Finite Subgroups of $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SL}(3, \mathbb{C})$

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"I hear and I forget,
I see and I remember,
I do and I understand."
Confucius

Notation.

Symbol	Meaning
$\mathrm{SL}(n,\mathbb{F})$	special linear group of degree $n \in \mathbb{N}$ over the field $\mathbb{F}$
$\mathrm{GL}(n,\mathbb{F})$	general linear group of degree $n \in \mathbb{N}$ over the field $\mathbb{F}$
$\mathrm{O}(n)$	orthogonal group of degree $n \in \mathbb{N}$ over $\mathbb{R}$
$\mathrm{SO}(n)$	special orthogonal group of degree $n \in \mathbb{N}$ over $\mathbb{R}$
$\mathrm{U}(n)$	unitary group of degree $n \in \mathbb{N}$ over $\mathbb{C}$
$\mathrm{SU}(n)$	special unitary group of degree $n \in \mathbb{N}$ over $\mathbb{C}$
$\mathrm{PSL}(n,\mathbb{F})$	projective special linear group of degree $n \in \mathbb{N}$ over the field $\mathbb{F}$
$\mathrm{PGL}(n,\mathbb{F})$	projective linear group of degree $n \in \mathbb{N}$ over the field $\mathbb{F}$
$\mathrm{Spin}(n)$	spin group, i.e., the double cover of $\mathrm{SO}(n)$
$\lvert G\rvert$	order of a group G
$C_n, \mathbb{Z}_n, \mathbb{Z}/n\mathbb{Z}$	cyclic group of order n
$\mathrm{BD}_{4n}, \mathbb{D}_n$	binary dihedral group of order $4n$
$\mathrm{BT}_{24}, \mathrm{BT}, \mathbb{T}$	binary tetrahedral group, of order 24
$\mathrm{BO}_{48}, \mathrm{BO}, \mathbb{O}$	binary octahedral group, of order 48
$\mathrm{BI}_{120}, \mathrm{BI}, \mathbb{I}$	binary tetrahedral group, of order 120
$R\,[x_1, \ldots, x_n]$	ring of polynomials in n indeterminates over R
M^t	transpose of the matrix M
M^*	adjoint (conjugate transpose) of the matrix M
I_n	$n \times n$ identity matrix
$\langle\cdot,\cdot\rangle$	Hermitian inner product
$\overline{z}$	conjugate of $z \in \mathbb{C}$
$\lvert z\rvert$	norm of $z \in \mathbb{C}$
$\mathbb{H}$	algebra of quaternions
q^*	conjugate of $q \in \mathbb{H}$
$\lVert q\rVert$	norm of $q \in \mathbb{H}$
q^{-1}	inverse of $q \in \mathbb{H}$
U	unit quaternions
$\cong$	is isomorphic to
S^n	n-sphere, $n \in \mathbb{N}$
$\mathbb{RP}^n$	real projective n-space
$\mathbb{CP}^n$	complex projective n-space
G^*	conjugate-imaginary group of a group G
$\overline{G}$	projection of the group G, $\pi\,(G)$
$\langle g\rangle$	(cyclic) group generated by an element of a group G, $g \in G$
A_n	alternating group on n letters
S_n	symmetric group on n letters
$(123\ldots n)$	permutation of n letters
$a \equiv b(c)$	a is congruent to b modulo c
$N \trianglelefteq G$	N is a normal subgroup of a group G
ρ, σ	representations of a group $G, \rho, \sigma : G \to \mathrm{GL}(n,\mathbb{C})$
ε_n	primitive nth root of unity, $n \in \mathbb{N}$

Symbol	Meaning
$C_{n,q}, \frac{1}{n}(1,q)$	cyclic group generated by $\begin{pmatrix} \varepsilon_n & 0 \\ 0 & \varepsilon_n^q \end{pmatrix}$
$\mathrm{D}_{n,q}$	dihedral group of order $4(n-q)q$
T_m	tetrahedral group of order $24m$
O_m	octahedral group of order $48m$
I_m	icosahedral group of order $120m$
N_p	normal subgroup of a group G with order a power of a prime number p
n_p	number of Sylow p-subgroups of a group G
$Z(G)$	center of a group G
O_x	orbit of an element $x \in X$, G group, X set
G_x	stabilizer of an element $x \in X$, G group, X set
$N_G(H)$	normalizer of H in G, H subset of a group G
$\oplus$	direct sum
$\otimes$	tensor product

1 Introduction

The classification of finite subgroups of the special linear group, $\mathrm{SL}(n, \mathbb{C})$, with $n \geq 2$, is a work initiated by Klein around 1870. He links the study of geometry with the properties of an invariant space under a given group action, namely the Erlangen Program: gives strong relations between geometry and group theory and representation theory. In his *Vorlesungen über das Ikosaeder* [Klein, 1993], published in 1884, Felix Klein gives the classification of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ up to conjugacy: he proves that they are the binary polyhedral groups or a cyclic group of odd order, i.e., the preimage of finite point groups under the double cover of the rotation group $\mathrm{SO}(3)$ by $\mathrm{Spin}(3) = \mathrm{SU}(2)$, the special unitary group.

For the case $n = 3$, the classification of finite subgroups of $\mathrm{SL}(3, \mathbb{C})$ is given, up to conjugacy, by the works of Blichfeldt in *Finite collineation groups* [Blichfeldt, 1917] and Miller, Blichfeldt and Dickson in *Theory and applications of finite groups* [MBD, 1916]. These classical works are used in the present and we have not needed to give a modern classification based on it, which means that they did it well, although they missed two classes of subgroups. Finally, Yau and Yu give in [YY, 1993, page 2] a completed classification of finite subgroups of $\mathrm{SL}(3, \mathbb{C})$, including the lost classes, although this book is hard to follow due to the amount of computations and a poor translation.

The aim of this paper is, in the first place, give a classification of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SL}(3, \mathbb{C})$ as in [MBD, 1916], providing some pictures of the regular solids with the corresponding axes of rotation and these rotations (this is the way to get the finite subgroups of $\mathrm{SL}(2, \mathbb{C})$) to do the construction easier to understand. In a natural way, we get finite subgroups of $\mathrm{SL}(3, \mathbb{C})$ from finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ via the group monomorphism:

$$\mathrm{GL}(2, \mathbb{C}) \hookrightarrow \mathrm{SL}(3, \mathbb{C})$$

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & \frac{1}{\det(g)} \end{pmatrix},$$

so we need to list the finite subgroups of $\mathrm{GL}(2, \mathbb{C})$. Notice that g is an element of G, a finite subgroup of $\mathrm{GL}(2, \mathbb{C})$.

One could think that trying to get the finite subgroups of $\mathrm{SL}(3, \mathbb{C})$ we have found a bigger problem since the general linear group contains the special, $\mathrm{SL}(2, \mathbb{C}) \subset \mathrm{GL}(2, \mathbb{C})$. But the finite groups of $\mathrm{GL}(2, \mathbb{C})$ are given by Behnke and Riemenschneider in [BR, 1995] in an easy way: they come from a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$ and a finite cyclic extension.

Now, after doing the classification of the finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SL}(3, \mathbb{C})$, we should ask ourselves what this is good for, apart from being interesting in itself. The answer is given in the beginning of this introduction: the finite subgroups correspond to the "given group actions".

So let G be a finite subgroup of $\mathrm{SL}(n, \mathbb{C})$, we want to study the quotient variety $\mathbb{C}^n/G$ and its resolutions $f : Y \to \mathbb{C}^n/G$, using G-invariant polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. For the case $n = 2$, we have an important result due to McKay, called the McKay correspondence: there is a correspondence between the ADE Dynkin diagrams (i.e., the resolution diagrams for Du Val singularities of $\mathbb{C}^2/G$) and the McKay graphs of the finite subgroups of $\mathrm{SL}(2, \mathbb{C})$. In other words, we get a bijection between finite subgroups and quotient singularities. Then arises an obvious question: could we extend this correspondence or one near to higher dimensions? This is a current trend in Algebraic Geometry: the problem is presented as giving bijections between the irreducible representations of G and the basis of $H^*(Y, \mathbb{Z})$, meanwhile in McKay correspondence we link the irreducible representations of G with the irreducible components of the exceptional locus $f^{-1}(0)$ of a minimal resolution Y of the Du Val singularity $\mathbb{C}^2/G$. For example, some important results are the work of Gonzalez-Sprinberg and Verdier in [G-SV, 1983] using sheaves on Y, or the work of Ito and Nakamura in [IN, 1999] using G-Hilbert schemes.

