}_0 z\partial_z F_0 + \boldsymbol{\mu}_0\partial_s F_1\big) + O(\varepsilon) \tag{5.15}$$

for the spatial derivative of f_ε close to γ_ε where $\nabla_{\Gamma_0}\boldsymbol{p}_1$, $\boldsymbol{\mu}_0$, $\boldsymbol{\tau}_0$, and $\nabla_{\Gamma_0}(\nabla_{\Gamma_0} d)$ are evaluated at $\tilde{\boldsymbol{x}}(s)$.

5.5. Matching conditions. In view of (5.6) and (5.14) we have two expansions of a field f_ε on Γ_ε, the first one being valid away from the interface γ_ε and the second one, expressed in terms of rescaled coordinates, close to γ_ε. These expansions are supposed to match, and this leads to some conditions on the functions f_i and F_i. Here, we only state these conditions and refer to the appendix of [18] for a careful derivation. As $z \to \pm\infty$

$$F_0(s,z) \sim f_0(\boldsymbol{x}(s,0^\pm)), \tag{5.16}$$

$$F_1(s,z) \sim f_1(\boldsymbol{x}(s,0^\pm)) + \big(\nabla_{\Gamma_0} f_0(\boldsymbol{x}(s,0^\pm)) \cdot \boldsymbol{\mu}_0(\boldsymbol{x}(s,0))\big)z, \tag{5.17}$$

$$\partial_z F_1(s,z) \sim \nabla_{\Gamma_0} f_0(\boldsymbol{x}(s,0^\pm)) \cdot \boldsymbol{\mu}_0(\boldsymbol{x}(s,0)) \tag{5.18}$$

where $0^\pm$ stands for the limit as $d \searrow 0$ (approaching γ_0 from $\Gamma_{0,1}$) and $d \nearrow 0$ (from $\Gamma_{0,2}$), respectively.

5.6. Inner expansions. Recalling that $\boldsymbol{\mu}_\varepsilon = \nabla_{\Gamma_\varepsilon} d_\varepsilon(\boldsymbol{x}_\varepsilon)$ where d_ε is the signed distance to γ_ε on Γ_ε, the expansions (5.1) and the smoothness of $\boldsymbol{p}_\varepsilon$ away from γ_0 yields that there is an expansion of $\boldsymbol{\mu}_\varepsilon$ of the form

$$\boldsymbol{\mu}_\varepsilon(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = \boldsymbol{\mu}_0(\boldsymbol{x}) + \varepsilon\boldsymbol{\mu}_1(\boldsymbol{x}) + O(\varepsilon^2).$$

With the Taylor-expansion of $\boldsymbol{\mu}_0(\boldsymbol{x}(s,r))$ in $\tilde{\boldsymbol{x}}(s) = \boldsymbol{x}(s,0)$ and replacing $r = \varepsilon z$ again we see that

$$\boldsymbol{\mu}_\varepsilon(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = \boldsymbol{\mu}_0(\tilde{\boldsymbol{x}}) + \varepsilon\big(\nabla_{\Gamma_0}(\nabla_{\Gamma_0} d(\tilde{\boldsymbol{x}}))\boldsymbol{\mu}_0(\tilde{\boldsymbol{x}})z + \boldsymbol{\mu}_1(\tilde{\boldsymbol{x}})\big) + O(\varepsilon^2). \tag{5.19}$$

In particular, since $\boldsymbol{\mu}_1(\tilde{\boldsymbol{x}}) = \frac{d}{d\varepsilon}\boldsymbol{\mu}_\varepsilon(\boldsymbol{p}_\varepsilon(\tilde{\boldsymbol{x}}))|_{\varepsilon=0}$ and since $|\boldsymbol{\mu}_\varepsilon(\boldsymbol{p}_\varepsilon(\tilde{\boldsymbol{x}}))| = 1$ for all ε the first order correction is orthogonal to the co-normal on γ_0, $\boldsymbol{\mu}_1(\tilde{\boldsymbol{x}}) \cdot \boldsymbol{\mu}_0(\tilde{\boldsymbol{x}}) = 0$. Similarly, the expansion of $\boldsymbol{\tau}_\varepsilon$ can be derived:

$$\boldsymbol{\tau}_\varepsilon(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = \boldsymbol{\tau}_0(\tilde{\boldsymbol{x}}) + \varepsilon\big(\nabla_{\Gamma_0}(\nabla_{\Gamma_0} s(\tilde{\boldsymbol{x}}))\boldsymbol{\mu}_0(\tilde{\boldsymbol{x}})z + \boldsymbol{\tau}_1(\tilde{\boldsymbol{x}})\big) + O(\varepsilon^2). \tag{5.20}$$

