INTRODUCTION TO TOWERS IN TOPOLOGY

SAUL SCHLEIMER

Abstract. These lecture notes were written for a GAGTA mini-course in the summer of 2018. Our aim is to give a gentle introduction to three-manifolds. We will start with basic notions, prove the disk theorem using the famous tower argument, and give a variety of applications. If time permits we will discuss the JSJ theory and give the statement of the geometrisation theorem.

1. Introduction

In these lecture notes we cover the following material:

- Introduction to surfaces and three-manifolds. Statement of the disk theorem, loop theorem, and Dehn's lemma.
- Towers and the proof of the disk theorem. Characterisation of the unknot. Statements of the sphere, annulus, torus, and SFS theorems.
- Stallings' theorem on fibred manifolds. Statement of the Scott core theorem. The JSJ theory. Geometrisation.

Accessible references on this material include the following.

- *3-manifolds* by John Hempel,
- *Notes on basic 3-manifold topology* by Allen Hatcher, and
- *3-dimensional topology up to 1960* by Cameron Gordon.

There are exercises throughout the notes. Some of these are easy, some are more difficult, and some become easy after reading ahead a bit.

Please do email me about any mistakes you find.

2. Topological background

We use S^n, B^n, and T^n to denote the n–dimensional sphere, ball, and torus. Note that S^1 is also called the circle. We also use D^2 to denote B^2, the disk. If M is a manifold with boundary then we take ∂M to be the subspace of boundary points. We often write M^m to remind ourselves of the dimension of the manifold, and afterwards omit

Date: November 29, 2019.

the superscript. We write $M \cong N$ if M and N are homeomorphic. We say that a manifold M is *closed* if it is compact and without boundary.

One way to understand a manifold M is to cut it into pieces, understand the pieces, and then reassemble. So, we will need something to cut along. Recall that a *map* f is a continuous function. A *topological embedding* of manifolds $f\colon N \to M$ is a map which is a homeomorphism onto its image. If $f\colon S^1 \to S^2$ is a topological embedding then we call the image of f a *Jordan curve*. Here, then, is the Jordan-Schönflies theorem.

Theorem 2.1. *[Theorem 3.1, Thomassen, 1992] Suppose that $A \subset S^2$ is a Jordan curve. Then there is an ambient isotopy of S^2 taking A to a round circle.* □

We turn this into a definition as follows. Suppose that A is a circle topologically embedded in a surface F. We say A is *trivial* if

- A separates F and
- one component of $F - A$ has closure in F homeomorphic to a disk with boundary A.

If either property fails then we call A *essential*. In particular, non-separating curves are essential. If F is a compact connected oriented surface then we may cut F repeatedly along disjoint essential curves $A_1, A_2, \ldots$. We require that A_{k+1} is non-separating in $F - \cup_{i=1}^{k} A_k$. The length of any such sequence is the *genus* of F [Clebsch, 1865]. Jordan's theorem (**??**) tells us that the genus of the two-sphere is zero.

The obvious generalisation of Theorem **??** to the three-sphere is false; one counterexample is the *Alexander horned sphere* [Alexander, 1924]. The difficulty is caused by linking happening on smaller and smaller scales. If we restrict ourselves to *locally flat* maps, then we get a theory of surfaces in three-manifolds that is "equivalent" to the piecewise linear and smooth theories. A closely related discussion, with references and exercises, can be found in the third chapter of [Thurston, 1997]. In these notes we will work in whichever setting is the most convenient.

To define local flatness, we recall the notion of a *pair of spaces*: given an $N \subset M$ we write (M, N). All of the usual notions (homeomorphism, embedding, and so on) generalise to pairs.

Definition 2.2. Suppose that $N^n \subset M^m$ is a topologically embedded manifold. We say that N is *locally flat* if for all $x \in N$ there is a chart $U \subset M$ about x so that $(U, U \cap N) \cong (\mathbb{R}^m, \mathbb{R}^n)$.

Local flatness rules out all bad behaviour, as shown by Alexander's theorem.

Theorem 2.3. *[Alexander, 1924] Suppose that $A \subset S^3$ is a locally flat two-sphere. Then there is an ambient isotopy of S^3 taking A to a round two-sphere.* □

Exercise 2.4. Work through the details of a proof of Alexander's theorem. For example, see Theorem 1.1 of [Hatcher 2001].

From now on we will assume without mention that objects are sufficiently nice (locally flat, piecewise linear, or smooth) to allow us to carry out cut-and-paste constructions.

Alexander's theorem says that an embedded two-sphere A in S^3 bounds three-balls on both "sides". To formalise this, recall that a *proper map* of compact pairs $f\colon (F, \partial F) \to (M, \partial M)$ has the property that $f^{-1}(\partial M) = \partial F$.