2 Finite Subgroups of SL(2,$\mathbb{C}$)

2.1 Some definitions and first steps

In this section, we are going to follow [MBD, 1916] to construct the classification of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$. Most times I try to give a present notation, but there are times in which I fall into the old notation used in this book. In both cases, the notation is explained above. Also we'll see the classification using quaternions, adding a modern point of view of the finite subgroups of $\mathrm{SL}(2, \mathbb{C})$. We start by defining $\mathrm{SL}(2, \mathbb{C})$:

Definition 1. The *special linear group* of degree 2 over the field $\mathbb{C}$ is the set of 2×2 matrices with determinant 1 and complex entries, which is a group with respect to the matrix multiplication and matrix inversion, denoted by:

$$\mathrm{SL}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb = 1 \right\}.$$

Remark 2. The special linear group of degree 2 is the group of the linear transformations of $\mathbb{C}^2$ with determinant 1, so for the geometric point of view it preserves the volume and the orientation:

$$\mathbb{C}^2 \to \mathbb{C}^2$$

$$(z_1, z_2) \mapsto (z_1', z_2') := \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)^t = (az_1 + bz_2, cz_1 + dz_2).$$

We aim to list the finite subgroups of $\mathrm{SL}(2, \mathbb{C})$, that is $G \subset \mathrm{SL}(2, \mathbb{C})$ such that the order of G is finite, denoted by$|G| < \infty$.

Theorem 3. *Let G be a finite subgroup of* $\mathrm{SL}(2, \mathbb{C})$. *Then G is one of the following cases (up to conjugacy):*

- *a cyclic group, of the form $\mathbb{Z}/n\mathbb{Z}$, with $n \in \mathbb{N}$;*
- *a binary dihedral group, of the form* BD_{4n}, *with $n \in \mathbb{N}$;*
- *a binary group corresponding to one of the Platonic solids, that is* BT_{24}, BO_{48} *or* BI_{120}.

This theorem characterizes any finite subgroup of $\mathrm{SL}(2, \mathbb{C})$, and hence it gives the classification of them. We do not prove it now because we are going to construct the classification, step by step, with the same mathematical rigour as in a regular proof. The theorem is enunciated to give a general idea of what we are going to get.

One notes that we have five Platonic solids but we only have three binary groups corresponding to each one of them. This is due to the dual identification:

Lemma 4. *The Platonic solids, namely tetrahedron, hexahedron, octahedron, dodecahedron and icosahedron are reduced, by duality, to tetrahedron, octahedron and icosahedron.*

Proof. Given a polyhedron, we define its *dual polyhedron* as the polyhedron whose vertices corresponde to the faces of the first one, and whose faces corresponde to the vertices, that is we interchange vertices and faces. Then the dual polyhedron of the tetrahedron is itself (this property is known as *self-duality*); the dual polyhedron of the hexahedron is the octahedron; the dual of the octahedron (obviously since the dual of the dual is the initial polyhedron) is the hexahedron; and the dodecahedron and the icosahedron are duals among themselves.

But two duals polyhedron have the same group of rotations which fixes a polyhedron (i.e., two duals polyhedron furnish the same set of axes of rotation), called the pure polyhedral group, and the binary polyhedral groups are going to be the inverse of the pure polyhedral groups via the canonical homomorphism from $\mathrm{SL}(2, \mathbb{C})$ to $\mathrm{PGL}(2, \mathbb{C})$. So for our purpose, it is enough to study three of the Platonic solids. $\square$

Now, we want to prove that every finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ is conjugate to a finite subgroup of $\mathrm{SU}(2) \subset \mathrm{SL}(2, \mathbb{C})$, simplifying the problem to give the classification of finite subgroups of $\mathrm{SU}(2)$. Before do it, we need to give some definitions and properties about Hermitian inner products and talk about the special unitary group $\mathrm{SU}(2)$ and the unitary form.

Definition 5. A *Hermitian inner product* on $\mathbb{C}$, $\langle \cdot, \cdot \rangle : \mathbb{C}^2 \to \mathbb{C}$, is a complex bilinear form on $\mathbb{C}$ which is positive definite and antilinear in the second variable. This is the same than satisfy the followig properties:

1. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$, for $u, v, w \in \mathbb{C}$;

2. $\langle wu, v\rangle = w\langle u, v\rangle$, for $u, v, w \in \mathbb{C}$;

3. $\overline{\langle u, v\rangle} = \langle v, u\rangle$ for $u, v \in \mathbb{C}$, where $\overline{z} \in \mathbb{C}$ denotes the conjugate of $z \in \mathbb{C}$;

4. $\langle u, u\rangle \geq 0$, and the equality holds if and only if $u = 0$.

Remark 6. From the four properties listed above, we get the following immediate properties:

1. $\langle u, v + w\rangle = \langle u, v\rangle + \langle u + w\rangle$, for $u, v, w \in \mathbb{C}$;

2. $\langle u, wv\rangle = \overline{w}\langle u, v\rangle$, for $u, v, w \in \mathbb{C}$.

Example 7. The classical examples of a Hermitian inner product on $\mathbb{C}$ are $\langle u, v\rangle := u\overline{v} + v\overline{u}$ and $\langle u, v\rangle := u\overline{u} + v\overline{v}$, with $u = u_1 + iu_2, v = v_1 + iv_2 \in \mathbb{C}$.

Definition 8. The *special unitary group* of degree 2 over the field $\mathbb{C}$ is the set of 2×2 unitary matrices with determinant 1 and complex entries, which is a group with respect to the matrix multiplication and matrix inversion. Obviously, it is a subgroup of $\mathrm{SL}(2,\mathbb{C})$, denoted by:

$$\mathrm{SU}(2) = \{U \in \mathrm{SL}(2,\mathbb{C}) : UU^* = U^*U = I_2\},$$

where U^* denotes the conjugate transpose of U.

Remark 9. With some computations, we get that

$$\mathrm{SU}(2) = \{U \in \mathrm{SL}(2,\mathbb{C}) : UU^* = U^*U = I_2\} = \left\{\begin{pmatrix} a & b \\ -\overline{b} & \overline{a}\end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1\right\}.$$

Proof. Let $U = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SU}(2)$, then $\det U = 1$ and $U^* = U^{-1}$, so

$$\begin{cases} ad - bc = 1 \\ a\overline{a} + c\overline{c} = 1 \\ b\overline{b} + d\overline{d} = 1 \\ a\overline{b} + c\overline{d} = 1 \end{cases}.$$

Solving these equations we get $c = -\overline{b}$ and $d = \overline{a}$, and since $\det U = 1$, $a\overline{a} + b\overline{b} = |a|^2 + |b|^2 = 1$. ☐

Proposition 10. *Every finite subgroup of* $\mathrm{SL}(2,\mathbb{C})$ *is conjugate to a finite subgroup of* $\mathrm{SU}(2)$*, since it leaves invariant a Hermitian inner product.*

Proof. Let $G \subset \mathrm{SL}(2,\mathbb{C})$ be finite and let $\langle\cdot,\cdot\rangle : \mathbb{C}^2 \to \mathbb{C}$ be a Hermintian inner product on $\mathbb{C}$. Then define

$$\langle\cdot,\cdot\rangle_G : \mathbb{C}^2 \to \mathbb{C}$$
$$(u, v) \mapsto \langle u, v\rangle_G := \frac{1}{|G|}\sum_{g\in G}\langle gu, gv\rangle.$$

Note that $\langle\cdot,\cdot\rangle_G$ is well defined since $\langle\cdot,\cdot\rangle$ is so and G is finite, so $|G| < \infty$ and it has a finite numer of elements $g \in G$.

Then $\langle\cdot,\cdot\rangle_G$ is a Hermitian inner product:

1. $\langle u+v, w\rangle_G = \frac{1}{|G|}\sum_{g\in G}\langle g(u+v), gw\rangle = \frac{1}{|G|}\sum_{g\in G}\langle gu+gv, gw\rangle$
   $= \frac{1}{|G|}\sum_{g\in G}\langle gu, gw\rangle + \langle gv, gw\rangle = \frac{1}{|G|}\sum_{g\in G}\langle gu, gw\rangle + \frac{1}{|G|}\sum_{g\in G}\langle gv, gw\rangle =$
   $\langle u, w\rangle_G + \langle v, w\rangle_G$, for $u, v \in \mathbb{C}$.

2. $\langle wu, v\rangle_G = \frac{1}{|G|}\sum_{g\in G}\langle gwu, gv\rangle = \frac{1}{|G|}\sum_{g\in G} w\langle gu, gv\rangle = w\langle u, v\rangle_G$, for $u, v, w \in \mathbb{C}$.

3. $\overline{\langle u, v\rangle}_G = \overline{\frac{1}{|G|}\sum_{g\in G}\langle gu, gv\rangle} = \frac{1}{|G|}\sum_{g\in G}\overline{\langle gu, gv\rangle} = \frac{1}{|G|}\sum_{g\in G}\langle gv, gu\rangle =$
   $\langle v, u\rangle_G$, for $u, v \in \mathbb{C}$.

4. $\langle u, u\rangle_G = \frac{1}{|G|}\sum_{g\in G}\langle gu, gu\rangle \geq \frac{1}{|G|}\sum_{g\in G} 0 = 0$, and the equality holds if and only if $0 = gu = u$.

