ON THE GIRTH OF GROUPS

SAUL SCHLEIMER

ABSTRACT. DISCLAIMER: Everything that follows is of a preliminary nature. We give a new invariant for finitely generated groups, called the *girth*. Several results which indicate that the girth of a group might possibly be a quasi-isometry invariant are proved. We also compute the girth in several instances and speculate on the relation of girth to the growth of groups.

1. Introduction

Given a set of combinatorial objects and a particular property of interest, it is natural to inquire as to the nature of those objects that extremise the property. For example, it is a well known fact that the girth of a k-regular graph on n vertices is at most $2 \cdot \log(n)/\log(k-1)$. Such graphs can be shown to exist via probablistic methods. However, it was not until 1981 that Margulis [4] gave a construction, which involved Cayley graphs of the groups $SL(2, \mathbb{Z}/q\mathbb{Z})$.

It seems natural then, fixing a group G, to examine the set of all Cayley graphs and to decide which graph properties may be safely extremised. Furthermore, we wish to investigate group theoretic consequences of requiring a universal bound on these properties.

2. The girth

In this section we give the basic definition and discuss a few simple examples.

Let G be a finitely generated group. Let $\mathbf{X}_G$ be the set of finite nonempty subsets of G which generate. For any $X \in \mathbf{X}_G$ we have a short exact sequence of groups:

$$1 \to R_X \to F_X \to G \to 1$$

where F_X is the free group on the set X.

Definition. The *girth* of X is the length of the shortest non-trivial word in R_X, which we denote by $U(X, G)$.

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When R_X is the trivial group then we adopt the convention that $U(X, G) = \infty$. Note that this agrees with the graph-theoretic notion of girth: $U(G, X)$ is the length of the shortest nontrivial loop in the Cayley graph of G with respect to X. Maximizing this quantity over all possible generating sets gives:

Definition. The *girth* of G is

$$U(G) = \max\{U(X, G) \mid X \in \mathbf{X}_G\}.$$

Again, we take the convention that the maximum of an unbounded set is infinity, as is the maximum of a set which contains infinity. We will say that a group is *thin* if it has finite girth and *fat* if it does not.

Example. The cyclic groups, $\mathbb{Z}/n\mathbb{Z}$, have girth n.

Example. Let G be a noncyclic Abelian group. Any generating set must contain at least two elements and these will commute. So we have $U(G) \leq 4$.

Example. Let D_k be the dihedral group with $2k$ elements. Note that $\mathbb{Z}/k\mathbb{Z}$ is a subgroup of index two and all elements not in this subgroup have order two. It follows that any generating set contains an element of order two. Thus $U(D_k) \leq 2$. It is easy to check that $U(D_k) = 2$.

Example. Finally, it follows from the definitions that any finite group G is thin. Also, our choice of conventions dictates that the free groups are all fat.

3. Finite extensions and surjections

In this section we examine how the girth is changed by taking a finite extension or surjection. We also define the *diameter* of a finite group.

Lemma 3.1. *If H is an index n subgroup of G and H is thin then G is also thin and $U(G) \leq (2n - 1) \cdot U(H)$.*

We could restate this as "Virtually thin groups are thin."

Proof. Fix a generating set X for G. Form the graph Γ_G with one vertex and $|X|$ edges, one for each generator. This is the same as choosing a one-skeleton of the Eilenberg-MacLane space $K(G, 1)$. Since $K(H, 1)$ is an n-fold cover of $K(G, 1)$ we may lift Γ_G to the graph Γ_H which will also be a one-skeleton.

Let T be a spanning tree for Γ_H with root $\tilde{e}$. Note that the edges of $\Gamma_H - T$ give a generating set for H which we will call Y. A typical element of Y is given by a path which takes the unique path from $\tilde{e}$

to the non-tree edge, runs along the non-tree edge and then returns to the root. Notice that this path has length at most $2n - 1$.

Now, since $U(H) < \infty$ we must have a nontrivial relation among these generators of H, say $\tilde{r} = y_1 y_2 \ldots y_k$. Project this path down to $r \subset \Gamma_G$. This path is not contractable because $\tilde{r}$ was not and the map $\Gamma_H \to \Gamma_G$ is a covering map. Hence the set of relations R_X contains a nontrivial element of length at most $(2n-1) \cdot U(H)$. □

When H is normal we can sharpen the conclusion slightly as follows:

Definition. Suppose Z is a finite group. Given $X \in \mathbf{X}_Z$ the *diameter* of Z with respect to X is the diameter of the Cayley graph of Z with respect to X. We define the *diameter* of Z to be

$$\mathrm{Diam}(Z) = \max\{\mathrm{Diam}(Z, X) \mid X \in \mathbf{X}_Z\}.$$

Corollary 3.2. *Suppose that H is a normal, index n, thin subgroup of G. It follows that $U(G) \leq (2 \cdot \mathrm{Diam}(G/H) + 1) \cdot U(H)$.*

Proof. The graph Γ_H is exactly a Cayley graph of the quotient group. The diameter of a well chosen spanning tree will be at most twice the diameter of this graph. □

Note that computing diameters of finite groups can be a difficult problem. For example, no exact answer is know for the symmetric group S_n (See [3], page 132.)