Definition 2.5. Suppose that F^2 is a connected surface, properly embedded in a three-manifold M^3. Let $N(F)$ be a regular neighbourhood of F, taken in M. If F separates $N(F)$ then we say that F is *two-sided* in M. Otherwise we say that F is *one-sided*.

As a simple example, there is a one-sided proper embedding of the Möbius strip into the solid torus $D^2 \times S^1$.

Exercise 2.6. Any proper embedding of S^2, into any three-manifold M, is two-sided.

Exercise 2.7. Any closed surface F^2 embedded in S^3 separates; thus F is two-sided and also orientable.

Remark 2.8. The property of $F^2 \subset M^3$ being two-sided is equivalent to F being *transversely orientable* in M. Thus for surfaces in three-manifolds there are three important notions of orientability: the orientability of M, the orientability of F, and the transverse orientability of F in M. It is interesting to think about how these three properties interact.

Suppose that (M^3, F^2) is a compact pair where F is connected, properly embedded, and separates. Let N be the closure of one of the two components of $M - F$. If N is disjoint from ∂M we say that F *bounds* N in M.

Exercise 2.9. If $S^2 \subset M^3$ bounds three-balls on both sides, then M is homeomorphic to S^3.

Here is a nice generalisation of Theorem **??**, also found in [Alexander, 1924].

Exercise 2.10. Suppose that $T^2 \subset S^3$ is an embedded torus. Then T bounds a solid torus on at least one side.

Exercise 2.11. Suppose that $T^2 \subset M^3$ is a separating torus which bounds solid tori on both sides. Then M is a *lens space*. Prove that M is either homeomorphic to $S^2 \times S^1$ or M is finitely covered by S^3.

3. Irreducible and prime

Cutting a three-manifold M along embedded two-spheres was first studied in [Kneser, 1929]. Suppose that $A^2 \subset M$ is a properly embedded two-sphere. We say that A is *trivial* in M if A bounds a three-ball on at least one side. If A is not trivial, we call A *essential*.

Definition 3.1. A three-manifold M is *irreducible* if all properly embedded two-spheres in M are trivial.

Definition 3.2. A three-manifold M is *prime* if all properly embedded, separating, two-spheres in M are trivial.

Thus, if M is irreducible it is prime.

Exercise 3.3. List all three-manifolds which are prime but not irreducible.

If M and N are connected three-manifolds, then we may

(1) remove a small open ball from each and
(2) glue the resulting boundary two-spheres.

This gives a new three-manifold $M\#N$: the *connect sum* of M and N. If both M and N are oriented, we may require that the gluing be orientation reversing, and so obtain an orientation on $M\#N$. Some work is required to prove that this is all well-defined. See for example [Gugenheim, 1953]. Note that $M\#S^3 \cong S^3$ and $L\#(M\#N) \cong (L\#M)\#N$.

Theorem 3.4. *[Kneser, 1929] Suppose that M^3 is a compact and connected. There there is a finite collection of prime manifolds $\{P_i\}$ so that $M \cong \#_i P_i$. Furthermore, the collection $\{P_i\}$ is unique up to reordering.* □

The opposite of connect sum is *sphere surgery*, as follows. Suppose that $A \subset M$ is a properly embedded two-sphere. Let $n(A)$ be the interior of $N(A)$. Form $M_A = M - n(A)$. Finally, attach three-balls to the resulting pair of two-sphere boundary components of M_A.

4. The disk theorem

Given a three-manifold M, Kneser tells us which spheres to surger: the essential ones. All attempts to understand cutting along surfaces of higher genus lead to the *disk theorem*. We give the statement here, but defer some of the definitions to Section **??** and the proof to Section **??**.

Theorem ??. *[Disk theorem] Suppose that (M^3, F^2) is a compact, connected pair with $F \subset \partial M$ non-empty. Suppose that $H \lhd \pi_1(F)$ is a normal subgroup. Suppose that $f\colon (D, \partial D) \to (M, F)$ is a general position map with $[f|\partial D] \notin H$. Then there is a proper embedding of a disk $g\colon (E, \partial E) \to (M, F)$ so that*

1. *g is made from sectors of f and*
2. *$[g|\partial E] \notin H$.*

The key idea in the proof, namely the *tower* argument, is due to [Papakyriakopoulos, 1957]. This formulation of the disk theorem is due to [Stallings, 1960]. See [Gordon, 1999] for a detailed history of the proofs (and false proofs!) of the disk theorem. There are two well-known variants of the disk theorem, which follow quickly from it.