And, as a Hermitian inner product, it satisfies $\langle Uu, Uv\rangle_G = \langle u, v\rangle_G$ for $u, v \in \mathbb{C}$ and $U \in \mathrm{SU}(2)$, so G is a unitary group with respect to $\langle\cdot,\cdot\rangle_G$, and from this fact, $\langle\cdot,\cdot\rangle_G$ has an orthonormal basis (taking the columns or the rows of some $g \in G$).

Let $U : (\mathbb{C}, \langle\cdot,\cdot\rangle_G) \to (\mathbb{C}, \langle\cdot,\cdot\rangle)$ such that U maps the orthonormal basis to the canonical basis. Then $\langle u, v\rangle_G = \langle Uu, Uv\rangle = \langle Ugu, Ugv\rangle = \left\langle UgU^{-1}u, UgU^{-1}v\right\rangle = \langle u, v\rangle$, for $g \in G$ and $u, v \in \mathbb{C}$. So we are done. $\square$

Remark 11. The previous result holds for all $n \in \mathbb{N}$, we have not used that $n = 1$.

Now, our aim is to list the finite subgroups of SU(2).

Also we define the quaternions and give some of their properties.

Definition 12. The *algebra of quaternions* is the 4-dimensional vector space over $\mathbb{R}$, denoted by (for Hamilton, who described them in 1843):

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\},$$

satisfying the multiplication law:

$$i^2 = j^2 = k^2 = -1,$$

which says that they extend the complex numbers, and:

$$ij = k, jk = i, ki = j,$$

$$ji = -k, kj = -i, ik = -j.$$

We can replace the last six expressions by one:

$$ijk = -1.$$

$\mathbb{H}$ has three operations: addition and scalar multiplication as in $\mathbb{R}^4$ and quaternion multiplication, called the Hamilton product, defined by the product of the basis $\{1, i, j, k\}$ and the multiplication law given above.

And we extend the conjugation from the complex numbers:

$$q = a + bi + cj + dk \mapsto q^* = a - bi - cj - dk,$$

which gives the *norm*:

$$\|q\| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

A quaternion q is called *real* if $b = c = d = 0$ and *pure imaginary* if $a = 0$, and q is *invertible* if and only if $\|q\| \neq 0$, in which case we denote $q^{-1} = \frac{1}{\|q\|^2} q^*$. From this division of $\mathbb{H}$ into real and pure imaginary quaternions, we can see them as:

$$\mathbb{H} = \mathbb{R}^4 = \{q \in \mathbb{H} : q = a\} \oplus \{q \in \mathbb{H} : q = bi + cj + dk\} = \mathbb{R} \oplus \mathbb{R}^3.$$

Last, note that $(qp)^* = p^* q^*$, for every $q, p \in \mathbb{H}$.

At this point we have defined all the objects that we need, so it is the time to link them:

Proposition 13. *$\mathrm{SU}(2) \cong S^3 \cong U$, where $U := \{q \in \mathbb{H} : qq^* = 1\}$. Note that all unit quaternions are invertible.*

Proof. By definition,

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\},$$

$$S^3 = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \right\},$$

$$U = \{q \in \mathbb{H} : qq^* = 1\}.$$

As $a, b \in \mathbb{C}$, we can write them as $a = (u_1, v_1), b = (u_2, v_2)$, and using $|a|^2 = (u_1 + iv_1)(u_1 - iv_1) = u_1^2 + v_1^2, |b|^2 = (u_2 + iv_2)(u_2 - iv_2) = u_2^2 + v_2^2$, we have

$$\mathrm{SU}(2) \xrightarrow{\cong} S^3$$

$$\begin{pmatrix} u_1 + iv_1 & u_2 + iv_2 \\ -u_2 + iv_2 & u_1 - iv_1 \end{pmatrix} \mapsto (u_1, v_1, u_2, v_2).$$

And using that $ij = k$, we construct the isomorphism

$$\mathrm{SU}(2) \xrightarrow{\cong} U = \{q \in \mathbb{H} : qq^* = 1\}$$

$$\begin{pmatrix} u_1+iv_1 & u_2+iv_2 \\ -u_2+iv_2 & u_1-iv_1 \end{pmatrix} \mapsto (u_1+iv_1)+(u_2+iv_2)\,j = u_1+iv_1+ju_2+kv_2.$$

□

We simplified the problem to list the finite subgroups of SU(2), but we can simplify it more: classify the finite isometry groups of $\mathbb{R}^3$.

Definition 14. The *special orthogonal group* of dimension 3 over $\mathbb{R}$ is the subgroup of the orthogonal group O(3) whose elements have determinant 1, denoted by

$$\mathrm{SO}(3) = \{R \in \mathrm{O}(3) : \det R = 1\},$$

where

$$\mathrm{O}(3) = \left\{Q \in \mathrm{GL}(3,\mathbb{R}) : \mathrm{Q}^{\mathrm{t}} = \mathrm{Q}^{-1}\right\}.$$

It is called the rotation group of $\mathbb{R}^3$ because its elements are rotations around an axis passing through the origin.

By the Normal Form of Orthogonal Matrix Theorem (given in all the handbooks of Linear Algebra and Geometry; for example see [Klein, 1993, pp. 14-15]), its elements can be described, in a suitable basis, as one of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

with $\theta \in [0, 2\pi)$. In the corresponding suitable basis, the director vector of the axis of rotation L is given by the first basis element.

Theorem 15. *There is a surjective group homomorphism*

$$\pi : \mathrm{SU}(2) \xrightarrow{2:1} \mathrm{SO}(3),$$

with

$$\ker(\pi) = \{\pm I_2\}.$$

Proof. This is a proof using quaternions. By the previous proposition, it is enough to prove that there is a surjective group homomorphism $h : U = S^3 \to \mathrm{SO}(3)$, and recall that

$$\begin{pmatrix} u_1+iv_1 & u_2+iv_2 \\ -u_2+iv_2 & u_1-iv_1 \end{pmatrix} \in \mathrm{SU}(2) \mapsto (u_1+iv_1)+(u_2+iv_2)\,j$$

$$= u_1+iv_1+ju_2+kv_2 \in U.$$

We also have to prove that $\ker(\pi) = \{\pm I_2\}$, equivalently that $ker(h) = \{\pm 1\}$.

Let the action of $U \cong \mathrm{SU}(2)$ on $\mathbb{R}^3$ defined by

$$r_{q_1}(x) = q_1 x q_1^*,$$

where $x \in \mathbb{R}^3 = \{q = bi + cj + dk\}$ and $q_1 \in U$. Then r_{q_1} defines a surjective homomorphism $h : U \to \mathrm{SO}(3)$ such that $h(q_1) = r_{q_1}$, with $ker(h) = \{\pm 1\}$:

- let $q_1 \in U$, then r_{q_1} is an isometry of $\mathbb{H}$, which maps $\{q \in \mathbb{H} : q = bi+cj+dk\} = \mathbb{R}^3$ into itself and is the identity on $\{q \in \mathbb{H} : q = a\} = \mathbb{R}$, so it fixes the origin, therefore it is a rotation of $\{q \in \mathbb{H} : q = bi + cj + dk\} = \mathbb{R}^3$, the pure imaginary quaternions:
  - let $q = a \in \mathbb{R}$ be a real quaternion, then $r_{q_1}(a) = q_1 a q_1^* = q_1 q_1^* a = a$. Obviously it fixes the origin $0 + 0i + 0j + 0k$;
  - let $q = bi + cj + dk \in \mathbb{H}$ be a pure imaginary quaternion, if and only if $q^* = -bi - cj - dk = -q$, then $r_{q_1}(q) = q_1 q q_1^*$, so $(r_{q_1}(q))^* = (q_1 q q_1^*)^* = (q_1^*)^* q^* q_1^* = q_1 q^* q_1^* = -q_1 q q_1^*$, so $r_{q_1}(q)$ is a pure imaginary quaterion;
- h is a group homomorphism:

$$h(q_1 p_1) = r_{q_1 p_1} = (x \mapsto q_1 p_1 x (q_1 p_1)^*) = (x \mapsto q_1 p_1 x p_1^* q_1^*)$$

$$= (x \mapsto q_1 x q_1^*) \circ (x \mapsto q p_1 x p_1^*) = h(q_1) \circ h(p_1) = r_{q_1} \circ r_{p_1};$$

