Curves and laminations

CIRM

[2015-08-03, Gazi, Turkey]

[Reference: Casson and Bleiler]

Goals: Define measured laminations, show why $PMF(S) \simeq S^{6g+2n-7}$, define Teichmüller space, discuss the Thurston compactification.

Motivation: Thurston introduced laminations as a tool for understanding the mapping class group, Teichmuller space, (the ends of) Kleinian groups, holomorphic dynamics, etc ...

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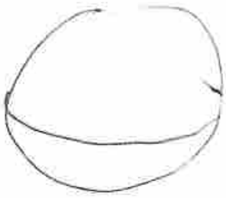
i) Surfaces S:

Classification Theorem for surfaces: Suppose S is a compact, connected, oriented surface. Then $g(S)$ = genus of S and $n(S) = |\partial S|$ [# of boundary components] determine S, up to homeomorphism.

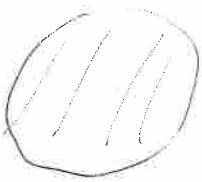
[That is if S, S' are surfaces, $g(S)=g(S')$, $|\partial S| = |\partial S'|$ then $S \cong S'$]

Standard pictures and names

$S_{g,n}$ = surface of genus g, $|\partial S| = n$



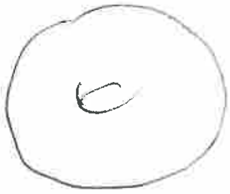
S^2, sphere



D^2, disk

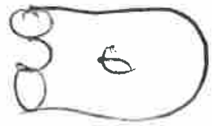
A^2, annulus

$S_{0,3}$, Pair of pants.



$T^2 \cong S^1 \times S^1$ torus

$S_{1,1}$ handle [once holed torus]



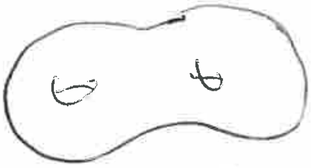
$S_{1,2}$ twice holed torus.

______

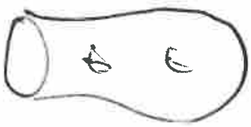
Def: Connect sum

$\chi(A \# B) = \chi(A) + \chi(B) - 2$

Restate classification thm: $S_{g,n} \cong gT^2 \# nD^2$



$S_2 = S_{2,0}$

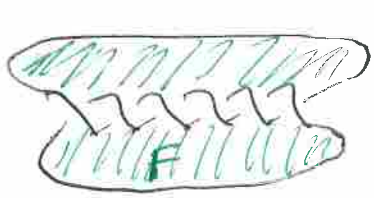


$S_{2,1}$

$S_{2,2}$

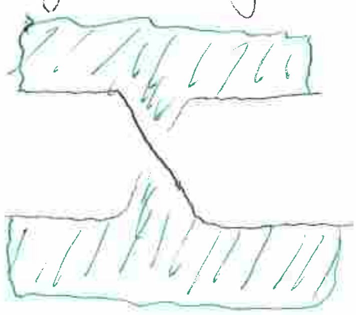
Lemma: $\chi(S_{g,n}) = 2 - 2g - n$. Pf: Connect sum //

Exercise: what is the genus of F



?

Closeup at crossing



Exercise: Give a homeomorphism $F \cong S_{g,n}$ where $S_{g,n}$ is the standard model above. [Harder!]

(2) Curves and arcs

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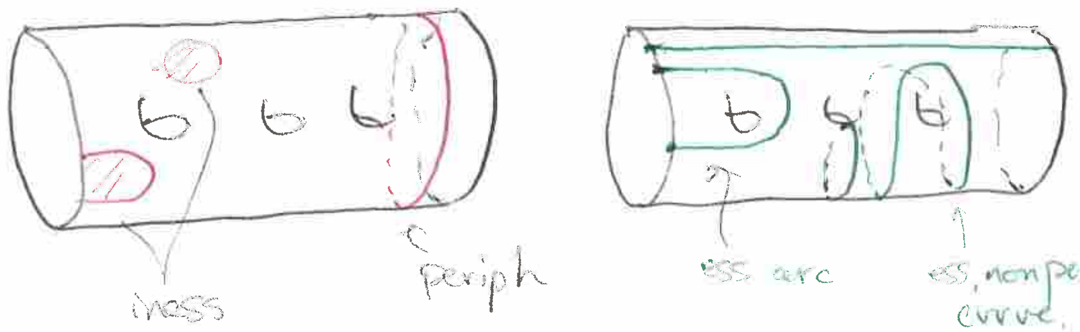
Suppose $\alpha: S^1 \to S$ (or $\alpha: I \to S$), $I = [0,1]$

is a proper embedding ($\alpha^{-1}(\partial S) = \partial\alpha$ and α is homeo to its image)

[Exercise: there are only finitely many proper arcs/curves up to homeo]

Def: α is inessential if α cuts a disk [of pairs] off of S. α is peripheral if α cuts an annulus off of S.

Pictures



Def: We call $F: X \times I \to S$ a proper isotopy if

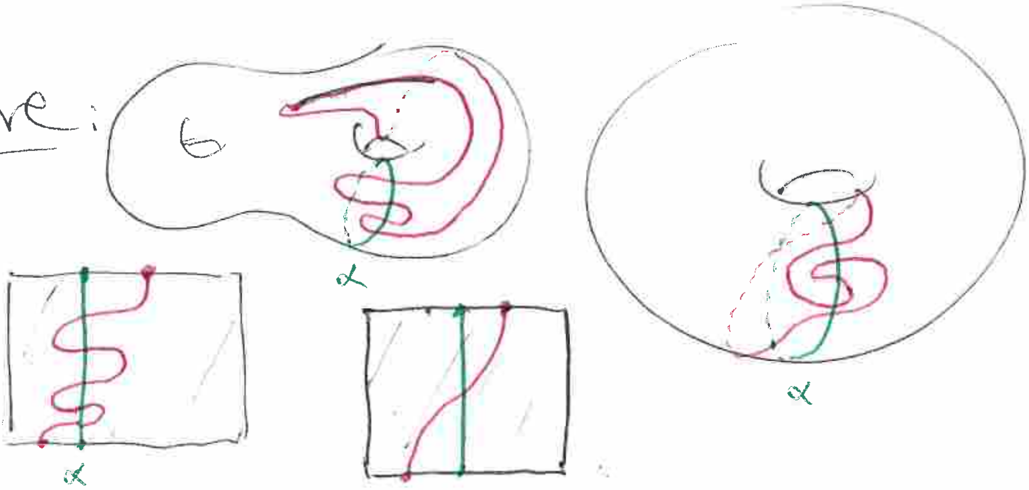
(*) $f_t: X \to S$ is a proper embedding for all $t \in I$ [$f_t(x) = F(x,t)$]

(*) F is continuous

If α, β are arcs/curves then write $\alpha \sim \beta$ if there is a proper isotopy F with $f_0 = \alpha$, $f_1 = \beta$.

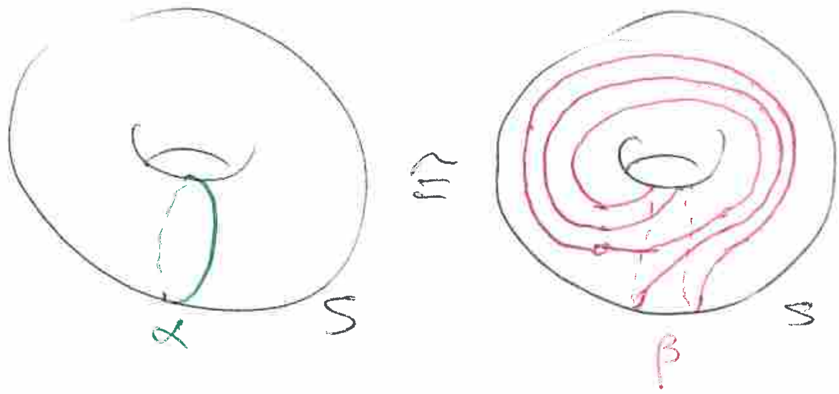
[Discussion of corners, of index, def of iness, isotopy rel corners]

Picture:



Exercise: There are only finitely many curves in S up to homeomorphism of pairs (S, α)

Exercise:



Find a homeomorphism of pairs!

The program begun by Dehn and continued by Thurston, is to understand the set of proper isotopy classes of arcs and curves

Def: $\mathcal{AC}(S) = \left\{ \begin{array}{c} \text{proper isot. classes of} \\ \text{arcs/curves } \alpha \subset S \end{array} \right\}$

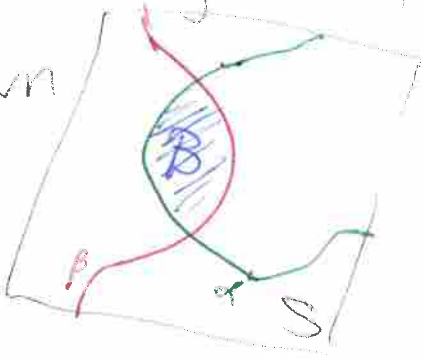
Define: $\mathcal{C}(S) = \{ [\alpha] \in \mathcal{AC}(S) \mid \alpha \text{ curve} \}$

$\mathcal{A}(S) = \{ \text{— — —} \ \alpha \text{ arc} \}$.