Theorem 4.1. *[Loop theorem] Suppose that (M^3, F^2) is a compact, connected pair with $F \subset \partial M$ non-empty. Suppose that the induced homomorphism $i_*\colon \pi_1(F) \to \pi_1(M)$ is not injective. Then there is an essential embedded loop $A \subset F$ which is null-homotopic in M.* □

Theorem 4.2. *[Dehn's lemma] Suppose that (M^3, F^2) is a compact, connected pair with $F \subset \partial M$ non-empty. Suppose that $A \subset F$ is an embedded curve which is null-homotopic in M. Then A bounds a properly embedded disk in M.* □

Here was Dehn's desired application.

Exercise 4.3. Suppose that K is *knot* in S^3: an embedded copy of S^1. Prove that K is ambiently isotopic to a round circle if and only if $\pi_1(S^3 - n(K)) \cong \mathbb{Z}$.

5. One-half lives, one-half dies

Suppose that Γ is a finite, connected, polygonal graph embedded in $\mathbb{R}^3$. So the fundamental group $\pi_1(\Gamma)$ is isomorphic to a free group, say $\mathbb{F}_g$. We define $V = N(\Gamma)$, a regular neighbourhood of Γ, to be a *handlebody* of *genus* g.

Exercise 5.1. Show that V contains a collection of disjoint properly embedded disks $D_1, D_2, \ldots$ so that D_{k+1} is non-separating in $V - \cup_{i=1}^k D_i$. Prove that the length of any such sequence is g, the genus of V.

If F^2 is a closed, connected, oriented surface of genus g then $H_1(F, \mathbb{Z}) \cong \mathbb{Z}^{2g}$. If V is a handlebody of genus g then $H_1(V, \mathbb{Z}) \cong \mathbb{Z}^g$. Also, ∂V is homeomorphic to F.

Exercise 5.2. Give a direct proof that $i_*\colon H_1(\partial V, \mathbb{Z}) \to H_1(V, \mathbb{Z})$ is surjective. Also, the kernel is isomorphic to $\mathbb{Z}^g$.

This generalises to give the "one-half lives, one-half dies" lemma.

Lemma 5.3. *Fix a field K. Suppose that M^3 is compact, connected, and oriented (over K). Let $i\colon \partial M \to M$ be the inclusion, which induces a homomorphism $i_*\colon H_1(\partial M, K) \to H_1(M, K)$. Then*

$$\frac{1}{2}\dim(H_1(\partial M)) = \dim(\ker(i_*)) = \dim(\mathrm{image}(i_*)). \qquad \square$$

Exercise 5.4. Suppose that M is a compact connected three-manifold. Can ∂M be homeomorphic to P^2, the real projective plane?

Here is the consequence we will need in our proof of the disk theorem.

Corollary 5.5. *Suppose that M is a compact, connected three-manifold. Suppose that M does not admit a double cover. Then M is orientable (over any field). Furthermore, all boundary components of M (if any) are two-spheres.* $\square$

6. Transversality and general position

6.1. **Transversality.** We follow [Guillemin and Pollack, 1974] in defining transversality. We abbreviate this to [GP].

Suppose that $f\colon N^n \to M^m$ is a smooth map. Suppose that $L^\ell \subset M$ is a submanifold. Recall that f is *transverse* to L if, for every point $x \in f^{-1}(L)$, we have $df(T_xN) + T_{f(x)}L = T_{f(x)}M$. As a consequence, the preimage $f^{-1}(L)$ is a submanifold of N. Also, the codimension of $f^{-1}(L)$ in N equals that of L in M [GP, page 28]. This generalises in a natural way to manifolds with boundary [GP, page 60]. Finally, if $f\colon N \to M$ is smooth, and $L \subset M$ is a submanifold, then there is a smooth perturbation of f making it transverse to L [GP, page 68].

6.2. **General position.** We now follow pages 8 to 13 of [Hempel, 1976] in defining general position, a weak form of self-transversality. Suppose that $f\colon (F^2, \partial F) \to (M^3, \partial M)$ is a proper map of compact pairs.

Definition 6.1. Suppose that, at $x \in F$, there is a neighbourhood $U_x \subset F$ so that $f|U_x$ is conjugate to the map

$$F\colon D^2 \to D^2 \times \mathbb{R}, \qquad z \mapsto (z^2, \mathrm{Im}\, z)$$

where we think of D^2 as the unit disk in the complex plane. Then we say that $f(x)$ is a *simple branch point* of f. (The image of F is sometimes called a *Whitney umbrella*.)

Definition 6.2. Suppose that, at $x \in F$, the map f is *self-transverse*:

- the set $f^{-1}(f(x))$ has size at most three and,
- for all $y, z \in f^{-1}(f(x))$ there are neighbourhoods $U_y, U_z \subset F$ so that $f|U_y$ and $f|U_z$ are embeddings with the image of each transverse to the image of the other.