- we can write $q_1 \in U$ in the form $q_1 = \cos\theta + \sin\theta I$, where $\theta \in [0, \pi]$ and $I = b'i + c'j + d'k \in U$, are uniquely determined (actually I is uniquelly determined only when $q_1 \notin \mathbb{R}$: if $q_1 \in \mathbb{R}$ then $q_1 = 1, -1$, thus $\theta = 0, \pi$ and we could choose any I): if $q_1 = a + bi + cj + dk$, then $\cos\theta = a, \sin\theta = \sqrt{b^2 + c^2 + d^2}$ (I satisfies that $I^2 = (b'i + c'j + d'k)(b'i + c'j + d'k) = -b'^2 - c'^2 - d'^2 + b'c'ij + b'd'ik + c'b'ji + c'd'jk + d'b'ji + d'c'kj = -(b'^2 + c'^2 + d'^2) + b'c'(k - k) + b'd'(j - j) + c'd'(i - i) = -\left(b'^2 + c'^2 + d'^2\right) = -1$, thus $\mathbb{R}[q_1] \cong \mathbb{C}$ and this gives $q_1 = \cos\theta + \sin\theta I$). Clearly if $I = b'i + c'j + d'k \in U$, then there exists a basis of $\mathbb{H}$, $\{1, I, J, K\}$, where $1, I, J, K$ satisfy the multiplicatoin law of the quaternions, $ijk = -1$. We need to prove that r_{q_1} fixes I and is the rotation of the plane spanned by J, K about I through the angel 2θ. Now we have

$$r_{q_1}(I) = q_1 I q_1^* = (\cos\theta + \sin\theta I)\, I\, (\cos\theta + \sin\theta I)^*$$

$$= (\cos\theta + \sin\theta I)\, I\, (\cos\theta - \sin\theta I) = \cos^2\theta I - \cos\theta I \sin\theta I + \sin\theta I^2 \cos\theta$$

$$- \sin\theta I^2 \sin\theta I = I(\cos\theta + \sin\theta) = I.$$

  Similarly

$$r_{q_1}(J) = q_1 J q_1^* = \left(\cos^2\theta - \sin^2\theta\right) J + 2\sin\theta\cos\theta K,$$

$$r_{q_1}(K) = q_1 K q_1^* = -2\sin\theta\cos\theta J + \left(\cos^2\theta - \sin^2\theta\right) K.$$

  Then r_{q_1} is the rotation of $\{q \in \mathbb{H} : q = bi + cj + dk\} = \mathbb{R}^3$ about the angle 2θ with axis defined by I. This also shows the surjectivity.

- $ker(h) = \{\pm 1\}$: using that $q_1 = \cos\theta + \sin\theta q_1'$,

$$h(q_1) = r_{q_1} = Id_{\mathbb{H}} \Leftrightarrow r_{q_1}(x) = q_1 x q_1^* = x, x \in \mathbb{R}^3$$

$$\Leftrightarrow (\cos\theta + \sin\theta q_1')\, x\, (\cos\theta + \sin\theta q_1')^* = (\cos\theta + \sin\theta q_1')\, x\, (\cos\theta - \sin\theta q_1')$$

$$= \cos^2\theta x + -\cos\theta\sin\theta x q_1' + \cos\theta\sin\theta q_1' x - \sin^2\theta q_1' x q_1' = x, x \in \mathbb{R}$$

$$\Leftrightarrow \theta = 0, \pi \Leftrightarrow q_1 = 1, -1.$$

□

We can give an alternative sketch of proof from the topological point of view:

Proof. The map is the natural projection

$$\pi : S^3 \xrightarrow{2:1} \mathbb{RP}^3$$

$$\mathbf{x} \mapsto [\mathbf{x}] = \{\mathbf{x}, -\mathbf{x}\},$$

seeing $\mathbb{RP}^3$ as S^3 with antipodal surface points identified, $\mathbf{x} \sim -\mathbf{x}$, and using that $\mathbb{RP}^3$ is diffeomorphic to $\mathrm{SO}(3)$. □

Another alternative sketch of proof, from representation theory:

Proof. From representation theory, we can see an element of $\mathrm{SU}(2)$ as $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, and map this to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix} \in \mathrm{SO}(3)$. □

Now we can apply the First Isomorphism Theorem, so

$$\mathrm{SU}(2)/\ker(\pi) \cong \mathrm{im}(\pi),$$

but since it is surjective, $\mathrm{im}(\pi) = \mathrm{SO}(3)$, then

$$\mathrm{SU}(2)/\{\pm I_2\} \cong \mathrm{SO}(3).$$

And using the properties of the group homomorphisms (specifically that the preimage of a subgroup is a subgroup) we get that the finite subgroups of $\mathrm{SU}(2)/\{\pm I_2\}$ are the preimage under the natural projection (the double cover $\mathrm{SU}(2) \xrightarrow{2:1} \mathrm{SO}(3)$) of the finite subgroups of $\mathrm{SO}(3)$ as desired. So let $G \subset \mathrm{SU}(2)$ be finite, it defines $\overline{G} \subset \mathrm{SO}(3)$ finite, and let $\overline{G} \subset \mathrm{SO}(3)$ be finite, it can be lifted to $G \subset \mathrm{SU}(2)$ finite with kernel of order ≤ 2. We denote the groups of $\mathrm{SO}(3)$ by $\overline{G}$ because they come from a projection.

All the finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ that we will find (except one, the cyclic group of odd order) are going to contain the subgroup of order 2 $\{\pm I_2\}$ since they will have even order and $-I_2$ is the only element of $\mathrm{SU}(2)$ with order 2:

Proposition 16. *Let $U \in \mathrm{SU}(2)$ of order 2, then this is I_2.*

Proof. Using Remark 9, $U = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$ such that $a, b \in \mathbb{C}$. Then

$$\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}^2 = \begin{pmatrix} a^2 - |b|^2 & ab + b\overline{a} \\ -\overline{b}a - \overline{a}\overline{b} & \overline{a}^2 - |b|^2 \end{pmatrix} = \begin{pmatrix} a^2 - |b|^2 & b\,(a + \overline{a}) \\ -\overline{b}\,(a + \overline{a}) & \overline{a}^2 - |b|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

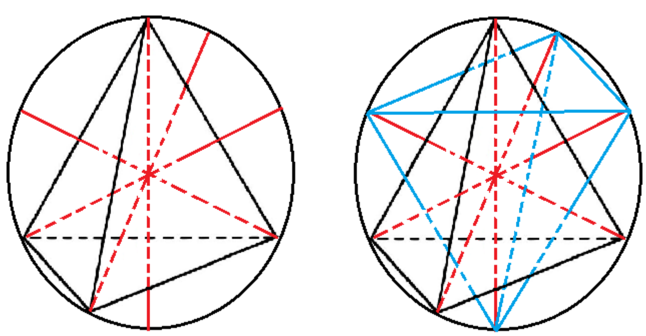
if and only if $a^2 = \overline{a}^2$, thus either $a \in \mathbb{R}$ or $a \in \mathbb{C} \backslash \mathbb{R}$, and if $a = 0$ then $-|b|^2 = 1$, so $a \neq 0$. Now, if $a \in \mathbb{R}$ then $-\overline{b}\,(a + \overline{a}) = -\overline{b}2a = 0$, so $b = 0$ and $a = \pm 1$ since $a \neq 0$. If $a \in \mathbb{C} \setminus \mathbb{R}$, then $a = ix, x \in \mathbb{R}$,so $a^2 - |b|^2 = -|x|^2 - |b|^2 < 0$. ☐

Consequently we are going to have that $|G| = 2\left|\overline{G}\right|$ if $-I_2 \in G$ and $|G| = \left|\overline{G}\right|$ otherwise (when G is a cyclic group of odd order).

2.2 Construction scheme

1. Let $G \subset \mathrm{SL}(2, \mathbb{C})$ be finite, we see G as a group of linear transformation of $\mathbb{C}^2$ of determinant 1, which maps $(z_1, z_2) \in \mathbb{C}^2$ to $(az_1 + bz_2, cz_1 + dz_2) \in \mathbb{C}^2$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Let $G^* \subset \mathrm{SL}(2, \mathbb{C})$ be the *conjugate-imaginary group* of G defined by the group whose elements map $(z_1, z_2) \in \mathbb{C}^2$ to $\left(\overline{a}\overline{z_1} + \overline{b}\overline{z_2}, \overline{c}\overline{z_1} + \overline{d}\overline{z_2}\right)$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ (clearly G^* isomorphic with G). Then we get $\overline{G} \subset \mathrm{SO}(3)$ finite by a change of variables, X, Y, Z, which are bilinear in $z_1, z_2, \overline{z_1}, \overline{z_2}$.

2. Let $S^2 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\}$ be the unit sphere in $\mathbb{R}^3$, for every rotation $\overline{g} \in \overline{G} \subset \mathrm{SO}(3)$ we have an axis of rotation L, which meets the sphere in two points $P_1, -P_1$. All the points of the corresponding rotations, say $\{P_1, \ldots, P_k\}$, under every rotation $\overline{g} \in \overline{G}$ are either the vertices of a regular polyhedron (as in the pictures), the vertices of a flat polygon, or there is only one axis of rotation.