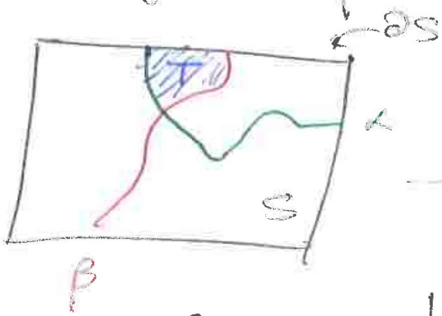
3 Bigon criterion:

Suppose $\alpha, \beta \subset S$ are arcs/curves.

We say α, β <u>share a bigon</u> if there is a disk B of $S - (\alpha \cup \beta)$ meeting each of α, β in a single arc, as shown



We say α, β <u>share a trigon</u> if there is a disk T meeting each of $\alpha, \beta, \partial S$ in an arc, as shown.



~~Bigon~~ Def: $i(\alpha,\beta) = \min \{ |\alpha' \cap \beta'| \;|\; \alpha' \sim \alpha, \beta' \sim \beta \}$ (proper isotopy)

$=$ geometric intersection # of α and β.

<u>Bigon Criterion</u>: ~~If $i(\alpha,\beta) < |\alpha \cap \beta|$ then~~

α and β share a bigon or trigon <u>if and only if</u> $i(\alpha,\beta) < |\alpha \cap \beta|$.

<u>Definition</u>. We say α, β are in <u>minimal position</u> if $i(\alpha,\beta) = |\alpha \cap \beta|$.

The proof of the bigon criterion uses the Jordan curve theorem in the disk, universal covers, and <u>hyperbolic geometry</u>.

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④ <u>Hyperbolic geometry in dimension 2</u> : [Cf. Scott's article]

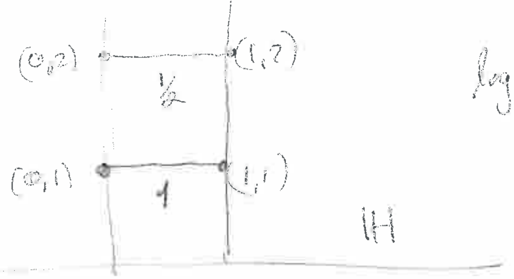
$\mathbb{H}^2$ = hyperbolic plane. We will discuss the <u>upper half plane</u> model: $\mathbb{H} = \{(x,y) \in \mathbb{R}^2 \mid y > 0\} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ equipped with the metric $ds_H = \frac{ds_E}{y} = \frac{\sqrt{dx^2+dy^2}}{y}$

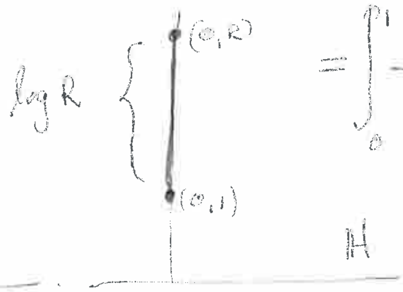
As usual if $\gamma : I \to \mathbb{H}$ is a path $|\gamma| = \int_\gamma ds_H = \int_\gamma \frac{ds_E}{y}$

$\gamma(t) = (u(t), v(t))$

$$= \int_0^1 \frac{\sqrt{(u')^2 + (v')^2}\, dt}{v(t)}$$

<u>Eg</u>:





<u>Facts</u>: ① $\mathrm{Isom}^+(\mathbb{H}^2) \cong PSL(2,\mathbb{R})$. Here a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R})$ acts via

$$z = x+iy \xmapsto{A} \frac{az+b}{cz+d}.$$

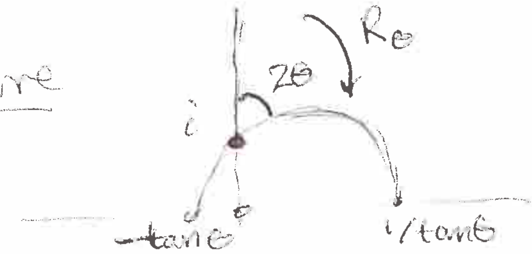
<u>Examples</u> ① if $|\mathrm{tr}\, A| < 2$ then A is <u>elliptic</u>

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ nicer!

Question: other characterization of the trichotomy?
Ans: ① Use Min sets
② use translation distances.

Picture



$$R_\theta(i) = \frac{i\cos\theta - \sin\theta}{i\sin\theta + \cos\theta} = \frac{i(\cos\theta + i\sin\theta)}{\cos\theta + i\sin\theta} = i \checkmark$$

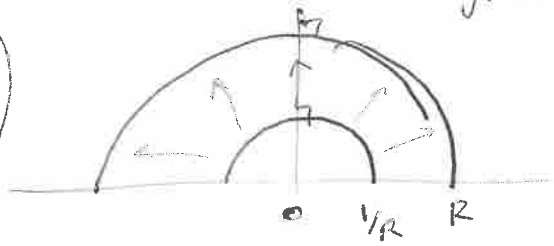
$$R_\theta(0) = -\tan\theta, \quad R_\theta(\infty) = \frac{1}{\tan\theta}$$

② If $|\mathrm{tr}\,A| = 2$ then A is parabolic

$$P_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

③ If $|\mathrm{tr}\,A| > 2$ then A is hyperbolic

$$H_R = \begin{pmatrix} R & 0 \\ 0 & 1/R \end{pmatrix}$$



$$H_R(1/R) = \frac{R \cdot 1/R + 0}{1/R} = R$$

Def: $\partial_\infty \mathbb{H}^2 = S^1_\infty = \mathbb{R} \cup \{\infty\}$ is the boundary at infinity of $\mathbb{H}^2$. $PSL(2,\mathbb{R})$ acts as usual with $A(\infty) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(\infty) = \frac{a}{c}$

Exercise: $PSL(2,\mathbb{R})$ is 3-transitive on anti clockwise triples of points in $\partial_\infty \mathbb{H}^2$

Picture of triples

3-transitive: For any (anticlockwise) triples (a,b,c) and (a',b',c') there is a unique $A \in PSL(2,\mathbb{R})$ st.

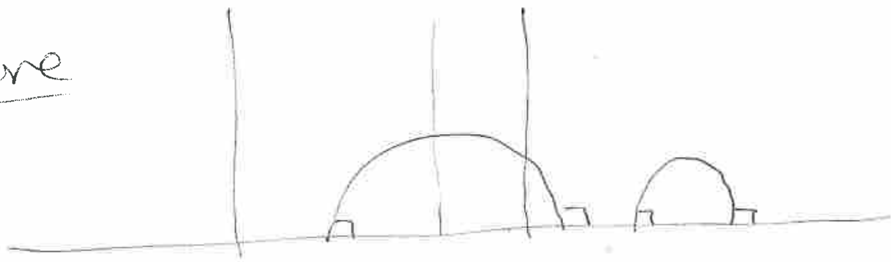
$$A(a) = a'$$
$$A(b) = b'$$
$$A(c) = c'$$

$PSL(2,\mathbb{R}) \cong$ space of triples $\cong UT\mathbb{H}^2$

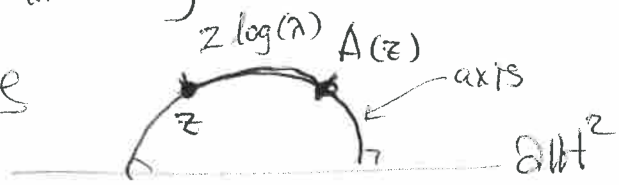
Pf: It suffices to send (a,b,c) to $(0,1,\infty)$.
[Now rotate c to ∞, translate a' to 0 and homothety b'' to 1.]

Fact ②: Geodesics in $\mathbb{H}^2$ are arcs of vertical lines or circles perp. to $\mathbb{R}$.

Picture



Fact ③: Hyperbolic elements have a unique axis. If $\text{trace}(A) = \lambda + \frac{1}{\lambda}$ then A translates along its axis by distance $2\log(\lambda)$. **Picture**



Fact ④: Uniformization: Any surface $\overset{\circ}{S}$, with $\chi(S) < 0$ has $\mathbb{H}$ as its universal cover. [Possibly in many ways!]

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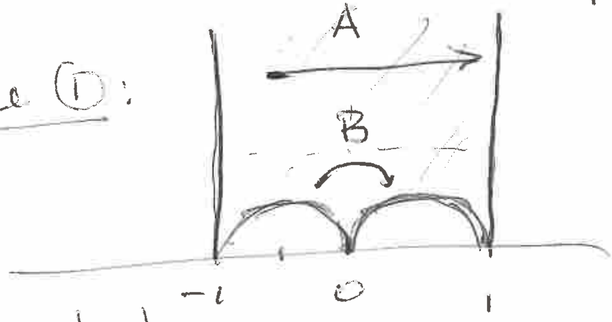
Here are two final examples

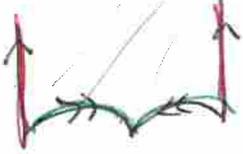
Example ①:



$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ parabolic about ∞

$B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ parabolic about 0.