In this case we call $f(x)$ a *triple point*, a *double point*, or a *regular point* as $f^{-1}(f(x))$ has size three, two, or one.

Definition 6.3. We say f is a *general position map* if it satisfies the following.

(1) It is an immersion away from a finite collection of simple branch points.
(2) It is at worst two-to-one away from a finite collection of triple points.
(3) It is an embedding away from a finite collection of arcs and curves of double points.

In particular, $f|\partial F$ is an immersion. The points of $f(\partial F)$ with two preimages are called *boundary double points*.

We define the *complexity* of a general position map f to be

$$c(f) = (s(f), t(f), d(f))$$

where s, t, and d count the number of simple branched points, triple points, and double arcs and curves of f, respectively. We order complexities lexicographically. If $s(f)$ is zero then f is an immersion. If $c(f) = (0, 0, 0)$ then f is an embedding.

Exercise 6.4. There is no proper embedding of the Mobius strip M^2 into the three-ball B^3. Find a general position map $f\colon M^2 \to B^3$. What is the minimal possible complexity for such a map?

We end this section with result similar to Theorem 1.14 of [Hempel, 1976].

Theorem 6.5. *Suppose that $f\colon (F, \partial F) \to (M, \partial M)$ is a proper map of compact pairs. Then there is a proper homotopy (with tracks as small as desired) making f into general position map.* □

7. Singularities, sectors, and swaps

Suppose that $f\colon (F, \partial F) \to (M, \partial M)$ is a general position map of compact pairs.

Definition 7.1. We define

$$\Sigma(f) = \{x \in F : |f^{-1}(f(x))| > 1\}$$

to be the *singular set* of f. This is a union of arcs, curves, and graphs. The latter have vertices of valence four, two, or one. Set $\mathcal{S}(f) = f(\Sigma(f))$. The vertices of $\mathcal{S}(f)$ have valence six or one: the former are triple points while the latter are simple branch points or boundary double points. We call the components of $f(F) - \mathcal{S}(f)$ the *sectors* of f.

Exercise 7.2. Draw $\Sigma(f) \subset F$ and $\mathcal{S}(f) \subset M$ for the general position map you found in Exercise **??**. Mark all simple branch points and triple points.

We now say what it means to be made from sectors of a general position map.

Definition 7.3. Suppose that $f\colon (F, \partial F) \to (M, \partial M)$ is a general position map with F oriented. For every vertex $v \in \mathcal{S}(f)$ let $B(v)$ be a sufficiently small three-ball about v so that $f(F) \cap B(v)$ is a model neighbourhood for $v \in f(F)$.

Set

$$\mathcal{B}(f) = \cup_v B(v).$$

Take $N(\mathcal{S}(f))$ a much smaller neighbourhood. Set

$$\mathcal{T}(f) = N(\mathcal{S}(f)) - \text{interior}(\mathcal{B}(f)).$$

For every component α of $\mathcal{S}(f) - \text{interior}(\mathcal{B}(f))$ there is a component $T(\alpha) \subset \mathcal{T}(f)$ containing it; note that $T(\alpha)$ comes with a homeomorphism to $D^2 \times \alpha$. A surface $R \subset T(\alpha)$ is *vertical* if R is of the form $\beta \times \alpha$ for some properly embedded arc $\beta \subset D^2$.

Suppose that $g\colon (G, \partial G) \to (M, \partial M)$ is a proper embedding. We say that g is *made from sectors* of f if

- every component C of $g(G) - n(\mathcal{S}(f))$ is contained in some sector of f,
- for every arc α of $\mathcal{S}(f) - \text{interior}(\mathcal{B}(f))$ the intersection $g(G) \cap T(\alpha)$ is a (perhaps empty) disjoint union of vertical surfaces, and
- for every vertex v of $\mathcal{S}(f)$ the intersection $g(G) \cap B(v)$ is a (perhaps empty) disjoint union of disks.

As a very special case of Definition **??**, suppose that $f\colon (F, \partial F) \to (M, \partial M)$ is a general position map without simple branch or triple points. That is, $c(f) = (0, 0, d)$ and $\mathcal{S}(f)$ is a one-manifold properly embedded in M. Suppose that $g\colon (G, \partial G) \to (M, \partial M)$ is a proper embedding. Suppose that g is made from sectors of f. Then we say that g is obtained from f by *swaps and discards*.

8. Beginning the proof of the disk theorem

From now on our general position map will have the form $f\colon (D, \partial D) \to (M, F)$ where $F \subset \partial M$. We recall the statement before tackling the proof.