3. We get five *types* (i.e., subgroups up to conjugacy) of finite subgroups of $\mathrm{SO}(3)$ and the corresponding five types of $\mathrm{SL}(2, \mathbb{C})$.

Proof. *Step 1*: it follows from Remark 2, 9, Proposition 10, 13 and Theorem 15: let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \subset \mathrm{SL}(2,\mathbb{C})$, then $g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1$ (by Remark 9 and Proposition 10), so g is the linear transformation $(z_1, z_2) \mapsto (az_1 + bz_2, -\bar{b}z_1 + \bar{a}z_2)$. Define $p := \sqrt{a\overline{a}}, q := \sqrt{b\overline{b}}$ and let α, β such that $a = p\alpha = \sqrt{a\overline{a}}\alpha, b = q\beta = \sqrt{b\overline{b}}\beta$ satisfying $p = \overline{p}, q = \overline{q}, p^2 + q^2 = 1$, thus $-\overline{b} = -q\overline{\beta}, \overline{a} = p\overline{\alpha}$ satisfying $\alpha\overline{\alpha} = \beta\overline{\beta} = 1$, so we get $g = \begin{pmatrix} p\alpha & q\beta \\ -q\overline{\beta} & p\overline{\alpha} \end{pmatrix}$. And define $\gamma := \sqrt{\alpha\beta}, \delta := \sqrt{\alpha/\beta}$ satisfying $\gamma\overline{\gamma} = \delta\overline{\delta} = 1$, so we get

$$g = \begin{pmatrix} p\gamma\delta & q\gamma\overline{\delta} \\ -q\overline{\gamma}\delta & p\overline{\gamma}\overline{\delta} \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ 0 & \overline{\gamma} \end{pmatrix} \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \begin{pmatrix} \delta & 0 \\ 0 & \overline{\delta} \end{pmatrix} =: g_1 g_2 g_3.$$

Similarly for $g^* \in G^*$, we get:

$$g^* = \begin{pmatrix} \overline{\gamma} & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \begin{pmatrix} \overline{\delta} & 0 \\ 0 & \delta \end{pmatrix} =: g_1^* g_2^* g_3^*.$$

We write $\gamma = \cos\theta_1 - i\sin\theta_1, \delta = \cos\theta_3 - i\sin\theta_3, p = \cos\theta_2, q = \sin\theta_2$ (we can do this because $\gamma\overline{\gamma} = \delta\overline{\delta} = 1, p^2 + q^2 = 1$ and $p, q \in \mathbb{R}, \gamma, \delta \in \mathbb{C}$), for some $\theta_1, \theta_2, \theta_3 \in [0, 2\pi)$, so

$$g_1 = \begin{pmatrix} \cos\theta_1 - i\sin\theta_1 & 0 \\ 0 & \cos\theta_1 + i\sin\theta_1 \end{pmatrix}, g_2 = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} \cos\theta_3 - i\sin\theta_3 & 0 \\ 0 & \cos\theta_3 + i\sin\theta_3 \end{pmatrix},$$

and we choose the new variables $X(z_1, z_2), Y(z_1, z_2), Z(z_1, z_2)$ such that $X(z_1, z_2) = z_1\overline{z_1} - z_2\overline{z_2}, Y(z_1, z_2) = z_1\overline{z_2} + z_2\overline{z_1}, Z(z_1, z_2) = i\,(z_1\overline{z_2} - z_2\overline{z_1})$, for $z_1, z_2 \in \mathbb{C}$, which satisfy ($g^*g(z_1, z_2)$ denotes the linear transformation, not the product of matrices):

$$X(g_1^*g_1(z_1, z_2)) = X(z_1, z_2),$$

$$Y(g_1^*g_1(z_1, z_2)) = Y(z_1, z_2)\cos 2\theta_1 - Z(z_1, z_2)\sin 2\theta_1,$$

$$Z(g_1^*g_1(z_1, z_2)) = Y(z_1, z_2)\sin 2\theta_1 + Z(z_1, z_2)\cos 2\theta_1,$$

$$X(g_3^*g_3(z_1, z_2)) = X(z_1, z_2),$$

$$Y(g_3^*g_3(z_1, z_2)) = Y(z_1, z_2)\cos 2\theta_3 - Z(z_1, z_2)\sin 2\theta_3,$$

$$Z(g_3^*g_3(z_1, z_2)) = Y(z_1, z_2)\sin 2\theta_3 + Z(z_1, z_2)\cos 2\theta_3,$$

$$X(g_2^*g_2(z_1, z_2)) = X(z_1, z_2)\cos 2\theta_2 + Y(z_1, z_2)\sin 2\theta_2,$$

$$Y(g_2^*g_2(z_1, z_2)) = -X(z_1, z_2)\sin 2\theta_2 + Y(z_1, z_2)\cos 2\theta_2,$$

$$Z(g_2^*g_2(z_1, z_2)) = Z(z_1, z_2).$$

These equalities come from a lot of computations and the trigonometric identities $\cos^2\theta+\sin^2\theta=1, \cos 2\theta=\cos^2\theta-\sin^2\theta, \sin 2\theta = 2\sin\theta\cos\theta$. For example, we show the first and the last one:

$$g_1 : (z_1, z_2) \mapsto \left(\begin{pmatrix}\gamma & 0\\ 0 & \overline{\gamma}\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}\right)^t = (\gamma z_1, \overline{\gamma} z_2),$$

$$g_1^* : (\gamma z_1, \overline{\gamma} z_2) \mapsto \left(\begin{pmatrix}\overline{\gamma} & 0\\ 0 & \gamma\end{pmatrix}\begin{pmatrix}\overline{\gamma z_1}\\ \overline{\overline{\gamma} z_2}\end{pmatrix}\right)^t = \left(\overline{\gamma}^2\overline{z_1}, \gamma^2\overline{z_2}\right),$$

$$X(g_1^* g_1(z_1,z_2)) = X\left(\overline{\gamma}^2\overline{z_1}, \gamma^2\overline{z_2}\right) = \overline{\gamma}^2\overline{z_1}\overline{\overline{\gamma}^2\overline{z_1}} - \gamma^2\overline{z_2}\overline{\gamma^2\overline{z_2}}$$
$$= \overline{\gamma}^2\gamma^2\overline{z_1}z_1 - \overline{\gamma}^2\gamma^2\overline{z_2}z_2 = \overline{z_1}z_1 - \overline{z_2}z_2 = X(z_1,z_2).$$

$$g_2 : (z_1, z_2) \mapsto \left(\begin{pmatrix}\cos\theta_2 & \sin\theta_2\\ -\sin\theta_2 & \cos\theta_2\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}\right)^t$$
$$= (\cos\theta_2 z_1 + \sin\theta_2 z_2, -\sin\theta_2 z_1 + \cos\theta_2 z_2),$$

$$g_1^* : (\gamma z_1, \overline{\gamma} z_2) \mapsto \left(\begin{pmatrix}\cos\theta_2 & \sin\theta_2\\ -\sin\theta_2 & \cos\theta_2\end{pmatrix}\begin{pmatrix}\cos\theta_2\overline{z_1} + \sin\theta_2\overline{z_2}\\ -\sin\theta_2\overline{z_1} + \cos\theta_2\overline{z_2}\end{pmatrix}\right)^t$$
$$= \left(\cos^2\theta_2\overline{z_1} - \sin^2\theta_2\overline{z_1} + 2\cos\theta_2\sin\theta_2\overline{z_2}, \cos^2\theta_2\overline{z_2} - \sin^2\theta_2\overline{z_2} - 2\cos\theta_2\sin\theta_2\overline{z_1}\right)$$
$$= (\cos 2\theta_2\overline{z_1} + \sin 2\theta_2\overline{z_2}, \cos 2\theta_2\overline{z_2} - \sin 2\theta_2\overline{z_1}),$$

$$Z(\overline{z_1}\cos 2\theta + \overline{z_2}\sin 2\theta, -\overline{z_1}\sin 2\theta + \overline{z_2}\cos 2\theta) =$$
$$= i\left(\overline{z_1}\cos 2\theta + \overline{z_2}\sin 2\theta\right)\left(-z_1\sin 2\theta + z_2\cos 2\theta\right)$$
$$-i\left(-\overline{z_1}\sin 2\theta + \overline{z_2}\cos 2\theta\right)\left(z_1\cos 2\theta + z_2\sin 2\theta\right)$$
$$= i\left(z_1\overline{z_1}\left(-\cos 2\theta\sin 2\theta + \sin 2\theta\cos 2\theta\right) + z_2\overline{z_2}\left(\cos 2\theta\sin 2\theta - \sin 2\theta\cos 2\theta\right)\right)$$
$$+i\left(z_1\overline{z_2}\left(\cos^2 2\theta + \sin^2 2\theta\right) + z_2\overline{z_1}\left(-\sin^2 2\theta - \cos^2 2\theta\right)\right) = i\left(z_1\overline{z_2} - z_2\overline{z_1}\right)$$
$$= Z(z_1, z_2).$$