We assume that within the interfacial layer around the curve γ_ε we have expansions of the form

$$h_{\boldsymbol{\nu},\varepsilon}(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = H_{\boldsymbol{\nu},0}(s) + \varepsilon H_{\boldsymbol{\nu},1}(s,z) + O(\varepsilon^2),$$
$$h_{d,\varepsilon}(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = H_{d,0}(s) + \varepsilon H_{d,1}(s,z) + O(\varepsilon^2),$$
$$h_{p,\varepsilon}(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = H_{p,0}(s,z) + \varepsilon H_{p,1}(s,z) + O(\varepsilon^2),$$
$$c_\varepsilon(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = C_0(s,z) + \varepsilon C_1(s,z) + O(\varepsilon^2).$$

Since by assumption the limiting surface Γ_0 is an admissible two-phase surface and, hence, C^1 the quantities $h_{\boldsymbol{\nu}}$ and h_d are continuous in the limit, or $h_{\boldsymbol{\nu},0}(\boldsymbol{x}(s,0^+)) = h_{\boldsymbol{\nu},0}(\boldsymbol{x}(s,0^-))$ and similarly for h_d. The matching condition (5.16) motivates to assume that $H_{\boldsymbol{\nu},0}$ and $H_{d,0}$ are independent of z.

The above expansions are plugged into the governing equations (4.15), (4.16), and (4.17) where the spatial derivatives are expanded as in (5.15).

5.7. Inner solutions: First order. The phase field equations (4.13) yields to order ε^{-1}

$$0 = \sigma\big(\partial_{zz} C_0 - \psi'(C_0)\big). \tag{5.21}$$

Recalling that $c_0 = \pm 1$ in $\Gamma_{0,i}$, $i = 1, 2$, respectively, the matching condition (5.16) yields the boundary conditions $C_0(s,z) \to \pm 1$ as $z \to \pm\infty$. By (5.2) and (5.3)

$c_\varepsilon(\boldsymbol{p}_\varepsilon(\boldsymbol{x})) = 0$ for $\boldsymbol{x} \in \gamma_0 = \{\boldsymbol{x}(s,0)\}$, hence we also have that $C_0(s,0) = 0$. The solution is given by $C_0(z) = \tanh(z)$ and, in particular, does not depend on s. Multiplying (5.21) with $\partial_z C_0$ and integrating with respect to sz from $-\infty$ to $\tilde{z}$ we obtain

$$\psi(C_0(\tilde{z})) = \tfrac{1}{2}|\partial_z C_0(\tilde{z})|^2 \tag{5.22}$$

which is commonly known as equipartition of energy. Let

$$Q_0 := k_\kappa(C_0)(H_{\boldsymbol{\nu},0} + H_{p,0} - \kappa_s(C_0)) + k_g(C_0)H_{\boldsymbol{\nu},0}.$$

The membrane equation (4.12) to order ε^{-2} reads $0 = \partial_{zz}Q_0$, hence Q_0 is a linear function in z. The matching condition (5.16) implies that $\partial_z Q_0 \to 0$ as $z \to \pm\infty$ whence $\partial_z\big(k_\kappa(C_0)(H_{\boldsymbol{\nu},0} + H_{p,0} - \kappa_s(C_0)) + H_{\boldsymbol{\nu},0}k_g(C_0)\big) = \partial_z Q_0 = 0$, and we further conclude that

$$Q_0 = k_\kappa^{(1)}(h_{\boldsymbol{\nu},0} + h_{p,0}^{(1)} - \kappa_s^{(1)}) + h_{\boldsymbol{\nu},0}k_g^{(1)} = k_\kappa^{(2)}(h_{\boldsymbol{\nu},0} + h_{p,0}^{(2)} - \kappa_s^{(2)}) + h_{\boldsymbol{\nu},0}k_g^{(2)}$$

so that, in particular, $k_\kappa(h_{\boldsymbol{\nu},0} + h_{p,0} - \kappa_s) + h_{\boldsymbol{\nu},0}k_g$ is continuous across γ_0 and we have recovered equation (3.16).