Theorem 8.1. *[Disk theorem] Suppose that (M^3, F^2) is a compact, connected pair with $F \subset \partial M$ non-empty. Suppose that $H \triangleleft \pi_1(F)$ is a normal subgroup. Suppose that $f\colon (D, \partial D) \to (M, F)$ is a general position map with $[f|\partial D] \notin H$. Then there is a proper embedding of a disk $g\colon (E, \partial E) \to (M, F)$ so that*

(1) *g is made from sectors of f and*
(2) *$[g|\partial E] \notin H$.*

We begin with two easy lemmas.

Lemma 8.2. *If f has a simple branch point, then there is another general position map $f'\colon (D, \partial D) \to (M, F)$ so that $c(f') < c(f)$ and $[f'|\partial D] \notin H$.*

Proof. Suppose that the vertex $u \in \mathcal{S}(f)$ is the given simple branch point. Let α be the double arc adjacent to u. Let v be the other endpoint of α.

- If v is a branch point then, as D is oriented, we can do a swap along α and discard the sphere component. This reduces s by at least two and d by at least one.
- If v is a triple point then we can homotope u along α and past v. This leaves s unchanged but reduces t by one and d by at least one.
- If v is a boundary double point, then we do a swap along α. This separates D into a pair of disks, D' and D''. Since the product of $[f'|\partial D']$ and $[f'|\partial D'']$ equals $[f|\partial D]$ at least one of the former is essential. We discard the other. This reduces s and d by at least one. □

Lemma 8.3. *Suppose that $c(f) = (0, 0, d)$. Then there is some collection of swaps and discards giving the desired embedding g.* □

Exercise 8.4. Provide a proof of Lemma **??**.

9. The tower

Suppose that $f\colon (D, \partial D) \to (M, F)$ is as in the statement of the disk theorem (**??**). Applying Lemma **??**, we may assume that $s(f) = 0$.

9.1. **At the base.** We build the base of the tower as follows. Set $(M_0, F_0) = (M, F)$. Set $H_0 = H$. Set $f_0 = f$.

9.2. **Climbing the tower.** Induction gives us a general position map $f_k\colon (D, \partial D) \to (M_k, F_k)$ so that $[f_k|\partial D] \notin H_k$. Let N_k be a regular neighbourhood of $f_k(D)$, taken in M_k. Let $G_k = N_k \cap F_k$. Let $i_k\colon (N_k, G_k) \to (M_k, F_k)$ be the inclusion map. Define $K_k = (i_k)_*^{-1}(H_k)$. Note that $K_k \triangleleft \pi_1(G_k)$

Let $h_k\colon (D, \partial D) \to (N_k, G_k)$ be the induced general position map. Note that $f_k = i_k \circ h_k$. Thus $[h_k|\partial D] \notin K_k$.

If N_k has no connected double cover then we set $n = k$ and the construction of the tower is complete. Suppose instead that $p_k\colon M_{k+1} \to N_k$ is a non-trivial double cover. Set F_{k+1} equal to the full lift of G_k. Define $H_{k+1} = (p_k)_*^{-1}(K_k)$. Let h' and h'' be the two lifts of h_k to M_{k+1}. Since $h_k(D)$ is a spine for N_k, the union $h'(D) \cup h''(D)$ is a spine for M_{k+1}. Thus the union is connected. Thus

$$c(h') = c(h'') < c(h_k) = c(f_k).$$

Set $f_{k+1} = h'$. Thus $f_{k+1}\colon (D, \partial D) \to (M_{k+1}, F_{k+1})$ is a general position map. Thus $[f_{k+1}|\partial D] \notin H_{k+1}$ and $c(f_{k+1}) < c(f_k)$. This completes the induction step.

Since the complexity decreases after every lift, we eventually reach the top of the tower.

9.3. **Enjoying the view.** Since $h_n(\partial D)$ is contained in G_n, the manifold N_n has non-empty boundary. Since N_n has no non-trivial double cover, by Corollary **??** all components of ∂N_n are two-spheres. Let $S_n \subset \partial N_n$ be the two-sphere containing G_n. Note that, as $[h_n|\partial D] \in \pi_1(G_n) - K_n$, the normal subgroup K_n is not all of $\pi_1(G_n)$. Since the boundary components of G_n normally generate $\pi_1(G_n)$, there is some boundary component, say $\gamma \subset \partial G_n$ so that $[\gamma] \notin K_n$. By Jordan's theorem (**??**) the curve γ bounds a disk $E \subset S_n - G_n$. Perform a small isotopy, moving E into N_n. After applying i_n we obtain a proper embedding $g_n\colon (E, \partial E) \to (M_n, F_n)$ which is made from sectors of f_n. Finally, $[g_n|\partial E] \notin H_n$.