Now we can see the linear transformations $g_1^*g_1, g_2^*g_2, g_3^*g_3$ in these new variables as:

$$\overline{g_1} := g_1 g_1^* : \begin{pmatrix}1 & 0 & 0\\ 0 & \cos 2\theta_1 & -\sin 2\theta_1\\ 0 & \sin 2\theta_1 & \cos 2\theta_1\end{pmatrix}, \overline{g_2} := g_2 g_2^* : \begin{pmatrix}\cos 2\theta_2 & \sin 2\theta_2 & 0\\ -\sin 2\theta_2 & \cos 2\theta_2 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$\overline{g_3} := g_3 g_3^* : \begin{pmatrix}1 & 0 & 0\\ 0 & \cos 2\theta_3 & -\sin 2\theta_3\\ 0 & \sin 2\theta_3 & \cos 2\theta_3\end{pmatrix}.$$

We identify

$$\begin{pmatrix}1 & 0 & 0\\ 0 & \cos 2\theta_1 & -\sin 2\theta_1\\ 0 & \sin 2\theta_1 & \cos 2\theta_1\end{pmatrix}, \begin{pmatrix}\cos 2\theta_2 & \sin 2\theta_2 & 0\\ -\sin 2\theta_2 & \cos 2\theta_2 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & \cos 2\theta_3 & -\sin 2\theta_3\\ 0 & \sin 2\theta_3 & \cos 2\theta_3\end{pmatrix}$$

with the rotations around the X-axis, Z-axis and X-axis through the angles $2\theta_1, 2\theta_2$ and $2\theta_3$ respectively. Thus given $g \in G \subset \mathrm{SL}(2, \mathbb{C})$ we get these three rotations in SO(3), which are equivalent to a single rotation and they are the identity if and only if $\theta_1 = 0, \pi, \theta_2 = 0, \pi, \theta_3 = 0, \pi$, if and only if (using definitions of γ, δ, p, q) $\gamma = 1, -1, \delta = 1, -1, p = 1, -1, q = 0$, if and only if (using $\gamma\delta = \alpha, \gamma/\delta = \beta$) $\alpha = 1, -1, \beta = 1, -1, p = 1, -1, q = 0$, if and only if (using $a = p\alpha, b = q\beta$) $g = I_2, -I_2$. So the isomorphism between the group G of linear transformations and the group $\overline{G}$ of the corresponding rotations is $2:1$ if $-I_2 \in G$ and $1:1$ else.

Step 2: let L be an axis of rotation of a group of rotations $\overline{G}$, which meets the sphere in two points $P_1, -P_1$, we say that L is an axis of *index* n if the multiples of $\frac{2\pi}{n}, n \in \mathbb{N}$ are the corresponding angles of rotations around L. P_1 is moved to $\{P_1, \ldots, P_k\}$ under the action of all the rotations $\overline{g} \in \overline{G}$, which are the points where every axis of index n of $\overline{G}$ meets the sphere.

We link the point P_1 with $\{P_2, \ldots, P_k\}$ using great circles (in this part we assume basic notions of trigonometry and spherical trigonometry) and we get the great-circle distance (henceforth, "distance" means "great-circle distance") of each pair of points given by the shortest arc of each of them. Let D be the smallest distance, then the corresponding arc is moved by rotations (multiples of $\frac{2\pi}{n}$) to n or a multiple of n arcs, preserving the distance, so we have n or a multiple of n arcs of length D. We claim that if we have more than two points actually there are five or less arcs of length D:

If we have only one point, then there are infinitely many great circles with length $D = 2\pi$, and if we have two points, they are antipodal points, so again there are infinitely many great circles, with length $D = \pi$. Now assume there are six or more, so at least two of them make an angle $\theta \leq \pi/3$ (since if they are six and equispaced, then every two of them which are consecutive make an angle $2\pi/6 = \pi/3$). Let P_i, P_j the points corresponding with the arcs that make an angle $\theta \leq \pi/3$. Let D_1 be the distance between P_i and P_j. The arcs linking P_i with P_j, P_1 with P_i and P_1 with P_j form a spherical triangle, so using spherical trigonometry and $\theta \leq \pi/3$ we have that $\cos D_1 > \cos D$:

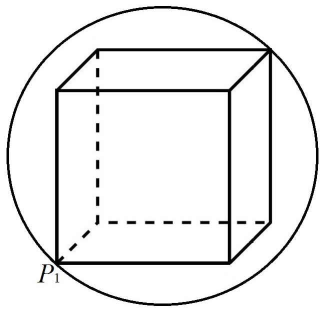
$$\cos D_1 = \cos^2 D + \sin^2 D \cos\theta \geq \cos^2 D + \frac{1}{2}\sin^2 D > \cos D,$$

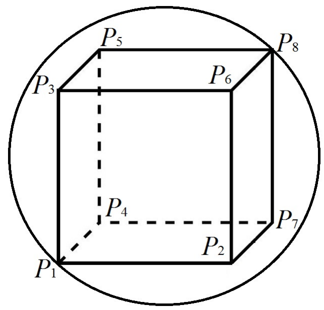
but $0 < D_1 < \pi/2$, so $D_1 < D$, in contradiction with the fact that the distribution of the points is the same view from any of the poins, so D_1 should be equal to D.

Now we have two cases to study, $n = 2, n > 2$ ($n = 1$ gives the identity):

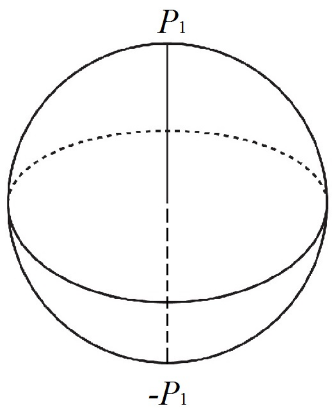
1. If $n = 3, 4, 5$ then for every point in $\{P_1, \ldots, P_k\}$ we have n arcs of smallest great-circle distance D and every two consecutive arcs make an angle $\frac{2\pi}{n}$. So we have divided the sphere in arcs of distance D linking the points $\{P_1, \ldots, P_k\}$. The way to get the platonic solids is the following: fix a point P_1 in the sphere, draw the n arcs of smallest distance D, getting n new points, and doing the same with these points until we get $\{P_1, \ldots, P_k\}$. Now joining these points using straight lines (not great circles) they are the

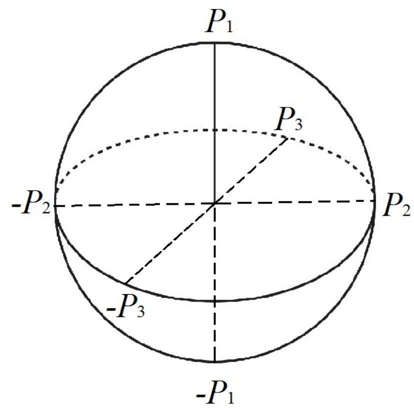
vertices of the platonic solids. In particular, if $n=3$ we get the tetrahedron ($D=\frac{2\pi}{3}$), hexahedron ($D=\frac{\pi}{2}$) and dodecahedron ($D=\frac{\pi}{3}$); if $n=4$ we get the octahedron ($D=\frac{\pi}{2}$); and if $n=5$ we get the icosahedron ($D=\frac{\pi}{5}$). For example we show the construction of the hexahedron, the rest are similar: fix a point P_1 in the sphere and draw $n=3$ arcs of length $D=\pi/2$, getting the points P_2, P_3, P_4. Now we draw two (the third is already determined by P_1) new arcs from every point, getting three new points (we would get six, but they coincide using spherical trigonometry or geometrical intuition), P_5, P_6, P_7. Finally, doing the same we get the last point, P_8.See the picture to help the geometrical intuition





2. If $n=2$, let L be an axis of rotation of index 2 and $\{P_1,\ldots,P_k\}$ the corresponding points, where P_1 is again the intersection between L and the sphere. As in the previous case, for every point in $\{P_1,\ldots,P_k\}$ we have at least two arcs of smallest great-circle distance D and every two consecutive arcs make an angle $\frac{2\pi}{n}=\pi$. If we extend the arcs of length D that meets in P_1, since they make an angle π, they form a great circle in the sphere. At least $P_1, -P_1$ lie in this great circle. Suppose we have $k>2$ points lying in the great circle, then $k=4$: the composition of two rotations (we are in index two, so they are rotations of π) around $P_1$and one of its adyacent points is equivalent to a rotation around the axis perpendicular to the plane generated by the great circle about the angle $\frac{4\pi}{k}$, but the axis is of index 2, so $k=4$: we have three axis of index 2, which are mutually perpendicular. So we have two cases: if $D=\pi$ then $\overline{G}$ is a single axis of index 2; and if $D=\pi/2$, then $\overline{G}$ contains three axis of index 2, which are mutually perpendicular:

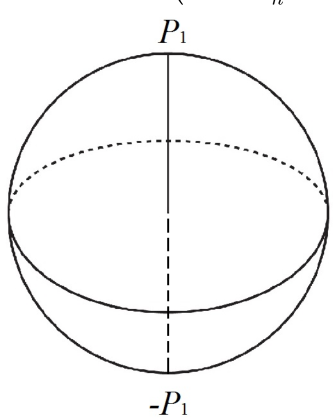




Step 3: for every rotation of every group we get determined values for $\theta_1, \theta_2, \theta_3$, and with them we get the corresponding values of a, b, the entries of the matrices. This step is developed in the following five subsections. $\square$

2.3 Cyclic groups: $\mathbb{Z}/n\mathbb{Z}$

The first case is the simplest case: we only have a single axis of rotation L, of index $n, n \in \mathbb{N}$, that is the rotation around L through $\frac{2\pi}{n}$. In suitable coordinates, we can assume that L is the X-axis. So $\overline{g}$ is the rotation given, in suitable coordinates, by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ 0 & \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}$:



Recall that we get the following $\overline{g_1}, \overline{g_2}, \overline{g_3}$ rotations around the X-axis, Z-axis and X-axis through the angles $2\theta_1, 2\theta_2$ and $2\theta_3$ respectively in the *Step 2*:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta_1 & -\sin 2\theta_1 \\ 0 & \sin 2\theta_1 & \cos 2\theta_1 \end{pmatrix}, \begin{pmatrix} \cos 2\theta_2 & \sin 2\theta_2 & 0 \\ -\sin 2\theta_2 & \cos 2\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta_3 & -\sin 2\theta_3 \\ 0 & \sin 2\theta_3 & \cos 2\theta_3 \end{pmatrix}.$$

Then we need the rotation around the Z-axis to be the identity:

$$\overline{g_2} = \begin{pmatrix} \cos 2\theta_2 & \sin 2\theta_2 & 0 \\ -\sin 2\theta_2 & \cos 2\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \Leftrightarrow \cos 2\theta_2 = 1, \sin 2\theta_2 = 0,$$

so $\theta_2 = 0$ and $g_2 = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix} = I_2$. We have then $g = g_1 g_2 g_3 = \begin{pmatrix} \gamma & 0 \\ 0 & \overline{\gamma} \end{pmatrix} I_2 \begin{pmatrix} \delta & 0 \\ 0 & \overline{\delta} \end{pmatrix} = \begin{pmatrix} \gamma\delta & 0 \\ 0 & \overline{\gamma}\overline{\delta} \end{pmatrix}$, with $\gamma := \sqrt{\alpha\beta}, \delta := \sqrt{\alpha/\beta}$ and $\alpha\overline{\alpha} = 1$, so $\gamma\delta = \pm\alpha, \overline{\gamma}\overline{\delta} = \pm\overline{\alpha} = \pm\frac{1}{\alpha}$, that is $g = \begin{pmatrix} \pm\alpha & 0 \\ 0 & \pm\alpha^{-1} \end{pmatrix}$. Now the axis of rotation has index n, thus $\overline{g}^n = I_3$ and $g^n = I_2$, hence $(\pm\alpha)^n = 1$. We have found the first type of finite subgroups (up to conjugacy):

- $\overline{G} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ 0 & \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}^k : n \in \mathbb{N}, k = 1, \ldots, n \right\}$, for every $n \in \mathbb{N}$;
- $G = \left\{ \begin{pmatrix} \pm\alpha & 0 \\ 0 & \pm\alpha^{-1} \end{pmatrix}^k : (\pm\alpha)^n = 1, n \in \mathbb{N}, k = 1, \ldots, n, (\pm\alpha)^k \neq 1, k < n \right\}$, for every $n \in \mathbb{N}$.

In the literature, the nth primitive root of unity is denoted by $\varepsilon = \exp\frac{2\pi i}{n}$, and for a fix n this group is called the *cyclic group of order n*, denoted by $C_n, \mathbb{Z}_n$ or $\mathbb{Z}/n\mathbb{Z}$. So

$$G = \left\{ \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}^k : \varepsilon^n = 1, n \in \mathbb{N}, k = 1, \ldots, n, (\pm\alpha)^k \neq 1, k < n \right\} = C_n, n \in \mathbb{N}.$$

Note that if $|C_n| = n$ is odd then it does not contain $-I_2$, so $ker(\pi) = I_2$ and the $2:1$ homomorphism is actually an isomorphism.

2.3.1 Cyclic groups in terms of quaternions

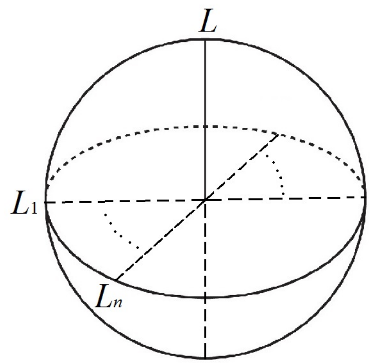
Recall that $\mathrm{SU}(2) \xrightarrow{\cong} U = \{q \in \mathbb{H} : qq^* = 1\}$ via $\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mapsto a + bj$ and that $\varepsilon = \exp\frac{2\pi i}{n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, so $\begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix} \mapsto \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ and

$$C_n = \left\{ \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \in \mathbb{H} : k = 1, \ldots, n \right\}, n \in \mathbb{N},$$

$$C_n = \left\langle \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} \right\rangle, n \in \mathbb{N}.$$

2.4 Binary Dihedral groups: BD_{4n}

The second case is when we have an axis of rotation L, of index $n, n \in \mathbb{N}$ and n axes of index 2, say $L_1, \ldots, L_n$, lying in a great-circle perpendicular to L, that is a rotation around L through $\frac{2\pi}{n}$ and n rotations around $L_1, \ldots, L_n$ (respectively) through π. In suitable coordinates, we can assume that L is the X-axis and one of the $L_1, \ldots, L_n$, say L_1, is the Z-axis. So $\overline{g}_L$ is the rotation given, in suitable coordinates, by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ 0 & \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}$ and $\overline{g}_{L_1}$ by $\begin{pmatrix} \cos\pi & \sin\pi & 0 \\ -\sin\pi & \cos\pi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$:



The rotation around the Z-axis is given by $\overline{g_2} = \begin{pmatrix} \cos 2\theta_2 & \sin 2\theta_2 & 0 \\ -\sin 2\theta_2 & \cos 2\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\overline{g_1} = \overline{g_3} = I_3$, and the rotation around the X-axis is given by $\overline{g_2} = I_3$ and

$$\overline{g_1 g_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta_1 & -\sin 2\theta_1 \\ 0 & \sin 2\theta_1 & \cos 2\theta_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta_3 & -\sin 2\theta_3 \\ 0 & \sin 2\theta_3 & \cos 2\theta_3 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2(\theta_1 - \theta_3) & -\sin 2(\theta_1 - \theta_3) \\ 0 & \sin 2(\theta_1 - \theta_3) & \cos 2(\theta_1 - \theta_3) \end{pmatrix}.$$

The axis of rotation L is like in the previous case, so $C_n, n \in \mathbb{N}$ is contained in the second type of groups. For L_1 we need $\overline{g_2}$ to be $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\overline{g_1 g_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2(\theta_1 - \theta_3) & -\sin 2(\theta_1 - \theta_3) \\ 0 & \sin 2(\theta_1 - \theta_3) & \cos 2(\theta_1 - \theta_3) \end{pmatrix} = I_3$: we need $\theta_2 = \pi/2$ and $\cos 2(\theta_1 - \theta_3) = 1, \sin 2(\theta_1 - \theta_3) = 0$. So for L_1 we have $g = g_1 g_2 g_3 =$

$\begin{pmatrix}\gamma & 0\\ 0 & \overline{\gamma}\end{pmatrix}\begin{pmatrix}\cos\theta_2 & \sin\theta_2\\ -\sin\theta_2 & \cos\theta_2\end{pmatrix}\begin{pmatrix}\delta & 0\\ 0 & \overline{\delta}\end{pmatrix}=\begin{pmatrix}\gamma & 0\\ 0 & \overline{\gamma}\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\delta & 0\\ 0 & \overline{\delta}\end{pmatrix}=\begin{pmatrix}0 & \gamma\overline{\delta}\\ -\overline{\gamma}\delta & 0\end{pmatrix}=$ $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ using the definitions of γ,δ and $\cos 2(\theta_1-\theta_3)=1, \sin 2(\theta_1-\theta_3)=0$:

$\gamma\overline{\delta}=(\cos\theta_1-i\sin\theta_1)(\cos\theta_3+i\sin\theta_3)=\cos 2(\theta_1-\theta_3)+i\sin 2(\theta_1-\theta_3)=1.$

But we have n axes $L_1,\dots,L_n$ of index two, and note that since each of them makes an angle of $\frac{2\pi}{n}$ with its adjacent axes, they are moved to the next axis under the rotation $\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n}\\ 0 & \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n}\end{pmatrix}$, so all of these rotations are given by $\begin{pmatrix}0 & \varepsilon\\ -\varepsilon^{-1} & 0\end{pmatrix}^k : k=1,\dots,n$, with ε being a nth primitive root of unity.