5.8. Inner solutions: Tangential force balance. Using that $\partial_s C_0 = 0$ equation (4.13) to order ε^0 reads

$$\begin{aligned}
0 = &-\sigma\big(\partial_{zz}C_1 - h_{g,0}\partial_z C_0 - 2\boldsymbol{\mu}_0 \cdot \nabla_{\Gamma_0}\boldsymbol{p}_1\boldsymbol{\mu}_0\partial_{zz}C_0 - \psi''(C_0)C_1\big) \\
&+ \tfrac{1}{2}\big(H_{\boldsymbol{\nu},0} + H_{p,0} - \kappa_s(C_0)\big)^2 k_\kappa'(C_0) - k_\kappa(C_0)\big(H_{\boldsymbol{\nu},0} + H_{p,0} - \kappa_s(C_0)\big)\kappa_s'(C_0) \\
&+ \lambda_{c,0}h'(C_0)
\end{aligned} \tag{5.23}$$

which can be considered as an equation for the correction C_1. We multiply with $\partial_z C_0$ and integrate with respect to z from $-\infty$ to ∞. The third summand then vanishes because $\int_{-\infty}^{\infty} \partial_{zz}C_0\partial_z C_0 = 0$. By (5.17) and (5.18) and since $c_1 = 0$ we see that $C_1(s,z) \to 0$ and $\partial_z C_1(s,z) \to 0$ as $z \to \pm\infty$, whence using (5.21)

$$\int_{-\infty}^{\infty}\big(\partial_{zz}C_1 - \psi''(C_0)C_1\big)\partial_z C_0 = \int_{-\infty}^{\infty}(-\partial_{zz}C_0 + \psi'(C_0))\partial_z C_1 = 0.$$

As a solvability condition for (5.23) we therefore obtain that

$$\begin{aligned}
0 = &\int_{-\infty}^{\infty} \big(\sigma h_{g,0}(\partial_z C_0)^2 + \lambda_{c,0}h'(C_0)\partial_z C_0\big) \\
&+ \int_{-\infty}^{\infty} \tfrac{1}{2}(H_{\boldsymbol{\nu},0} + H_{p,0} - \kappa_s(C_0))^2\partial_z k_\kappa(C_0) \\
&- \int_{-\infty}^{\infty} k_\kappa(C_0)(H_{\boldsymbol{\nu},0} + H_{p,0} - \kappa_s(C_0))\partial_z \kappa_s(C_0)
\end{aligned} \tag{5.24}$$

Consider the function

$$f(h_p, c, h_{\boldsymbol{\nu}}, h_d) := \tfrac{1}{2}k_\kappa(c)(h_{\boldsymbol{\nu}} + h_p - \kappa_s(c))^2 + k_g(c)(h_{\boldsymbol{\nu}}h_p - h_d^2)$$

which clearly is convex in h_p. The relation between the partial derivative

$$q(h_p, c, h_{\boldsymbol{\nu}}, h_d) := \partial_{h_p}f(h_p, c, h_{\boldsymbol{\nu}}, h_d) = k_\kappa(c)(h_{\boldsymbol{\nu}} + h_p - \kappa_s(c)) + k_g(c)h_{\boldsymbol{\nu}}$$

and h_p may be inverted and we may write $h_p = h_p(q, c, h_{\boldsymbol{\nu}}, h_d)$. The Legendre transform of f with respect to h_p is

$$j(q, c, h_{\boldsymbol{\nu}}, h_d) := f(h_p(q, c, h_{\boldsymbol{\nu}}, h_d), c, h_{\boldsymbol{\nu}}, h_d) - q h_p(q, c, h_{\boldsymbol{\nu}}, h_d)$$

and fulfills $\partial_c j = \partial_c f$. We remark that j is given by

$$j(q(h_p, c, ...), ...) = \tfrac{1}{2} k_\kappa(c)(h_{\boldsymbol{\nu}} + h_p - \kappa_s(c))^2 + k_g(c)(h_{\boldsymbol{\nu}} h_p - h_d^2)$$
$$- (k_\kappa(c)(h_{\boldsymbol{\nu}} + h_p - \kappa_s(c)) + k_g(c) h_{\boldsymbol{\nu}}) h_p \quad (5.25)$$