We define $\Gamma = \langle A, B \rangle < PSL(2, \mathbb{R})$ and form $\mathbb{H}^2 / \Gamma$ =



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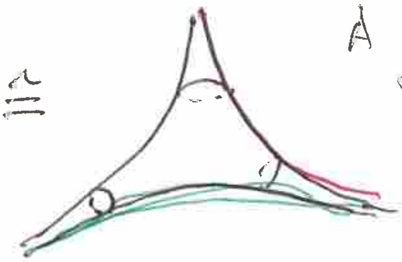
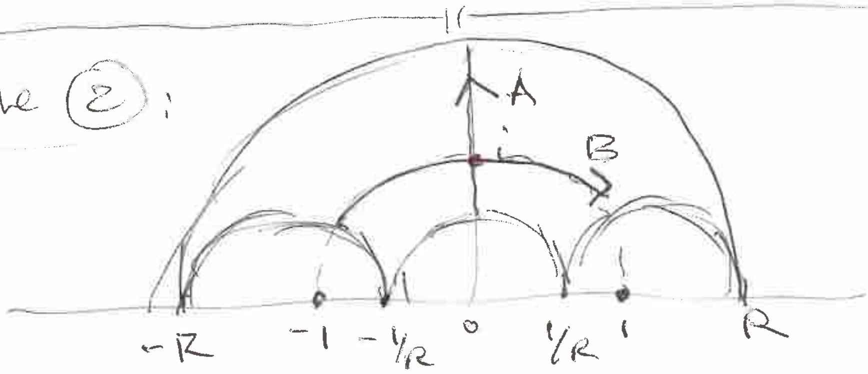


A pair of pants.

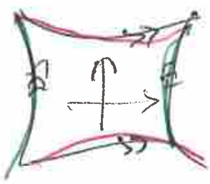
Example ②:



$A = \begin{pmatrix} R & 0 \\ 0 & 1/R \end{pmatrix}$ $B = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}$ where $a = \frac{R + 1/R}{2}$

If $\Gamma = \langle A, B \rangle$ then $\mathbb{H}^2 / \Gamma$

B



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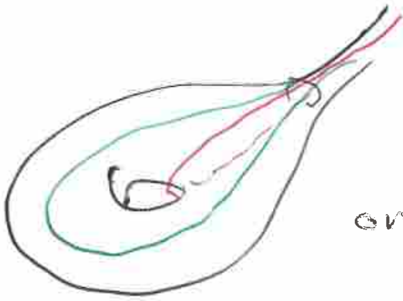
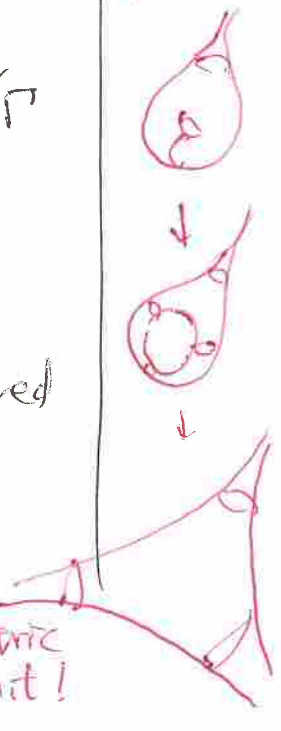


once punctured torus.

As $R \to 1$ from above the axis of A "pinches"



geometric limit!

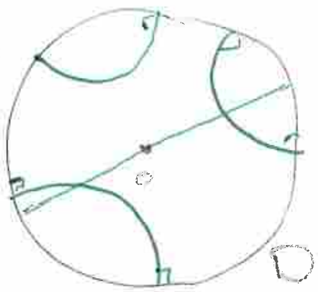
Curves and laminations III

[2015-08-04]

(4') One more model of $\mathbb{H}^2$. The Poincaré disk model is $D = \{ z \in \mathbb{C} \mid |z| < 1 \}$. Set $z = x + iy$, $r^2 = x^2 + y^2$. The metric is $ds_D = \dfrac{2 \cdot ds_E}{1 - r^2}$

[compare to the spherical metric $ds_S = \dfrac{2 ds_E}{1 + r^2}$ obtained via stereographic projection]

Picture



Now geodesics are lines through zero and circles perpendicular to $S^1 = \partial_\infty D$.

Exercise: Show $(H, ds_H) \cong (D, ds_D)$ are isometric.

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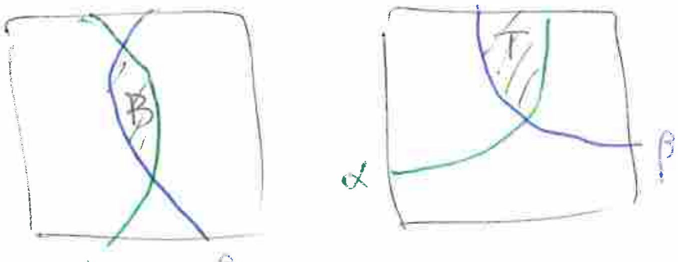
(5) Proof of Bigon criterion (Sketch)

Recall: Arcs / curves α, β are in minimal position if $|\alpha \cap \beta| = i(\alpha, \beta)$ (= geom. intersection number).

Thm: (Bigon criterion) α, β in minimal position iff α, β do not share a bigon (or trigon).

Pictures

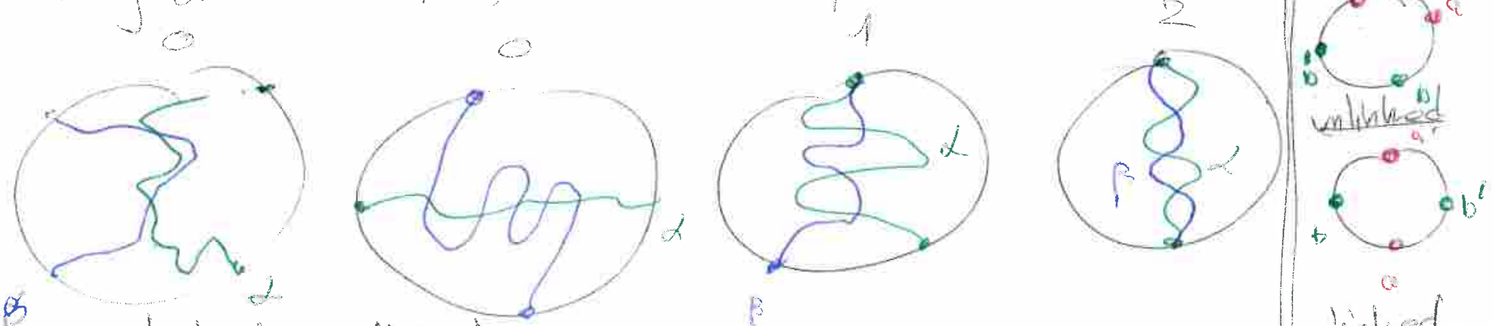


bigon trigon

The proof is not hard, but requires some new ideas. We'll just give an out-line.

~~Step~~ Pf Suppose α, β are arcs in the disk D.

They share 0, 1, or 2 endpoints. There are 4 cases



unlinked linked linked.

If $\partial\alpha \cap \partial\beta = \emptyset$ then define

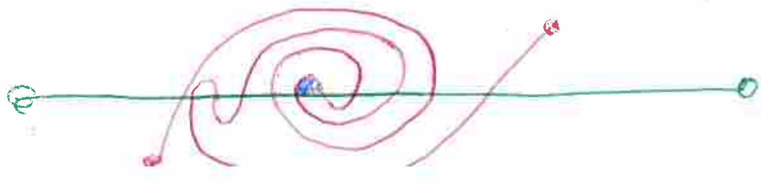
(*) α, β are $\left\{\begin{array}{c}\text{linked}\\ \text{unlinked}\end{array}\right\}$ if $\partial\alpha$ $\left\{\begin{array}{c}\text{separates}\\ \text{doesn't sep}\end{array}\right\}$ $\partial\beta$ in ∂D.

Claim (A) Suppose $\partial\alpha \cap \partial\beta = \emptyset$

α, β are $\left\{\begin{array}{c}\text{linked}\\ \text{unlinked}\end{array}\right\}$ iff $i(\alpha,\beta) = \left\{\begin{array}{c}1\\ 0\end{array}\right\}$.

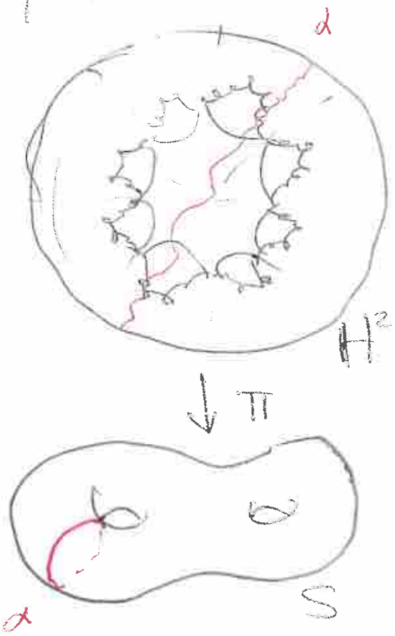
Pf: ($\Leftarrow$) Jordan curve theorem

($\Rightarrow$) Jordan curve theorem and "innermost bigon" argument. //



Now suppose $\chi(S) < 0$
and $\rho : \pi_1(S) \to PSL(2,\mathbb{R})$ is
a discrete and faithful rep.

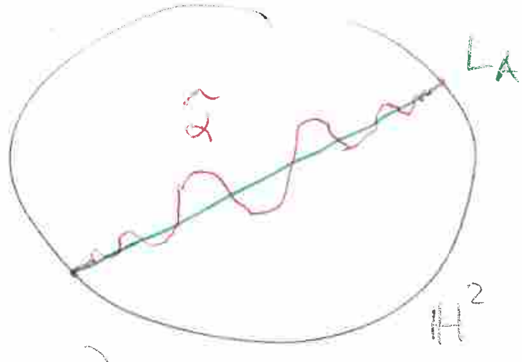
[i.e. $\mathbb{H}^2/\rho(\pi_1) \cong \mathring{S}$]   Picture



Claim (B): If $\alpha \subset S$ is ess, nonperip curve then $A = \rho(\alpha)$ is hyperbolic

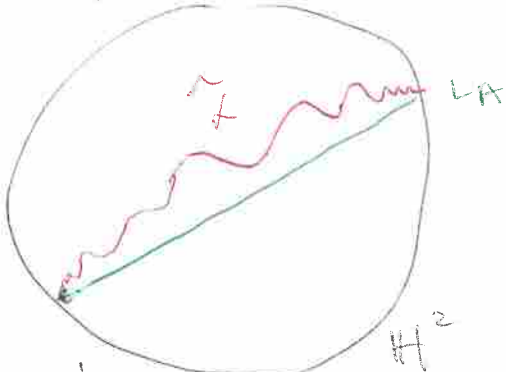
[Exercise: Give a proof!]