We also can show:

Lemma 3.3. *If $\pi : G \to H$ is a surjection of groups with $|ker(\pi)| = n$ and H is thin then G is thin, with $U(G) \leq n \cdot U(H)$.*

Proof. Fix a generating set X for G. Then $Y = \pi(X)$ generates H and gives us a nontrivial relation, say $r = y_1 y_2 \ldots y_k$. It follows that $\tilde{r} = x_1 x_2 \ldots x_k$ is an element of $\ker(\pi)$ and so has order $l \leq n$. So $\tilde{r}^n$ is a nontrival relation of length at most $n \cdot U(H)$. □

4. Groups satisfying identities

In this section we show that any group satisfying a global identity is thin. Here I would like to thank George Bergman for several highly edifying conversations. In particular, he has shown me a proof by "abstract nonsense" of Lemma 4.2. His proof relies only on the existence of the relatively free group on two generators in the variety over the given identity.

We say that G *satisfies an n-variable identity* if there is a cyclically reduced word $Z(z_1, \ldots, z_n)$ in the abstract variables $z_1, \ldots, z_n$ where, for any $g_1, \ldots, g_n \in G$, we have

$$Z(g_1, \ldots, g_n) = 1.$$

Theorem 4.1. *If G satisfies an identity and is noncyclic then G is thin.*

We will use the following lemma:

Lemma 4.2. *If G satisifies a nontrival n-variable identity of length k then G satisifies a two-variable identity of length less than or equal to $(2n-3)k$.*

Proof. If $n = 1$ then we may replace z_1 by the product $a \cdot b$ to obtain desired result. If $n = 2$ then there is nothing to prove.

Assume that $n \geq 3$. Let B be the figure eight graph, that is, the graph with exactly one vertex and two edges. Let B_{n-1} be any connected $n-1$ fold covering of B. Fix a rooted spanning tree (T, v_0) for B_{n-1}. Since B_{n-1} has $n-1$ vertices, T has $n-2$ edges and thus $\pi_1(B_{n-1}) = F_n$.

We use the non-tree edges to form a basis for $\pi_1(B_{n-1})$ as in Lemma 3.1. The n-variable identity Z gives us a path of length less than or equal to $(2n-3) \cdot k$ in B_{n-1}. We project this path down to B and note that the image corresponds to a non-trivial word $W(a, b)$ where a and b are the standard generators for $\pi_1(B)$. Finally, $W(a, b)$ is an identity because it was obtained from $Z(z_1, \ldots, z_n)$ by substituting words in a and b in for the z_i's. □

Remark 4.3. The upper bound of the lemma can be improved from $(2n-3)k$ to $Ck \log(n)$, where C is an appropriately chosen constant, by chosing a cover B_{n-1} with a low diameter spanning tree.

Theorem 4.1. Since G is not cyclic any generating set contains at least two elements. This, together with Lemma 4.2, proves the theorem. □

The main use we have for this theorem is the following:

Corollary 4.4. *Nilpotent and solvable groups are thin.*

We end this section by posing the following vexing question: does there exist a thin group which does *not* satisfy an identity? Clearly, such an example (or non-existence proof) is required in order to clarify the relationship between thin groups and those which satisfy identities.

Here is a proposal for such a group. Let $F = F_{x,y}$ be the free group on the two letters x and y. Let a generate the finite cyclic group of order p, a very large integer. Let A_p be the free product of $< a >$ and F. To obtain the desired group, B_p, add the following relations to A_p:

if a word W in x, y, and a has total exponent in a not a multiple of p then $W^p = 1$.

Note that this group is thin as every element outside the normal closure of F has order p. One also suspects that x and y continue to generate a free group inside of B_p. See Lemma 34.4 of [5].

5. Free groups

It is a consequence of the definitions that any finitely generated free group has infinite girth. It reassuring to note that this convention is not strictly necessary. That is, if F is a noncyclic free group then $\mathbf{X}_F$ contains generating sets with arbitarily large but finite girth.

To see this for this for F_2 we proceed as follows: Suppose that $\langle a, b\rangle = F_2$. Fix $k \in \mathbb{N}$. Fix w, a word in a and b, such that $\text{length}(w) \geq 2k$ and the presentation $\langle a, b, c \mid c = w\rangle$ has the small cancellation property $C'(1/6)$.