Observe that $Q_0 = q(H_{p,0}, C_0, H_{\boldsymbol{\nu},0}, H_{d,0})$. Using that $\partial_z Q_0 = 0$ and the matching condition (5.16)

$$\int_{-\infty}^{\infty} \partial_c f(H_{p,0}, C_0, H_{\boldsymbol{\nu},0}, H_{d,0}) \partial_z C_0$$
$$= \int_{-\infty}^{\infty} \partial_c j(Q_0, C_0, H_{\boldsymbol{\nu},0}, H_{d,0}) \partial_z C_0 + \partial_q j(Q_0, C_0, H_{\boldsymbol{\nu},0}, H_{d,0}) \partial_z Q_0$$
$$= \int_{-\infty}^{\infty} \partial_z j(Q_0, C_0, H_{\boldsymbol{\nu},0}, H_{d,0})$$
$$= j(Q_0, c_0, h_{\boldsymbol{\nu},0}, h_{d,0})(s, 0^+) - j(Q_0, c_0, h_{\boldsymbol{\nu},0}, h_{d,0})(s, 0^-)$$

With $\int_{-\infty}^{\infty} (\partial_z C_0)^2 = \int_{-\infty}^{\infty} (\tanh'(z))^2 = \tfrac{4}{3}$ and (1.4) and observing that by (5.9) $2\lambda_{c,0} = \lambda_A^{(1)} - \lambda_A^{(2)}$ we obtain from (5.24) that

$$0 = \int_{-\infty}^{\infty} \sigma h_{g,0}(\partial_z C_0)^2 + \lambda_{c,0} h'(C_0) \partial_z C_0 + \partial_c f(H_{p,0}, C_0, H_{\boldsymbol{\nu},0}, H_{d,0}) \partial_z C_0$$
$$= \sigma h_{g,0} \left(\int_{-\infty}^{\infty} (\partial_z C_0)^2 \right) + 2\lambda_{c,0} + \int_{-\infty}^{\infty} \partial_z j(Q_0, C_0, H_{\boldsymbol{\nu},0}, H_{d,0})$$
$$= \bar{\sigma} h_{g,0} + \lambda_A^{(1)} - \lambda_A^{(2)} + \left[j(Q_0, c_0, h_{\boldsymbol{\nu},0}, h_{d,0}) \right]_{(2)}^{(1)}.$$

Thanks to (5.25) we see that we have recovered (3.18). Furthermore, the identification (5.9) now is fully justified.

5.9. Inner solutions: Normal force balance. Equation (4.15) to order ε^{-1} is

$$0 = \partial_{zz} \left(k_\kappa'(C_0) C_1 (H_{\boldsymbol{\nu},0} + H_{p,0} - \kappa_s(C_0)) + k_\kappa(C_0)(H_{\boldsymbol{\nu},1} + H_{p,1} - \kappa_s'(C_0) C_1) \right)$$
$$+ \partial_{zz} \left(H_{\boldsymbol{\nu},0} k_g'(C_0) C_1 \right) + \partial_z \left(H_{\boldsymbol{\nu},1} \partial_z k_g(C_0) \right) - \partial_s H_{d,0} \partial_z k_g(C_0)$$
$$+ 2(H_{d,0} \boldsymbol{\mu}_0 \cdot \nabla_{\Gamma_0} \boldsymbol{p}_1 \boldsymbol{\tau}_0 - H_{d,0} \boldsymbol{\tau}_1 \cdot \boldsymbol{\mu}_0) \partial_{zz} k_g(C_0)$$
$$+ \sigma(\partial_z C_0)^2 H_{p,0} - \sigma \left(\tfrac{1}{2} (\partial_z C_0)^2 + \psi(C_0) \right) (H_{\boldsymbol{\nu},0} + H_{p,0}) \quad (5.26)$$

where as usual $\boldsymbol{\tau}_0$ and $\boldsymbol{\mu}_0$ are evaluated at $\tilde{\boldsymbol{x}}(s)$ and we used that $\partial_s C_0 = 0$ and $\partial_z Q_0 = 0$. In the last line we may use the equipartition of energy (5.22) to replace $\psi(C_0)$ by $\tfrac{1}{2} (\partial_z C_0)^2$ and altogether arrive at $-\sigma(\partial_z C_0)^2 H_{\boldsymbol{\nu},0}$ in that line.