9.4. **Descending the tower.** Induction gives us a proper embedding $g_k\colon (E, \partial E) \to (M_k, F_k)$, made from the sectors of f_k with $[g_k|\partial E] \notin H_k$. We form

$$g' = i_{k-1} \circ p_{k-1} \circ g_k \qquad \text{with} \qquad g'\colon (E, \partial E) \to (M_{k-1}, F_{k-1}).$$

So g' is a proper immersion without triple points, made from sectors of f_{k-1} and having $[g'|\partial E] \notin H_{k-1}$. We apply Lemma **??** to obtain an proper embedding $g_{k-1}\colon (E, \partial E) \to (M_{k-1}, F_{k-1})$, made from the sectors of f_{k-1}, and again having $[g_{k-1}|\partial E] \notin H_{k-1}$.

Thus $g = g_0$ is the desired proper embedding. This completes the proof of the disk theorem. □

10. The sphere, annulus, torus, and SFS theorems

The compact connected oriented surfaces of non-negative Euler characteristic are the sphere S^2, the disk D^2, the annulus A^2, and the torus T^2. It is a theme of low-dimensional topology that these surfaces, embedded in a three-manifold, lead to "flexibility". To obtain three-manifolds with more rigid structures (namely unique hyperbolic metrics) we should cut along such surfaces.

It is often easier to detect homotopy versions of these surfaces. The disk theorem tells us that a homotopy essential disk implies the existence of a embedded essential disk. The following theorems give the same result for the sphere, the annulus, and the torus. We first follow [Hempel 1976].

Theorem 10.1. *[Sphere theorem] Suppose that M^3 is closed, connected, and oriented. Suppose that $\pi_2(M)$ is non-trivial. Then M contains an essential embedded two-sphere.* □

Now we follow [Scott, 1980].

Theorem 10.2. *[Annulus theorem] Suppose that M^3 is compact, oriented, and irreducible with incompressible boundary. Suppose that $f\colon (A^2, \partial A) \to (M, \partial M)$ is an essential map. Then there is an essential embedded annulus in M.* □

Theorem 10.3. *[Torus theorem] Suppose that M^3 is compact, oriented, and irreducible, with incompressible boundary. Suppose that $f\colon T^2 \to M$ is an essential proper map. Then either*

- *there is an essential embedded torus in M or*
- *$\mathbb{Z} \triangleleft \pi_1(M)$.*

This last should be combined with work of [Gabai, 1992] and [Casson-Jungreis, 1994].

Theorem 10.4. *[Seifert fibred space theorem] Suppose that M^3 is closed, connected, oriented, and irreducible. Suppose that $\mathbb{Z} \triangleleft \pi_1(M)$. Then M is a Seifert fibred space.* □

11. Incompressible surfaces

When dealing with surfaces of higher complexity, there are two basic definitions.

Definition 11.1. We say that a connected surface F^2 embedded in a connected three-manifold M^3 is π_1*–injective* if the inclusion induces an injection on fundamental groups.

Before giving the second definition, we need a new concept. Suppose that $F^2 \subset M^3$ is properly embedded. Suppose that $(D, \partial D) \subset (M, F)$ is an embedded disk, disjoint from ∂M, and with $D \cap F = \partial D$. Then we call D a *surgery disk* for F. This is because we can carry out the following operation. Let $N(D)$ be a regular neighbourhood of D, taken in M. Set $A = F \cap N(D)$. So A is an annulus neighbourhood of ∂D taken in F. Let D' and D'' be the disk components of $\partial N(D) - F$. The *disk surgery* of F along D is the surface

$$F_D = (F - A) \cup (D' \cup D'').$$

Note that F_D is again properly embedded in M. If ∂D is essential in F then we call D a *compressing disk* for F.

Definition 11.2. Suppose that $F^2 \subset M^3$ is properly embedded. We say that F is *incompressible* if F has no compressing disk.

We extend the definition of incompressibility to subsurfaces of ∂M by isotoping them slightly into M, while keeping the boundary in the boundary.

Exercise 11.3. If $F \subset M$ is properly embedded, connected, and π_1–injective, then F is incompressible.

Exercise 11.4. Find an example of a connected surface F^2, properly embedded in some three-manifold M^3, so that F is incompressible but not π_1–injective.

Definition 11.5. Suppose that $A \subset D^2$ is a disjoint union of embedded curves. By Jordan's theorem (**??**) every curve $\alpha \subset A$ bounds an embedded disk $D_\alpha \subset D$. If $D_\alpha \cap A = \alpha$ then we call α an *innermost curve* and D_α an *innermost disk* for A in D.

Exercise 11.6. Suppose that $F \subset M$ is properly embedded and incompressible. Then every connected component $F' \subset F$ is incompressible.

Exercise 11.7. Suppose that M^3 is compact, connected, and irreducible. Suppose that $F, G \subset \partial M$ are compact and disjoint subsurfaces. Suppose that $\phi\colon F \to G$ is a homeomorphsim. Then the quotient three-manifold $M_\phi = M/(x \sim \phi(x))$ is irreducible.