Then we have a map $F_{\{a,b,c\}} \to F_{\{a,b\}}$ obtained by sending c to w. This gives the above presentation for F_2. Greendlinger's Lemma (see [1]) implies that any relation of length less than k is trivial. Thus $k < U(\{a, b, w\}, F_2) \leq |w| + 1$.

Extending this argument slightly gives:

Lemma 5.1. *If $f : G \to F_n$ is a surjection with finitely generated kernal and $n \geq 2$ then G is fat.*

Proof. Again, we will only consider the case where $n = 2$. Fix $k \in \mathbb{N}$. Let $H = \ker(f)$. Note that G can be written as a semidirect product $G = H \rtimes F_2$. Let $\{a, b\}$ be a generating set for F_2 and let $X = \{x_i\}$ be a generating set for H.

Pick words $w_i \in F_2$ such that the presentation

$$\langle a, b, c_1, \cdots, c_n \mid c_1 = w_1, \cdots, c_n = w_n\rangle$$

of F_2 has girth greater than k. This can be done using Greendlinger's Lemma, as above. Let $\pi_1 : F_{\{a,b,c_i\}} \to F_2$ be the homomorphism corresponding to this presentation.

We must procure a generating set for G which has girth greater than k. To this end set $Y = \{a, b, x_i w_i\}$. Clearly Y generates G. Let $\pi_2 : F_{\{a,b,d_i\}} \to G$ be the corresponding map from the free group on Y. That is, $\pi_2(a) = a$ and $\pi_2(b) = b$ while $\pi_2(d_i) = x_i w_i$. This map factors through the group $H \rtimes F_{\{a,b,c_i\}}$ via the maps π_3 and $\text{id}_H \rtimes \pi_1$.

$$\begin{array}{ccccc} F_{\{a,b,d_i\}} & \xrightarrow{\pi_3} & H & \rtimes & F_{\{a,b,c_i\}} \\ \downarrow & & \downarrow & & \downarrow \\ G & \xrightarrow{\text{id}_G} & H & \rtimes & F_2 \end{array}$$

Suppose that $r \in F_{\{a,b,d_i\}}$ is in the kernal of π_2 and $|r| < k$. Taking $r' = \pi_3(r)$ and applying the relations coming from the semidirect product structure on $H \rtimes F_{\{a,b,c_i\}}$ we find that r' is identical to the word obtained by replacing all the d_i's occuring in r by c_i's. It follows from our hypothesis on the girth of F_2 (as generated by $\{a, b, c_i\}$) that r' is in fact the trivial word and thus so is r. □

6. On growth

Here we speculate on the relation between the growth of finitely generated groups and their girth.

Proposition 6.1. *If G has polynomial growth then G is thin.*

Proof. This follows directly from Gromov's theorem [2] which states that every group of polynomial growth is virtually nilpotent. Simply apply Corollary 4.4 and Lemma 3.1. □

Perhaps optimistically we have:

Conjecture 6.2. *Fat groups have exponential growth.*

Note that the converse does not hold. For example, the Baumslag-Solitar group $BS(1, 2) = \langle a, b \mid aba^{-1} = b^2\rangle$ contains the free semigroup $\langle a^{-1}, a^{-1}b\rangle$ and thus has exponential growth. However $BS(1, 2)$ is solvable and thus thin by Corollary 4.4.

As a step in the direction of Conjecture 6.2 we introduce the *Hausdorff metric* on $\mathbf{X}_G$, the space of generating sets:

Definition. If $X, Y \in \mathbf{X}_G$ then

$$d_H(X, Y) = \min\{\log(r) \mid X \subset B_r(Y) \text{ and } Y \subset B_r(X)\}$$

where $B_r(X)$ denotes the ball of radius r, centered at the identity element, in the Cayley graph of G on X.

Lemma 6.3. *d_H is a metric on $\mathbf{X}_G$.*

We leave this to the reader, noting that the only nontrivial requirement is the triangle inequality. This allows us to state a somewhat technical

Lemma 6.4. *Suppose that $\{X_i\} \subset \mathbf{X}_G$ has the properties*

1. *$d_H(X_0, X_k) \leq \log(k)$ and*
2. *$U(X_k, G) \geq k$.*

Then there is a positive constant C such that $|B_k(X_0)| \geq \exp(Ck^{1/2})$.

All of the above can be summed-up into the following:

Question 6.5. *Is thinness a quasi-isometry invariant?*

Note that an affirmative answer to this question would trivally imply Lemmata 3.1 and 3.3 as well as their converses.

References

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Saul Schleimer, Department of Mathematics, University of California, Berkeley, California, 97420, U.S.A.

E-mail address: saul@math.berkeley.edu