We integrate with respect to z from 0 to a variable that we, for convenience,

denote by z again and obtain

$$
\begin{aligned}
0 = {} & \partial_z\big(k'_\kappa(C_0)C_1(H_{\boldsymbol{\nu},0} + H_{p,0} - \kappa_s(C_0)) + k_\kappa(C_0)(H_{\boldsymbol{\nu},1} + H_{p,1} - \kappa'_s(C_0)C_1)\big) \\
& + \partial_z\big(H_{\boldsymbol{\nu},0}k'_g(C_0)C_1\big) + \big(H_{\boldsymbol{\nu},1}\partial_z k_g(C_0)\big) - \partial_s H_{d,0}k_g(C_0) \\
& + 2(H_{d,0}\boldsymbol{\mu}_0 \cdot \nabla_{\Gamma_0}\boldsymbol{p}_1\boldsymbol{\tau}_0 - H_{d,0}\boldsymbol{\tau}_1 \cdot \boldsymbol{\mu}_0)\partial_z k_g(C_0) \\
& - \sigma\int_0^z (\partial_z C_0)^2 dz\, H_{\boldsymbol{\nu},0} + A
\end{aligned}
\tag{5.27}
$$

with an integration constant A. We are interested in the limit of this identity when $z \to \pm\infty$ and use the matching conditions (5.16)–(5.18) to draw conclusions. The fields $C_0, C_1, H_{\boldsymbol{\nu},0}, H_{p,0}, H_{d,0}$ are bounded and their derivatives with respect to z converge to zero. Since $\kappa'_s(\pm 1) = 0$ we also have that $\kappa'_s(C_0) \to 0$. Furthermore as $z \to \pm\infty$

$$
H_{\boldsymbol{\nu},1}(s,z) \sim h_{\boldsymbol{\nu},1}(\boldsymbol{x}(s,0^\pm)) + \nabla_{\Gamma_0}h_{\boldsymbol{\nu},0}(\boldsymbol{x}(s,0^\pm)) \cdot \boldsymbol{\mu}_0(\tilde{\boldsymbol{x}}(s))z,
$$

but since $\kappa_i = h_{\boldsymbol{\nu},i} + h_{p,i}$, $i = 0,1$, have bounded limits on γ_0 (see assumptions after (5.7)) we see that $H_{\boldsymbol{\nu},1}$ is at most of linear growth in z as $z \to \pm\infty$. But $\partial_z k_g(C_0)$ exponentially decays so that

$$
H_{\boldsymbol{\nu},1}\partial_z k_g(C_0) \to 0 \quad \text{as } z \to \pm\infty.
$$

With $\int_0^{\pm\infty}(\partial_z C_0)^2 = \pm 2/3$ and the relation (1.4) we obtain

$$
\begin{aligned}
(5.27) &\to k_\kappa^{(1)}\nabla_{\Gamma_0}(h_{\boldsymbol{\nu},0} + h_{p,0}) \cdot \boldsymbol{\mu} - k_g^{(1)}\partial_s h_{d,0} - \tfrac{\bar\sigma}{2}h_{\boldsymbol{\nu},0} - A && \text{as } z \to \infty, \\
(5.27) &\to k_\kappa^{(2)}\nabla_{\Gamma_0}(h_{\boldsymbol{\nu},0} + h_{p,0}) \cdot \boldsymbol{\mu} - k_g^{(2)}\partial_s h_{d,0} - \tfrac{\bar\sigma}{2}h_{\boldsymbol{\nu},0} - A && \text{as } z \to -\infty,
\end{aligned}
$$

where the functions are evaluated at $\boldsymbol{x}(s,0^+)$ in the first row and at $\boldsymbol{x}(s,0^-)$ in the second row. We remark that $\partial_s h_{d,0} = \nabla_{\gamma_0}h_{d,0} \cdot \boldsymbol{\tau}_0$. Subtracting the two rows we recover the only remaining equation (3.17) which finishes the asymptotic analysis.

Concluding, we have formally shown that the limiting membrane $\Gamma_0 = \Gamma_{0,1} \cup \gamma_0 \cup \Gamma_{0,2}$ together with $\lambda_{V,0}$, $\lambda_{A,0} + \lambda_{c,0}$, and $\lambda_{A,0} - \lambda_{c,0}$ solves Problem 3.10.

Appendix A. Sharp interface equilibrium equations in the axisymmetric case.