So let L_A be the axis of A
let $\tilde{\alpha}$ be the lift of α to $\mathbb{H}^2$ that fellow travels with L_A.



Define $\alpha^* = \pi(L_A)$.

Claim (C): ~~[illegible]~~
α^* is simple. (and homotopic to α)   or

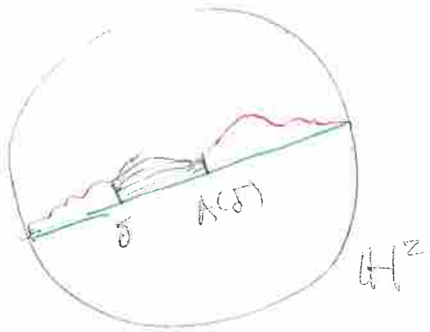


Pf: α simple $\Rightarrow \pi^{-1}(\alpha)$ is unlinked $\Rightarrow \rho(\pi_1) \cdot L_A$ is unlinked $\Rightarrow \alpha^*$ simple //

Claim (D): α is isotopic to α^*.

Pf: we induct on $|\alpha \cap \alpha^*|$.

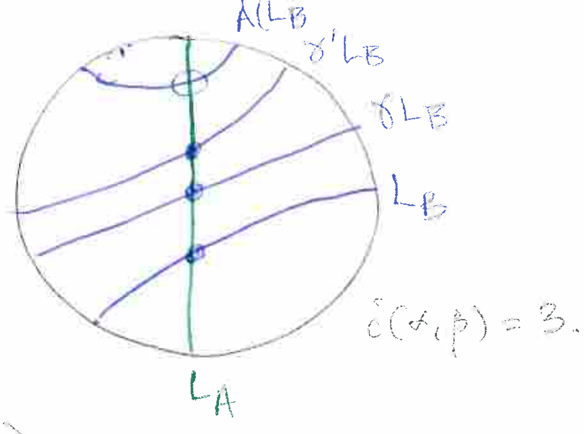
[Skip: If $|\alpha \cap \alpha^*| = 0$ then pick $\delta \subset \mathbb{H}^2$ geodesic arc from L_A to $\tilde{\alpha}$, and choose δ to be shortest such. Then $\delta \cap A(\delta) = \emptyset$ and $\pi(\delta)$ embeds, meets α, α^* at endpts. So get rectangle → get annulus → $\alpha \simeq \alpha^*$.
If $\alpha \cap \alpha^* \neq \emptyset$ find bigon in $\mathbb{H}^2$, this embeds so reduce $|\alpha \cap \alpha^*|$. //]



We call α^* the <u>geodesic representative</u> of α.

<u>Finishing the pf of bigon criterion:</u>

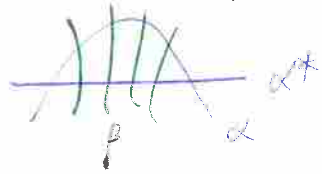
Given α, β, isotope them to α^*, β^*. These have no bigons (lift to $\mathbb{H}$ and recall classification of geodesics). Let $L_A = \widetilde{\alpha^*}$, $L_B = \widetilde{\beta^*}$.
We ~~[illegible]~~ count the number of orbit reps of L_B that <u>link</u> L_A.

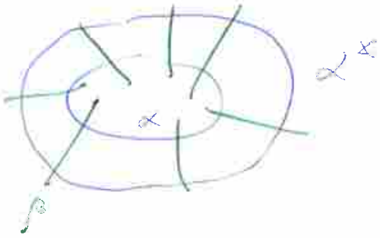


Note that isotopies of α^* do not move $\partial_\infty(\Gamma \cdot \widetilde{\alpha^*})$
So # of linking pairs is
(a) a lower bound for $i(\alpha, \beta)$
(b) realized by $|\alpha^* \cap \beta^*|$ //

Exercise: Minimal position is unique, up to isotopy.

[Pf:





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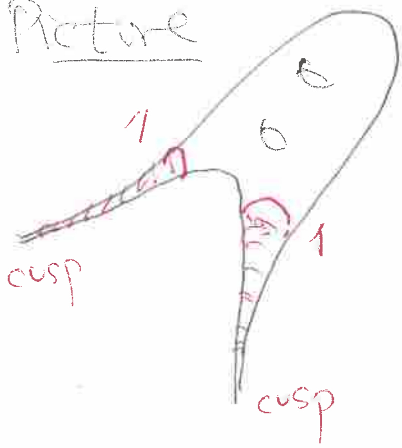
(6) Laminations

Fix $\rho : \pi_1(S) \to PSL(2,\mathbb{R})$, $\Gamma = \rho(\pi_1)$

$\mathring{S} \cong \mathbb{H}^2/\Gamma$, surface of finite ~~volume~~ area

Note $\chi(S) < 0$.

Picture



If $\partial S \neq \emptyset$ then $\mathbb{H}^2/\Gamma$ is noncompact. choose horocycles of length 1.

Picture

Skipped

Let $C \subset S$ be the shaded cusp neighborhoods.

So $\mathbb{H}^2/\Gamma - \mathring{C}$ is compact and homeomorphic to S.

Exercise: If $\alpha^* \subset S$ is a simple geodesic then

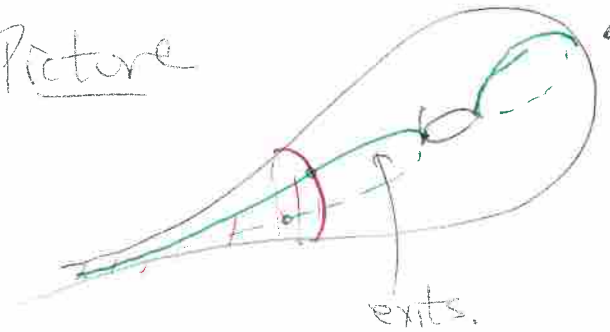
either (i) $\alpha^* \perp \partial C$ (perpend)

or (ii) $\alpha^* \cap C = \emptyset$.

In case (i) we say α^* exits S.

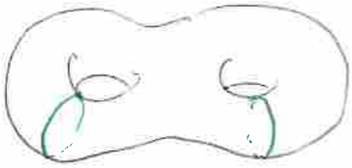
Picture

does not exit

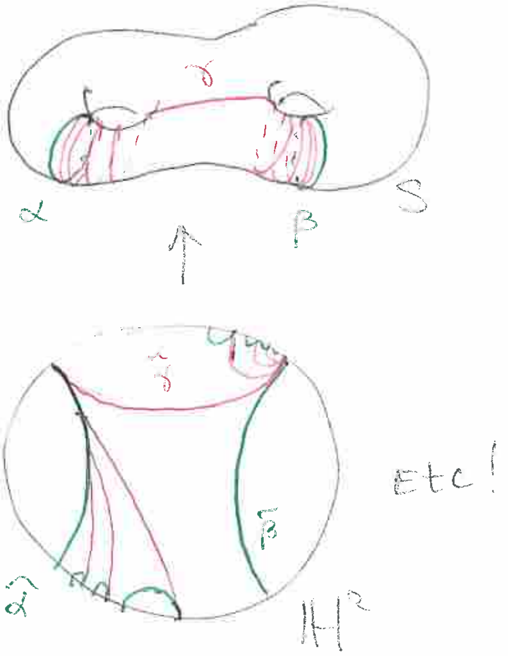


Def: A lamination $\Lambda \subset \dot{S} = \mathbb{H}/\Gamma$ is any closed set that can be realized as a disjoint union of simple geodesics, call leaves.

Ex



Ex



Before we think about the details lets think about an example.

Def: A leaf $L \subset \Lambda$ is isolated if $\forall x \in L\ \exists \varepsilon > 0$ s.t. [$B_\varepsilon(x) = \varepsilon$-ball]

$$B_\varepsilon(x) \cap \Lambda = B_\varepsilon(x) \cap L.$$

All examples above have isolated leaves.

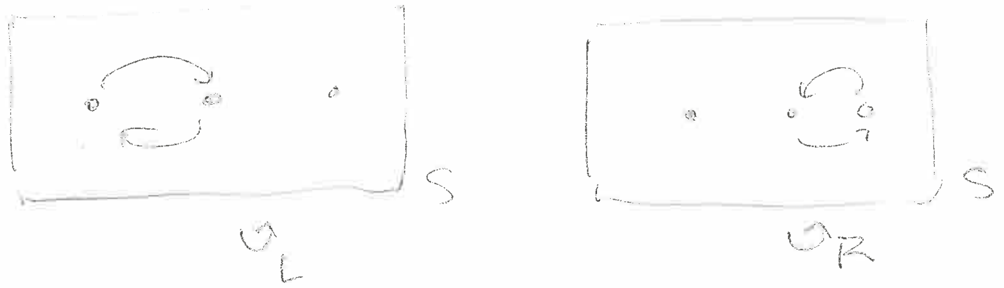
Def: Call Λ perfect if it has no isolated leaves.