In contrast to Exercise **??** we have the following.

Lemma 11.8. *Suppose that $F^2 \subset M^3$ is properly embedded, connected, two-sided, and incompressible. Then F is π_1–injective.*

Proof. We prove the contrapositive. Suppose that γ is an immersed loop in F so that $[\gamma]$ lies in the kernel of i_*. Let $f\colon (D, \partial D) \to (M, F)$ be a null-homotopy of γ. Homotope f so that

- $f|\partial D$ is an immersion,
- on a collar $C \subset D$ of ∂D, the image $f(C)$ lies on one side of F, and
- $f|\text{interior}(D)$ is transverse to F.

So $A = f^{-1}(F) \subset D$ is a disjoint union of curves.

Pick α, bounding D_α, innermost in A. If $[f|\alpha] = 1 \in \pi_1(F)$ then we can replace $f|D_\alpha$ by the null-homotopy of $f|\alpha$ in F. We then push off to get a null-homotopy of γ in M meeting F fewer times.

So we may assume, instead, that $[f|\alpha] \neq 1 \in \pi_1(F)$. Take $M_F = M - n(F)$. Note that there are two copies of F, say F' and F'', lying in ∂M_F. The image of D_α meets exactly one of these, say F'. So $f|D_\alpha$ is a null-homotopy, in M_F, of a loop in F'. By the disk theorem (**??**) there is a compressing disk E for F' in M_F. Thus F compresses in M. □

Exercise 11.9. Suppose that F is a closed, connected surface, of genus at least one, properly embedded in the three-ball B^3. Show that F is compressible on at least one side.

12. Stallings' theorem

Suppose that F^2 is compact, connected, and oriented. Suppose that $\phi\colon F \to F$ is an orientation preserving homeomorphism. We form M_ϕ, the *surface bundle* over the circle with *fibre* F and *monodromy* ϕ, as follows.

$$M_\phi = F \times [0,1] \big/ (x,1)\ (\phi(x),0)$$

The associated map $f\colon M_f \to S^1 = [0,1]/(0\ 1)$ is defined by $f(x,t) = t$. Note that if F is not a two-sphere then, by Exercise **??**, the three-manifold M_ϕ is irreducible. Also, every fibre $F_t = F \times \{t\}$ is π_1–injective in M_ϕ.

We follow [Stallings, 1961].

Theorem 12.1. *Suppose that M^3 is closed, connected, oriented, and irreducible. Suppose that $f\colon M \to S^1$ is a map so that*

- $K = \ker(f_*\colon \pi_1(M) \to \mathbb{Z})$ *is finitely generated and*
- f_* *is surjective.*

Then M is a surface bundle over the circle, f is homotopic to the bundle map, and K is isomorphic to the fundamental group of the fibre.

Proof. Fix $p \in S^1$. Homotope f to make it transverse to p. Define $F = f^{-1}(p)$. By transversality, F is a closed, oriented, two-sided surface.

Suppose that F is compressible, say by the disk $(D, \partial D) \subset (M, F)$. Let $N(D)$ be a regular neighbourhood of D. We define a homotopy from $f_0 = f$ to f_1 which is fixed on $M - n(D)$ and which moves $f(D)$ past p. This done, the new map f_1 has a new preimage $F_1 = f_1^{-1}(p)$ which is isotopic to F_D, the result of compressing F along D. This reduces the genus of some component of F. So after finitely many such homotopies we arrive at a map f_n with preimage F_n being a disjoint union of incompressible surfaces (Exercise **??**) and two-spheres. We abuse notation and again call the map f and the preimage F.

Suppose that F is a disjoint union of incompressible surfaces and two-spheres. Since M is irreducible, each two-sphere bounds a ball in M. Let $S \subset F$ be an innermost two-sphere, bounding an innermost three-ball B. We again define a homotopy from $f_0 = f$ to f_1 which is fixed on $M - n(B)$ and which moves B past p. This done, the new map f_1 has a new preimage $F_1 = f_1^{-1}(p)$ with one fewer two-sphere than F. After finitely many such homotopies we arrive at a map f_n with preimage F_n being a disjoint union of incompressible surfaces. We abuse notation and again call the map f and the preimage F.

Thus F is closed, orientable, two-sided, and incompressible. By Lemma **??** and Exercise **??** every component of F is π_1–injective in M.

Now, set $M_F = M - n(F)$. Note that $\partial M_F \cong F_- \cup F_+$ is two copies of F. Let M^K be the cover of M corresponding to $K = \ker(f_*)$. So M^K is obtained from $M_F \times \mathbb{Z}$ by gluing $F_+ \times \{n\}$ to $F_- \times \{n+1\}$.