We briefly show that the interface conditions (3.15)–(3.18) coincide with the conditions in the axisymmetric case as stated, for instance, in [11].

The boundary Γ of the vesicle is obtained by rotating a curve around a symmetry axis. Denoting by s the arc-length parameter of the curve and by $\theta \in [0, 2\pi]$ the rotation angle the membrane surface may be parameterized in the form $(s,\theta) \mapsto (x(s), r(s)\cos(\theta), r(s)\sin(\theta))$ where $x(s)$ the coordinate along the symmetry axis and $r(s) \geq 0$ is the distance to it. For any s we may consider the curve

$$
\tilde{\gamma}(s) = \{(x(s), r(s)\cos(\theta), r(s)\sin(\theta)) \,|\, \theta \in [0, 2\pi]\},
$$

and the phase interface γ is such a curve at a specific position s^*.

Let $\psi(s)$ denote the angle between the plane perpendicular to the symmetry axis and the surface Γ so that $h_g(s) = \cos(\psi(s))/r(s)$ and $h_{\boldsymbol{\nu}}(s) = -\sin(\psi(s))/r(s)$ for the geodesic and the normal curvature of a curve $\tilde{\gamma}$. We here have chosen the orientation

to be such that $\boldsymbol{\nu}$ points outward of the vesicle. Since the curvature vector of a $\tilde{\gamma}$ is given by

$$\boldsymbol{h}_{\tilde{\gamma}}(s,\theta) = \Big(0, -\frac{\cos(\theta)}{r(s)}, -\frac{\sin(\theta)}{r(s)}\Big)$$

the normal curvature then has the opposite sign than in [11]. Furthermore, $h_p(s) = -\psi'(s)$ (again with the opposite sign than in [11]) and $h_d = 0$.

Some lengthy calculations involving rewriting the surface gradient ∇_Γ in terms of the coordinates (s,θ) show that for any quantity $f = \tilde{f}(s)$ that depends on s but not on θ we have that $\Delta_\Gamma f = \tilde{f}''(s) + r'(s)\partial_s \tilde{f}'(s)/r(s)$ (we will drop the tilde sign in the following when considering the field in the new coordinates for convenience). Furthermore, thanks to (2.8), some short calculations show that

$$|\nabla_\Gamma \boldsymbol{\nu}|^2(\kappa - \kappa_s) - \tfrac{1}{2}(\kappa - \kappa_s)^2\kappa = \tfrac{1}{2}\kappa(\kappa^2 - \kappa_s^2) + 2\big(\tfrac{\sin(\psi)}{r}\kappa + \tfrac{\sin^2(\psi)}{r^2}\big)(\kappa - \kappa_s).$$

Defining $Q := -k_\kappa \kappa'$ (where the prime denotes the derivative with respect to s; observe that $\kappa = h_{\boldsymbol{\nu}} + h_p$ depends on s only), equation (3.15) then becomes

$$0 = -Q' - \frac{r'}{r}Q + \frac{k_\kappa}{2}\kappa\big(\kappa^2 - \kappa_s^2\big) + 2k_\kappa(\kappa - \kappa_s)\Big(\frac{\sin(\psi)}{r}\kappa + \frac{\sin^2(\psi)}{r^2}\Big) + \lambda_A\kappa - \lambda_V$$

which is equation (1) of [11]. Using that $\nabla_\Gamma \kappa \cdot \boldsymbol{\mu} = -\partial_s \kappa$ we obtain from the normal force balance (3.17) equation (3) of [11], $0 = -\big[Q'\big]_{(1)}^{(2)} + \bar\sigma\frac{\sin(\psi)}{r}$.

The continuity condition (3.16) gives

$$0 = \big[k_\kappa(\kappa - \kappa_s)\big]_{(1)}^{(2)} - \big[k_g\big]_{(1)}^{(2)}\frac{\sin(\psi)}{r} \tag{A.1}$$

which is just (4) of [11]. With respect to the tangential force balance (3.18) we observe that

$$\big[\tfrac{k_\kappa}{2}(h_{\boldsymbol{\nu}} + h_p - \kappa_s)^2 + k_g(h_{\boldsymbol{\nu}} h_p - h_d^2)\big]_{(1)}^{(2)} - \big[(k_\kappa(h_{\boldsymbol{\nu}} + h_p - \kappa_s) + k_g h_{\boldsymbol{\nu}})h_p\big]_{(1)}^{(2)}$$