Can we find a perfect lamination!?

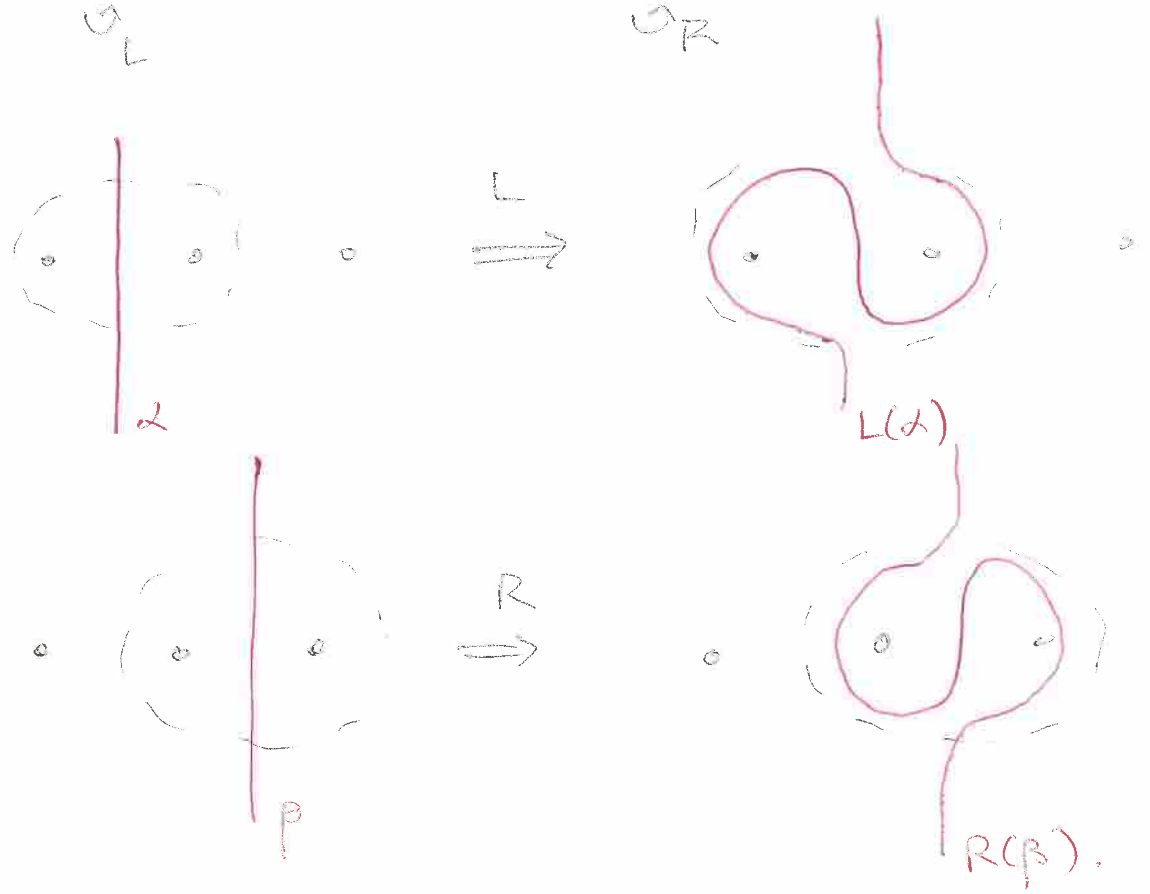
7 An example.

$S = \mathbb{C} - \{0,1,2\}$ and

$L, R : S \to S$ homeo as follows

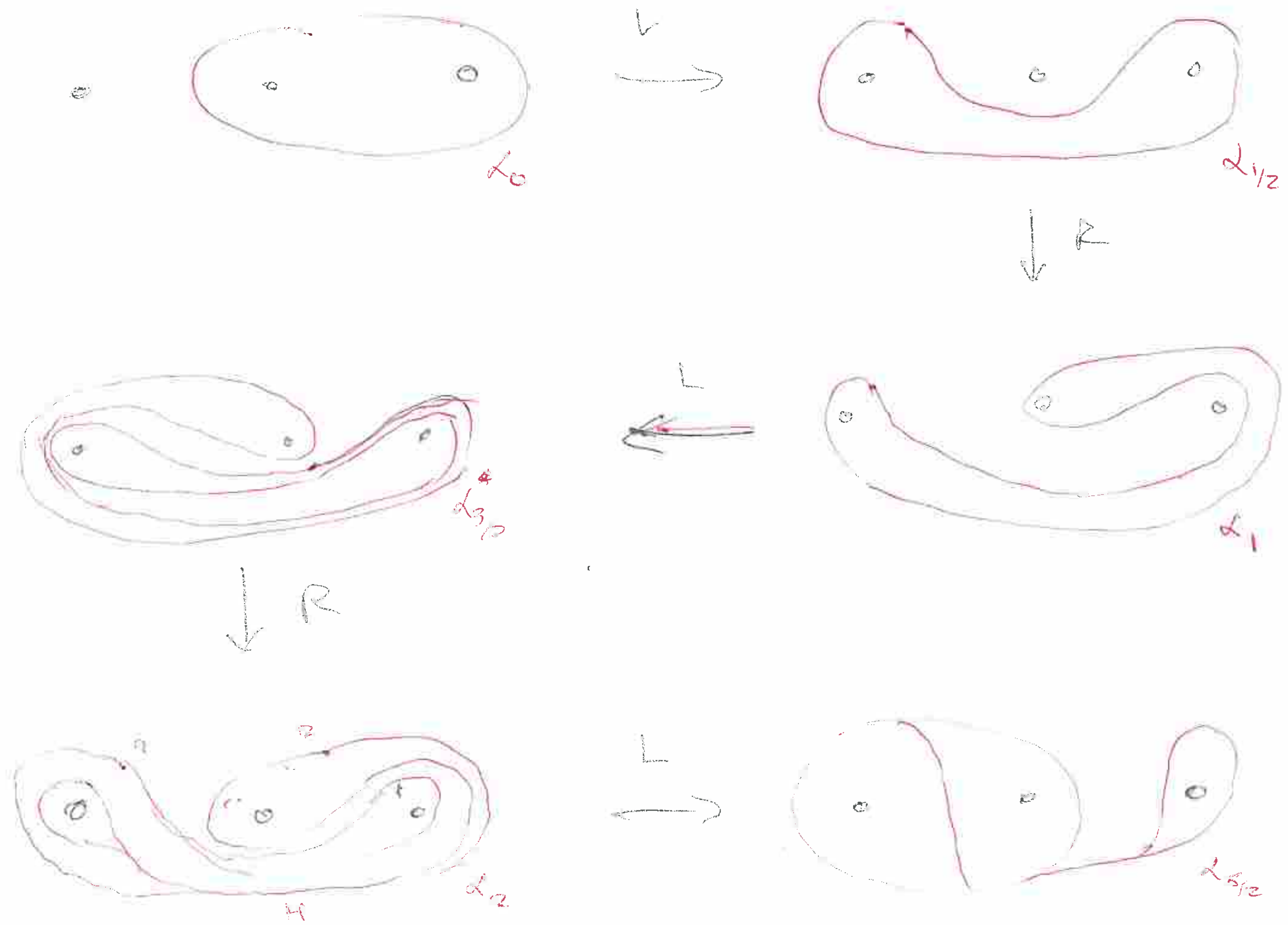


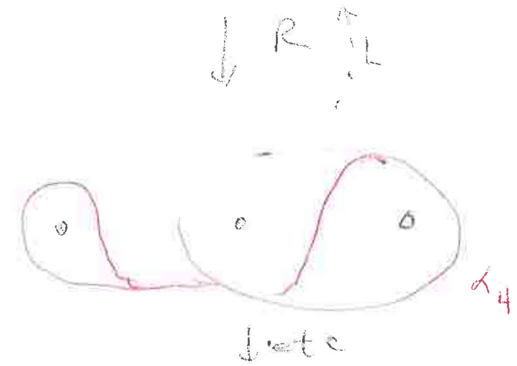
Eg:



Lets fix a geodesic curve α_0 and define $\alpha_k = \left((RL)^k(\alpha_0)\right)^*$

Here is the sequence of pictures.





The sequence of geodesics is converging (in the Hausdorff ~~topo~~ metric!) to a perfect lamination!

We write $\alpha_n \longrightarrow \Lambda$ and note $RL(\Lambda)=\Lambda$.

[Skip: Train tracks, weights, eigenvector gives transverse measure.]

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[2015-08-04]

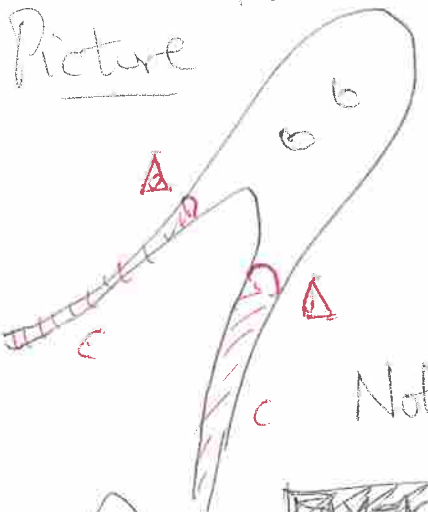
[skipped a lot of the material in an attempt to simplify. See notes at end.]