Claim. M_F and F are both connected.

Proof. Every component of M_F has boundary, as M was connected. Suppose some component N of M_F only meets components of F_+ (say). Then we may homotope f to move N past p, and so reduce the number of components of the preimage. Thus we may assume that every component of M_F meets both F_+ and F_-.

Suppose that some component N of M_F meets two components of F_+. Then the pigeonhole principle implies that there is some component N' of N_F that meets two components of F_-. Since M is connected, we can construct a loop γ in M so that

- $f_*([\gamma]) = 0$ and
- γ has algebraic intersection number one with some component of F.

Thus γ lifts to M^K. The translates of γ under the deck group gives a subgroup of $H_1(M^K, \mathbb{Z})$ of infinite rank, which is a contradiction.

So every component of M_F meets exactly one component of F_+ and F_-. If M_F is not connected, then f_* is not surjective, a contradiction. □

Claim. The homomorphism $i_*\colon \pi_1(F_-) \to \pi_1(M_F)$ is surjective.

Proof. Fix any loop $\alpha \subset M_F$. Recall that $K \cong \pi_1(M^K)$ is finitely generated. We obtain M^K from $M_F \times \mathbb{Z}$ by gluing copies of F_+ to the corresponding copies of F_-. Let $M_F(n)$ be the n^{th} copy. Let $M_F[0,n]$ be the union $\cup_{i=0}^{n-1} M_F(i)$. We take n large enough so that $M_F[0,n]$ contains a generating set for $\pi_1(M^K)$. Let $\alpha(n)$ be the copy of α in $M_F(n)$. Thus there is free homotopy $h\colon A^2 \to M^K$ with one end in $M_F[0,n]$ and the other on $\alpha(n)$. We make h transverse to $F(n)$, the copy of F between $M_F[0,n]$ and $M_F(n)$. Thus $h^{-1}(F(n))$ is a collection of loops in A. We eliminate all trivial curves, innermost first, via homotopies of h, relative to the boundary of A. Since $F(n)$ separates there is now a sub-annulus of A giving a homotopy of $\alpha(n)$ into $F(n)$, as desired. □

Thus inclusion induces isomorphisms $\pi_1(F_-) \cong \pi_1(M_F) \cong \pi_1(F_+)$. By Exercise **??** we have that $M_F \cong F \times [0,1]$. This completes the proof of Theorem **??**. □

Exercise 12.2. Suppose that M^3 is a compact connected oriented irreducible three-manifold. Suppose that $\pi_1(M) \cong \mathbb{F}_g$ is a free group. Prove that M is homeomorphic to a genus g handlebody.

Exercise 12.3. Suppose that F^2 is closed, connected, oriented, and not a two-sphere. Suppose that M^3 is a compact connected oriented irreducible three-manifold. Suppose that $\pi_1(M) \cong \pi_1(F)$. Prove that M is homeomorphic to $F \times [0,1]$.

13. Further topics

Similar to the Seifert fibred space theorem (**??**), Stallings' theorem (**??**) takes very minimal algebraic hypotheses and converts them to a very strong topological conclusion. Here is another such, similar in spirit to Stallings' theorem, due to [Scott, 1973]

Theorem 13.1. *[Scott core theorem] Suppose that $H < \pi_1(M)$ is finitely generated. Then H is finitely presented. In fact, in the cover M^H corresponding to H there is a compact submanifold N so that $i_*\colon \pi_1(N) \to \pi_1(M^H) \cong H$ is an isomorphism.* □

Setting aside the study of subgroups of three-manifold groups, we return to the theory of decomposing manifolds. After reducing along essential spheres, and cutting along essential disks, there is the JSJ theory due to [Jaco-Shalen, 1979] and [Johannson, 1979]. Every compact, connected, oriented irreducible three-manifold has a *canonical submanifold* which contains all essential, non-peripheral annuli and tori.

After removing the canonical JSJ sub-manifold, the pieces of M that remain are irreducible, acylindrical, atoroidal, and have incompressible boundary. These pieces are the subject of Thurston's geometrisation programme, proven by [Perelman, 2002, 2003, 2003].

Theorem 13.2. *Suppose that M is compact, connected, oriented, irreducible, acylindrical, atoroidal, and all boundary components are incompressible.*

- *If $\pi_1(M)$ is finite then M is homeomorphic to either B^3 or a spherical space form.*
- *If $\pi_1(M)$ is infinite, and all components (if any) of ∂M are tori, then the interior of M is homeomorphic to a finite volume hyperbolic manifold.* □

Note that the first conclusion includes the Poinaré conjecture as a special case. The second conclusion asserts that hyperbolic geometry is, unavoidably, an integral part of three-manifold topology.

Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom
E-mail address: s.schleimer@warwick.ac.uk