$$= \big[\tfrac{k_\kappa}{2}(h_{\boldsymbol{\nu}} + h_p - \kappa_s)(h_{\boldsymbol{\nu}} + h_p - \kappa_s - 2h_p)\big]_{(1)}^{(2)}$$

$$= \big[k_\kappa(h_{\boldsymbol{\nu}} + h_p - \kappa_s)\big]_{(1)}^{(2)} h_{\boldsymbol{\nu}} + \big[\tfrac{k_\kappa}{2}(h_{\boldsymbol{\nu}} + h_p - \kappa_s)(-h_{\boldsymbol{\nu}} - \kappa_s - h_p)\big]_{(1)}^{(2)}$$

which, using (A.1), is

$$= -\big[k_g\big]_{(1)}^{(2)}\frac{\sin^2(\psi)}{r^2} - \big[\tfrac{k_\kappa}{2}(\kappa^2 - \kappa_s^2)\big]_{(1)}^{(2)}. \tag{A.2}$$

Altogether, (3.18) yields

$$0 = -\big[k_g\big]_{(1)}^{(2)}\frac{\sin^2(\psi)}{r^2} - \big[\tfrac{k_\kappa}{2}(\kappa^2 - \kappa_s^2)\big]_{(1)}^{(2)} - \bar\sigma\frac{\cos(\psi)}{r} + \big[\lambda_A\big]_{(1)}^{(2)}$$

from which we recover the remaining equation (5) of [11].

Appendix B. Relation to results for open membranes.

Equilibrium conditions for two-phase membranes are stated in [32], Section 4.3. Using $|\nabla_\Gamma \boldsymbol{\nu}| = \kappa^2 - 2g$ and with the change of notation $\kappa \to 2H$, $g \to K$, $\lambda_V \to p$,

$\lambda_A^i \to 2\mu^i$, and $\kappa_s \to -c_0$ we see that our condition (3.15) coincides with equation (88) in that paper. Furthermore, with the change of notation $\boldsymbol{\mu} \to \boldsymbol{e}_2$, $h_{\boldsymbol{\nu}} \to k_n$, $h_d \to \tau_g$, $\nabla_\gamma(\cdot)\cdot\boldsymbol{\tau} \to \frac{d}{ds}(\cdot)$, $k_g \to \bar{k}$, and $h_g \to k_g$ we also see that (3.16) coincides with condition (90) and (3.17) with (89). But our condition (3.18) reveals the additional term

$$-\big[(k_\kappa(h_{\boldsymbol{\nu}} + h_p - \kappa_s) + k_g h_{\boldsymbol{\nu}})h_p\big]_{(1)}^{(2)} \tag{B.1}$$

in comparison with the corresponding condition (91).

In [32], the conditions for two-phase membranes are derived from the conditions for open membranes and can be traced back to the identities (71)–(76) where the variation of the energy of an open membrane is stated. The authors then assume that the two-phase membrane is smooth (see above identity (87) in [32]). Therefore, the quantity Ω_{323}, which is $\nabla_\Gamma(\boldsymbol{\nu} \cdot \boldsymbol{w}) \cdot \boldsymbol{\mu}$ in our notation, is continuous across the phase interface. Under this assumption it then would not be necessary to split up the term $\nabla_\Gamma(\boldsymbol{\nu} \cdot \boldsymbol{w})$ in our calculation (3.4). And since $h_p = -\boldsymbol{\mu} \cdot \nabla_\Gamma \boldsymbol{\nu}\boldsymbol{\mu}$ then would be continuous across γ the term (B.1) would vanish thanks to (3.16) so that (3.18) would coincide with (91).

We stress that condition (3.16) implies that, in general, the mean curvature is discontinuous across γ. Furthermore, in our calculation (A.2) it can be seen that we need the additional term (B.1) in order to obtain the equilibrium conditions for axisymmetric shapes stated in [11].

In turn, for an open membrane (i.e., Γ_1 is not present) without volume constraint ($\lambda_V = 0$) we obtain the equations (81), (83), (85), (86) in [32] (or (87)–(90) in [34]) since then $k_\kappa(\kappa - \kappa_s) + k_g h_{\boldsymbol{\nu}} = 0$ on γ by (3.16) so that the second term in (3.18) vanishes.

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