(8) Exiting leaves. As usual, fix $S = S_{g,n}$ and $\rho : \pi_1(S) \to PSL(2,\mathbb{R})$. Set $\Gamma = \rho(\pi_1)$ and suppose $S^\circ \cong \mathbb{H}/_\Gamma$ has finite area. $(\chi(S) < 0)$.

If $\partial S \neq \emptyset$ then S° is non-compact.

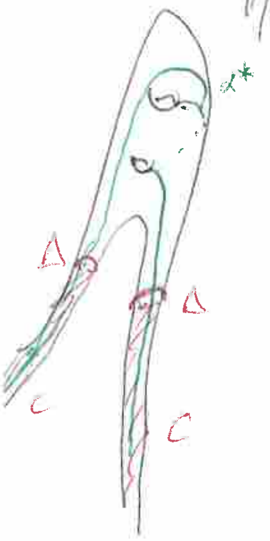
Let Δ be the union of the length one horocycles. $\overset{\shortparallel}{\{\delta_i\}}$

Picture



Let $C_{\{c_i\}}$ be the components of $S^\circ - \Delta$ with non-compact closure. [shaded]. These are the cusps of S.

Note $S^\circ - C^\circ \cong S$ is compact.



Def: If $\alpha^* \subset S^\circ$ is a simple geodesic, perpendicular to Δ, then we say α^* exits S°

Exercise: If $\alpha^* \subset S^\circ$ is a simple geodesic either (i) α^* exits S° or (ii) $\alpha^* \cap C = \emptyset$.

[This is an easy exercise in the definitions and requires one idea!]

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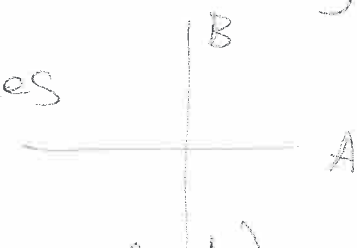
⑨ Hausdorff metric:

Suppose (X,d_X) is a compact metric space.

Define $H=H(X)=\{C\subset X \mid C \text{ closed}\}$. Define

$$d_{Haus}(A,B)=\inf\{\varepsilon\in\mathbb{R} \mid A\subset N_\varepsilon(B),\ B\subset N_\varepsilon(A)\}$$

Example: If $A,B\subseteq\mathbb{R}^2$ are the axes



then $d_{Haus}(A,B)$ is undefined.

~~Example~~: Thm [???] If (X,d_X) is compact, so is (H,d_H).

Define $\mathcal{GL}(S)=\{\Lambda\subset S \mid \Lambda \text{ lamination without exiting leaves}\}$.

Note $\mathcal{GL}(S)\subseteq \mathrm{Haus}(S^\circ - C^\circ)$ by the exercise.

Theorem ⊛: $(\mathcal{GL}(S), d_{Haus})$ is compact. [See Casson-Bleiler]

Rmk: This justifies the limit $\alpha_k \longrightarrow \Lambda(f)$ taken yesterday.

To prove ⊛ several ideas are required.

① Fix a partition $\Lambda=\sqcup\gamma$. Then directions of Λ vary continuously. [i.e. $\Lambda \longrightarrow \mathbb{P}TS$ is cts, $x \longmapsto [\gamma_x']$]

② The decomposition $\Lambda=\sqcup\gamma$ is unique.

③ Λ is nowhere dense

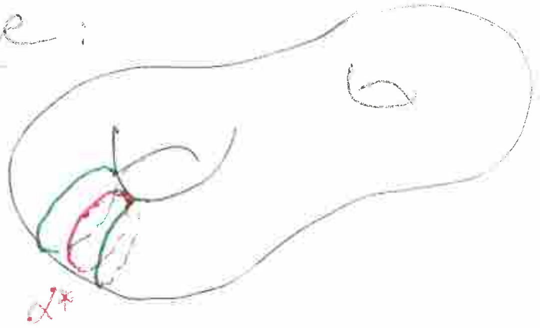
④ The closure of any disj. union of simple geodesics is a lamination.

Cantor × I

Rmk: If Λ minimal all leaves are dense.

Exercise: Suppose α^* is a simple closed geodesic. Then $\alpha^* \in (\mathcal{GL}(S), d_{Haus})$ is an isolated point.

Picture:



Choose $\varepsilon > 0$ so that $N_\varepsilon(\alpha^*) \cong A^2$ is an annulus!

In fact

Thm [Bonahon, Zhu] $\dim_{Haus} (\mathcal{GL}(S), d_H) = 0$.

Corollary: $(\mathcal{GL}(S), d_H)$ is totally disconnected. [conn cpts are pts]

[This is... unpleasant!]

---

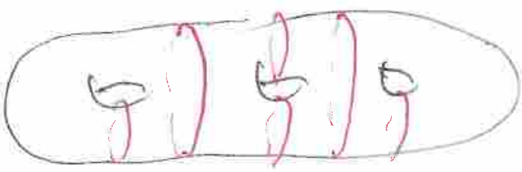
(10) Measured laminations:

Def: Suppose α is an arc/curve system

$\alpha = \{\alpha_i\}_{i=0}^n$ where $\alpha_i \neq \alpha_j$ $\forall i \neq j$

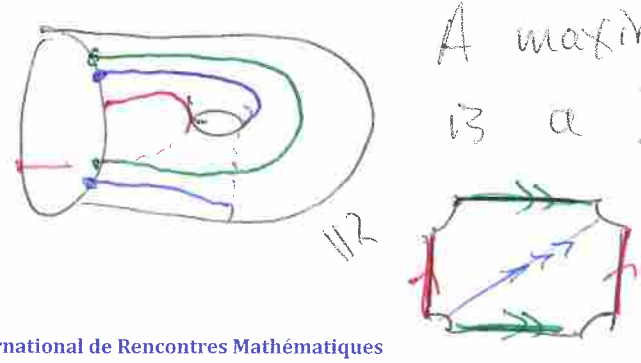
and $i(\alpha_i, \alpha_j) = 0$ $\forall i, j$.

Picture:



A maximal curve system is a pants decomposition

Picture



A maximal arc system is a hexagon decomposition

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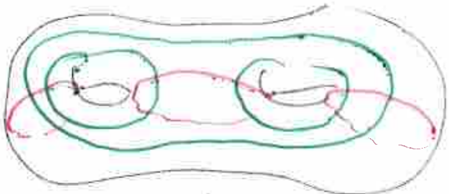
Def: If α, β are systems, define $i(\alpha,\beta) = \sum_{i,j} i(\alpha_i, \beta_j)$

Picture



$i(\alpha,\beta) = 6.$

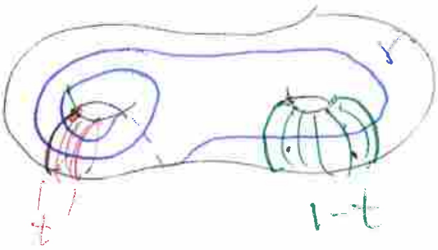
Def: A ~~weighted~~ transversely measured system (α,μ) is a system and a function $\mu : \alpha \longrightarrow \mathbb{R}_{>0}$.

Picture

(α, μ_t) is a "path" in the "space" of measured systems.

Def: $i((\alpha,\mu),(\beta,\nu)) = \sum_{i,j} \mu(\alpha_i)\nu(\beta_j)\, i(\alpha_i,\beta_j)$

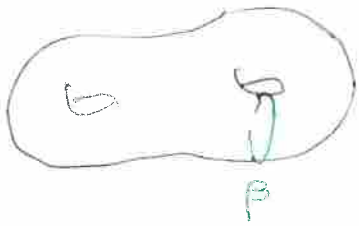
Morally: $\mu(\alpha_i)$ measures the "thickness" of α_i



$i((\alpha,\mu),\gamma) = 1+t.$

Basic example Consider the seq–

$(\alpha_n, 1/n)$



Exercise: $\forall \gamma \in \mathcal{C}(S)$ $i(\gamma, (\alpha_n, 1/n)) \longrightarrow i(\gamma,\beta)$

That is: β is isolated in Haus top. but is not isolated in the "measure topology"

[I draw the geometric and measure limits next to each other – good board work –]

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Definition [Penner-Harer]

Suppose $\Lambda \in \mathcal{GL}(S)$ is a geodesic lamination. Let $T(\Lambda) = T$ be the set of (not necess proper) simple arcs and curves in S transverse to Λ. A *transverse measure* μ on Λ is a function

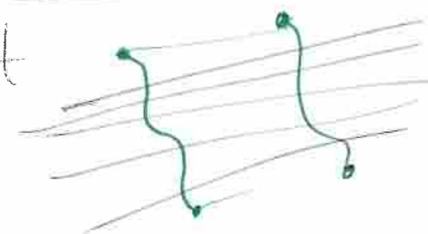
$$\left.\begin{aligned} \mu : T(\Lambda) &\longrightarrow \mathbb{R}_{\geq 0} \\ \alpha &\longmapsto \mu(\alpha) \end{aligned}\right\} \text{ so that.}$$

(1) If $\alpha, \beta \in T$ are flow equivalent ~~then~~ $\mu(\alpha) = \mu(\beta)$

(2) If $\alpha, \beta, \gamma \in T$ with $\gamma = \alpha \cup \beta$, $\alpha \cap \beta = \{pt\}$ $\mu(\gamma) = \mu(\alpha) + \mu(\beta)$

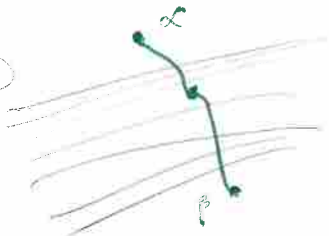
(3) If $\alpha \in T$ $\alpha \cap \Lambda \neq \emptyset$ then $\mu(\alpha) > 0$.
(and conversely!)

Picture (1) of flow equivalent

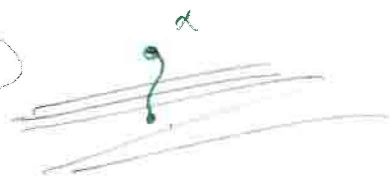


} isotopy fixing Λ set-wise

Picture (2)



Picture (3)



Define $\mathcal{ML}(S) = \{0\} \cup \{(\Lambda, \mu) \text{ trans meas lam}\}$.

Note there is a natural action of $\mathbb{R}_{>0}$

$$r \cdot (\Lambda, \mu) = (\Lambda, r \cdot \mu).$$

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So define $\mathcal{PML}(S) = \dfrac{\mathcal{ML}(S) - \{0\}}{\mathbb{R}_{>0}}$.

(11) The "measure" topology:

Suppose $\lambda = (\Lambda, \mu) \in \mathcal{ML}(S)$, $\alpha \in \mathcal{C}(S)$

we define $i(\alpha, \lambda) = \mu(\alpha)$.

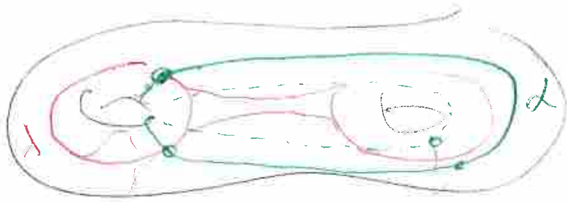
Define

$$i_\alpha : \mathcal{ML}(S) \longrightarrow \mathbb{R}_{\geq 0}$$
$$\lambda \longmapsto i(\alpha, \lambda)$$

The measure topology on $\mathcal{ML}(S)$ makes all of the functions i_α continuous.

[This is not the definition]

Picture:



As we've seen in examples above $\mathcal{ML}(S)$ is not totally disconnected.

Thm [Thurston] $\mathcal{ML}(S) \cong \mathbb{R}^{6g+2n-6}$

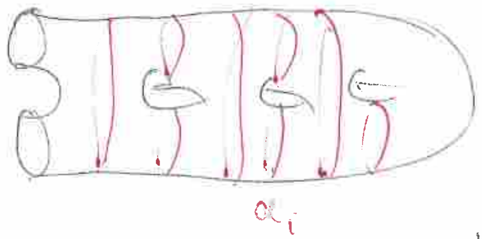
$\mathcal{PML}(S) \cong S^{6g+2n-7}$.

Thm [Thurston, Rees, Bonahon, Luo ...]

Geometric intersection $i : \mathbb{R}_{\geq 0}\mathcal{C}(S) \times \mathbb{R}_{\geq 0}\mathcal{C}(S) \to \mathbb{R}$ has a continuous extension to $\mathcal{ML} \times \mathcal{ML} \to \mathbb{R}$

(12) Dehn-Thurston coordinates:

Let $\{a_i\} \subseteq \mathcal{C}(S)$ be a pants decomposition of S.

Picture 

$S=S_{g,n}$

Since $\chi(S)=2-2g-n$ and since $\chi(S_{0,3})=-1$ there are $2g+n-2$ pairs of pants, and so there are

$$\left[\frac{(6g+3n-6)+n}{2}-n\right]$$

red curves i.e. $\xi(S)$ = pants number

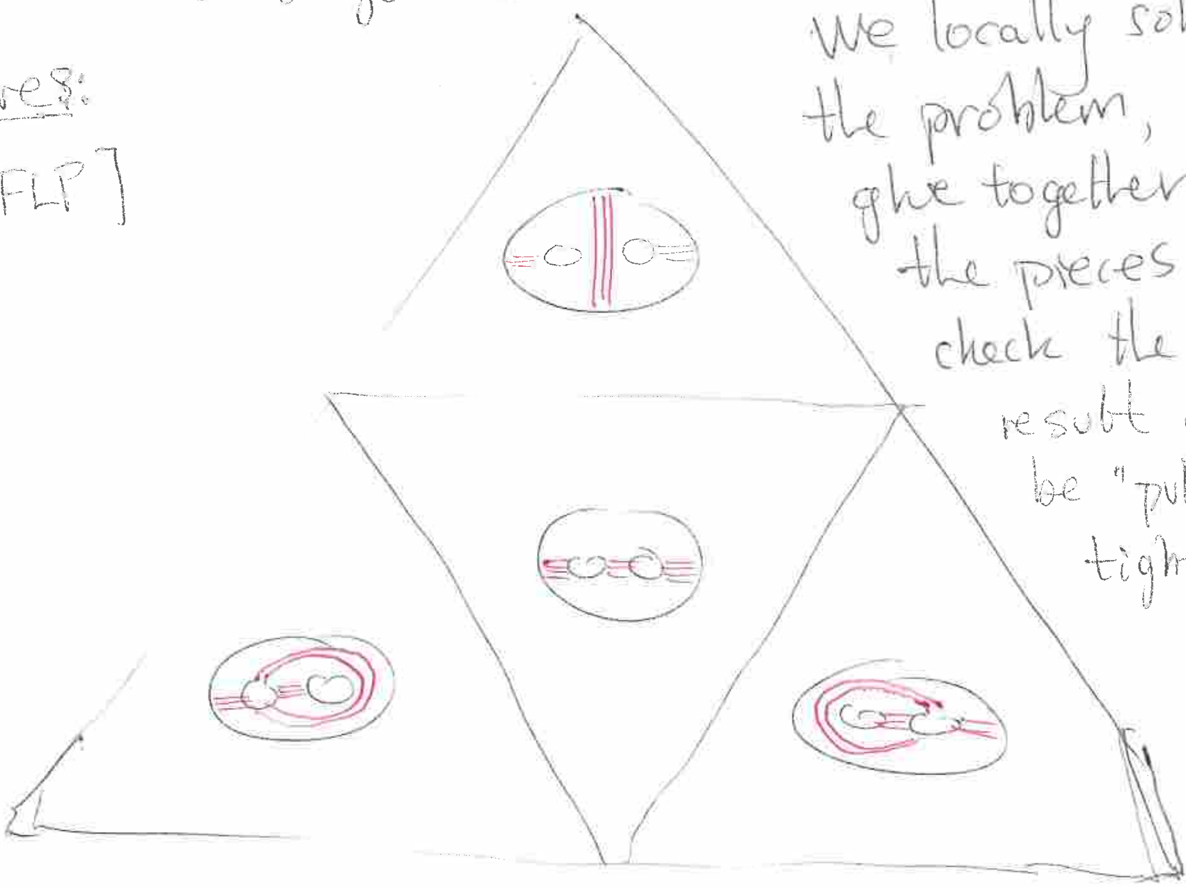
$$\boxed{\xi(S)=3g+n-3}$$

Lemma: The map

$$\mathcal{MF}(S)\longrightarrow \mathbb{R}^{3g+n-3}_{\geq 0}$$
$$\lambda \longmapsto (i(\alpha_i,\lambda))$$

is surjective.

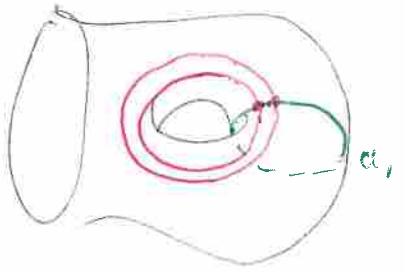
Pictures: [cf FLP]



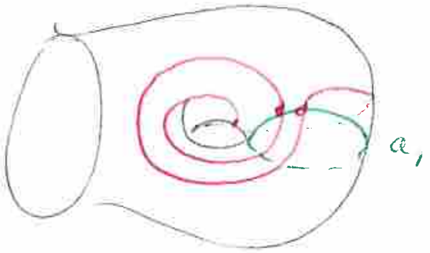
We locally solve the problem, glue together the pieces and check the result can be "pulled tight".

However, this map is not an injection.

Example



No twisting



Some twisting

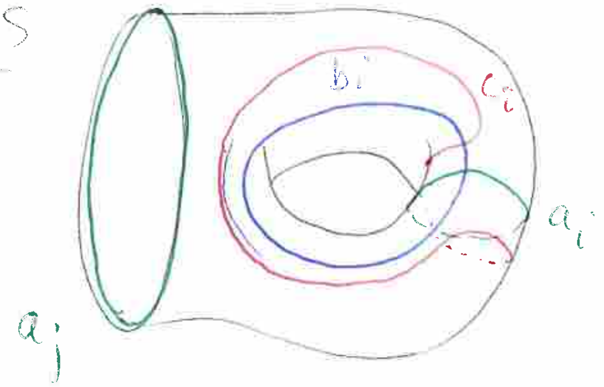
The following discussion of the twist coordinate is confused and wrong Please see FLP (or my notes for my talks in 2018 at the Fields) for a correct discussion.

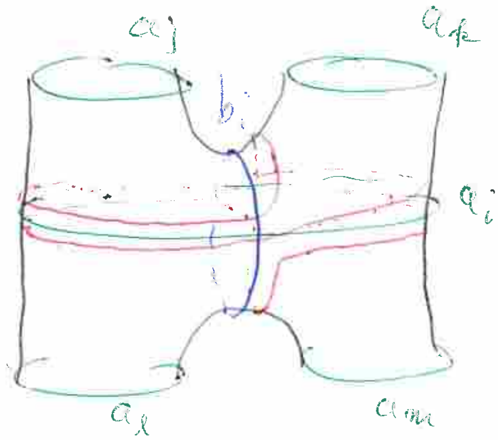
Following FLP: For each a_i choose curves b_i, c_i st.

(i) $i(a_i, b_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } a_i \text{ in handle} \\ 2 & \text{if } a_i \text{ in } S_{0,4} \end{cases}$

(ii) $c_i = T_{a_i}(b_i)$

Pictures



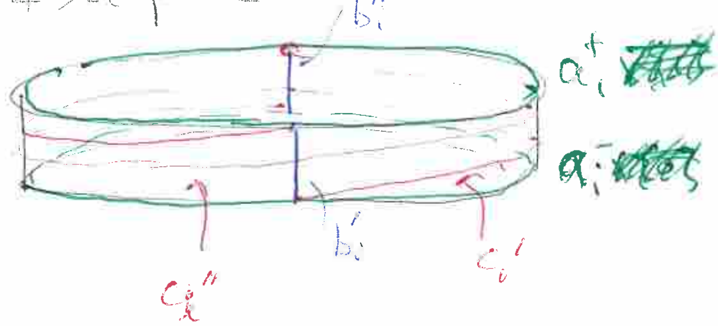


Pick annuli $N_i = N(a_i)$. Define ~~[crossed out]~~ $\sqcup P_j = S - \sqcup \mathring{N}_i$ union of pants.

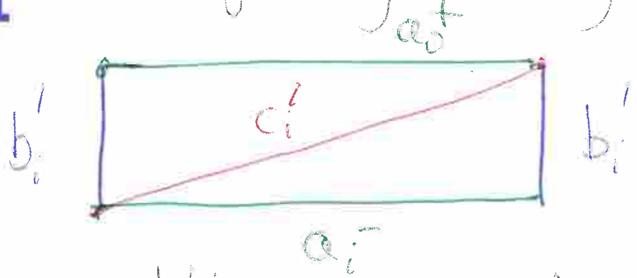
Isotope c_i so $b_i \cap P_j = c_i \cap P_j$ $(\forall i, j)$.

Fix $\lambda = (\Lambda, \mu)$. Isotope Λ to be standard in P_j.

In N_i we see

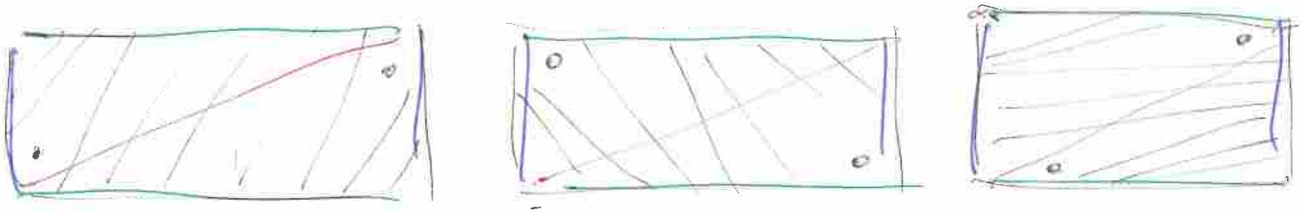


Now: Consider the rectangle R we get by cutting along b_i'



Define
$r_i = \mu(a_i)$
$s_i = \mu(b_i')$
$t_i = \mu(c_i')$

There are 3 possibilities for Λ as it flows thru R



① $r = s + t$ (pos)

③ $t = r + s$ (neg)

② $s = t + r$ (pos)

---

we define the twist $tw_i = \begin{cases} +s_i & ① \text{ or } ② \\ -s_i & ③ \end{cases}$

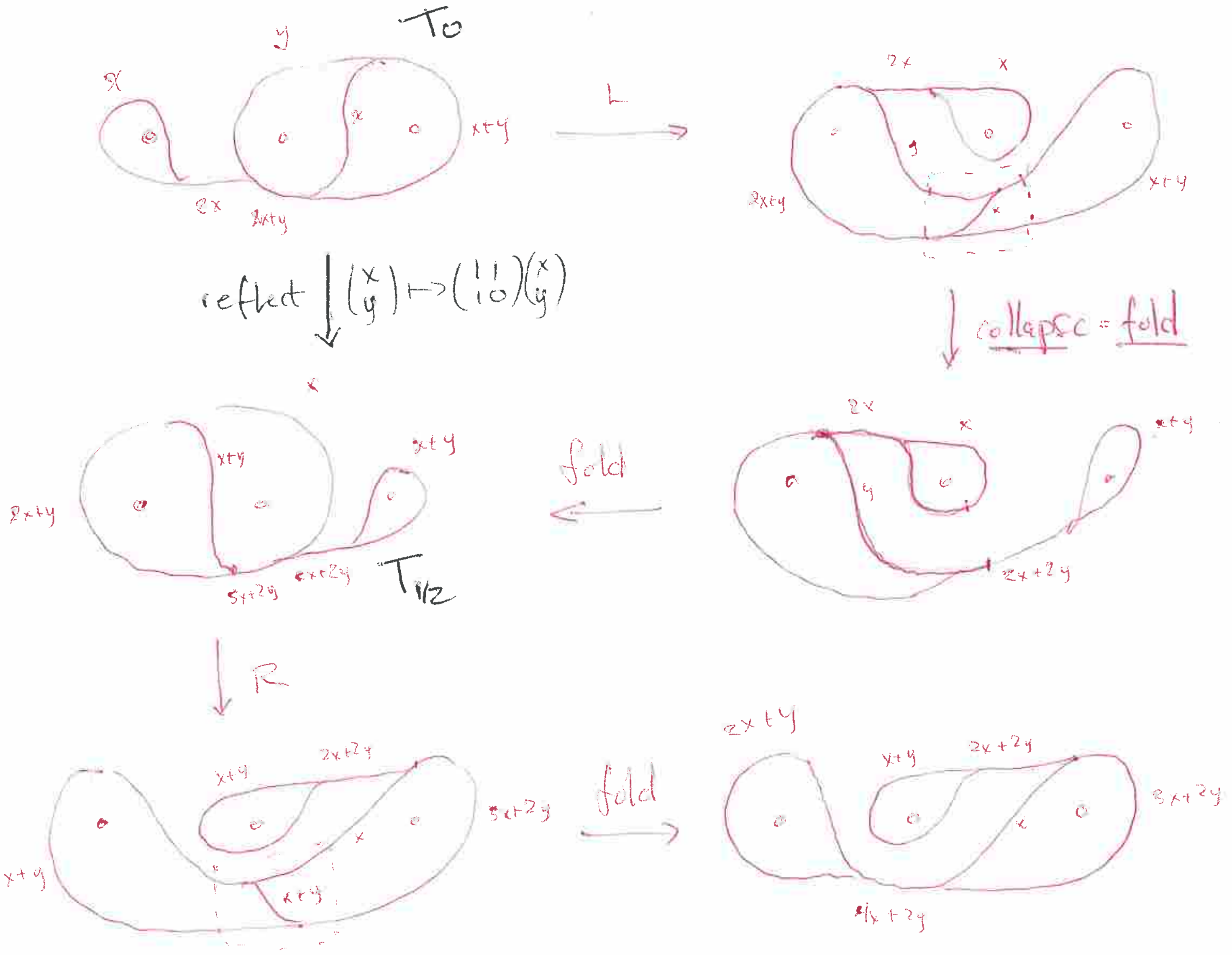
Note that (r_i, tw_i) recovers (r_i, s_i, t_i)

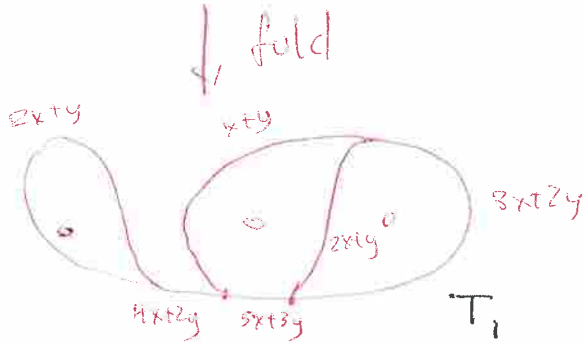
Theorem $\mathcal{ML}(S) \longrightarrow \mathbb{R}_{\geq 0}^{3g+n-3} \times \mathbb{R}^{3g+n-3}$

$\lambda \longmapsto (r_i, tw_i)_i$

is a homeomorphism (equivar wrt scaling)

[I failed on this talk: instead I talked about pA maps, ending laminations, Thurston trichotomy, comparing limits in $\mathcal{GL}(S)$ to limits in $\mathcal{ML}(S)$]





$$\begin{pmatrix}x\\y\end{pmatrix} \longmapsto \begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$\lambda^{+} = \frac{3+\sqrt{5}}{2}$ top eigenvalue

$\lambda^{-} = \frac{3-\sqrt{5}}{2}$ bottom eigenvalue

$$\begin{pmatrix}1\\y\end{pmatrix} \longmapsto \begin{pmatrix}2+y\\1+y\end{pmatrix} = \lambda\begin{pmatrix}1\\y\end{pmatrix} = \begin{pmatrix}\lambda\\\lambda y\end{pmatrix}$$ so

$2+y = \lambda$, $1+y = \lambda y$ } $y = \lambda - 2$

$1+\lambda-2 = \lambda(\lambda-2)$

$\lambda - 1 = \lambda^2 - 2\lambda$

$\Rightarrow \lambda^2 - 3\lambda + 1 = 0$ ✓

So: $\begin{pmatrix}1\\\lambda-2\end{pmatrix}$ is the eigenvector.