Therefore G is generated by $\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}$ and $\begin{pmatrix}0 & \varepsilon\\ -\varepsilon^{-1} & 0\end{pmatrix}$, with ε being a nth primitive root of unit, which is equivalent to be generated by $A=\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}$ and $B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, where ε is a $2n$th primitive root of unity: $G=\langle A,B\rangle$, for every $n\in\mathbb{N}$. The order of G is $|G|=4n$ since ε is a $2n$th primitive root of unity and we have the following relations:

$$\begin{cases}A^n=B^2\\ B^4=\left(B^2\right)^2=(-I_2)^2=I_2\\ BAB^{-1}=A^{-1}\end{cases}.$$

The groups of this type are called the *binary dihedral* (or *dicyclic*) *groups of order 4n*, denoted by BD_{4n} or $\mathbb{D}_n$.

2.4.1 Binary Dihedral groups in terms of quaternions

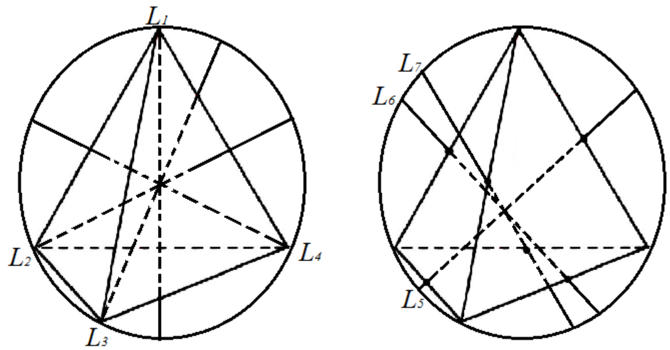
Proceeding as in 3.3.1 and using $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mapsto j$ we get

$$\mathrm{BD}_{4n}=\left\langle \cos\frac{\pi}{n}+i\sin\frac{\pi}{n}, j\right\rangle, n\in\mathbb{N}.$$

For the rest types of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ we have to study the groups of rotations of the Platonic solids: the groups of rotations of the tetrahedron, octahedron and icosahedron. Recall that the hexahedron has the same group of rotations as the octahedron, and the dodecahedron has the same group of rotation as the icosahedron.

2.5 Binary Tetrahedral group: BT_{24}

The third type is given by the group of rotations of the tetrahedron. We have two kinds of axes of rotation: four axes of rotation $L_1,\dots,L_4$ of index 3 provided by linking every vertex with the origin (which links every vertex with the middle of the opposite face); and three axes of rotation L_5, L_6, L_7 of index 2 provided by joining the middle of every edge with the opposite, as in the pictures:



The axes of rotation L_5, L_6, L_7 are mutually perpendicular thus in suitable coordinates we can assume that L_5, L_6, L_7 are the X-,Y-,Z-axes and hence the rotations around them through π are given by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ respectively. One could think that we have a little problem since our rotations $\overline{g_1}, \overline{g_2}, \overline{g_3}$ are around X-,Y-,Z-axes, but

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

i.e., the rotation around the Y-axis through π is equivalent to the composition of rotations around the X-,Z-axes through π. In 3.4 we saw that the rotation around the Z-axis through π gives us $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Similarly we get $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. And from the fact that $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ we have $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, so it is enough to consider the generators $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Now if we denote the vertices of the tetrahedron by $P_1, \ldots, P_4$, these three rotations correspond to the following permutations of the vertices:

$$(P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4), (P_1P_4)(P_2P_3),$$

and the other four rotations correspond to the permutations:

$$(P_1P_2P_3), (P_1P_2P_4), (P_1P_3P_4), (P_2P_3P_4).$$

It means that the group of rotations of the tetrahedron is isomorphic to the alternating group of degree 4, A_4, of order 12. Now we have to give the linear transformation corresponding to one of $(P_1P_2P_3), (P_1P_2P_4), (P_1P_3P_4), (P_2P_3P_4)$, for example $(P_1P_2P_3)$, since composing $(P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4), (P_1P_4)(P_2P_3)$ with $(P_1P_2P_3)$ we get the other three. But $(P_1P_2P_3)$ can be calculated analytically since it has order 3 and transforms the rotations $(P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4)$, $(P_1P_4)(P_2P_3)$ cyclically. We have then that $(P_1P_2P_3)$ corresponds to

$$\frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}.$$

Therefore G is generated by $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $C = \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$: $G = \langle A, B, C\rangle$. The order of G is $|G| = 24$ since A, B, C satisfy the following relations:

$$\begin{cases} A^2 = B^2 = (AB)^2 = C^3 = -I_2 \\ (AC)^3 = (BC)^3 = I_2 \end{cases}.$$

This group is called the *binary tetrahedral group* and is denoted by $\mathrm{BT}_{24}, \mathrm{BT}$ or $\mathbb{T}$.

2.5.1 Binary Tetrahedral group in terms of quaternions

Proceeding as above and using $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto 1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto j, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mapsto i, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \mapsto k$ we get

$$\mathrm{BT}_{24} = \left\langle i, j, \frac{1}{2}(1 + i - j + k) \right\rangle.$$

2.6 Binary Octahedral group: BO_{48}

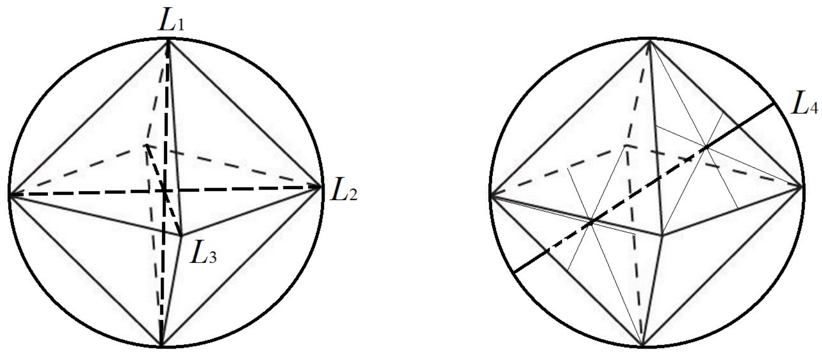
The next type of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ is given by the group of rotations of the octahedron. The group of rotations includes the rotations of the tetrahedron due to the following identification: if $P_1, \ldots, P_4$ represent the pairs of opposite faces, then we have the rotations given by the permutations

$$(P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4), (P_1P_4)(P_2P_3),$$

coming from the rotations around the axes provided by joinning opposite vertices, L_1, L_2, L_3, through π; and the permutations

$$(P_1P_2P_3), (P_1P_2P_4), (P_1P_3P_4), (P_2P_3P_4),$$

coming from the rotations around the axes provided by joinning the middle points of each opposite faces, $L_4, \ldots, L_7$. See the pictures to help the geometrical intuition (the first represents the axes L_1, L_2, L_3 and the second represents only one of $L_4, \ldots, L_7$):



We have one rotation more: the permutation $(P_1P_2P_3P_4)$ (actually the permutations $(P_aP_bP_cP_d)$ with $a, b, c, d \in \{1, 2, 3, 4\}$ pairwise different), which is produced by the rotations around L_1, L_2, L_3 through $\frac{\pi}{2}$. It is enough to get one of these rotations since the others can be see as the composition of one of them and $(P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4), (P_1P_4)(P_2P_3)$. Now the corresponding linear transformation, say D, has order 4 and has to satisfy $D^2 = A$, where A is the generator of BT_{24} described above. Thus $D = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$. Note that $i, -i$ are inverse fourth primitive roots of unity, so $\frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}$ are inverse eighth primitive roots of unity.

Therefore G is generated by BT_{24} and $D = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$, that is G is generated by $D = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $C = \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$: $G = \langle B, C, D\rangle$. The order of G is $|G| = 48$ since B, C, D satisfy the following relations:

$$\begin{cases} D^4 = B^2 = \left(D^2B\right)^2 = C^3 = -I_2 \\ (D^2C)^3 = (BC)^3 = I_2 \\ D^8 = I_2 \end{cases}.$$

This group is called the *binary octahedral group* and is denoted by BO_{48}, BO or $\mathbb{O}$.

Notice that the group of rotations of the octahedral is generated by:

$$(P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4), (P_1P_4)(P_2P_3),$$

$$(P_1P_2P_3), (P_1P_2P_4), (P_1P_3P_4), (P_2P_3P_4),$$

$$(P_1P_2P_3P_4),$$

so it is isomorphic to the symmetric group of order 4, S_4, which has 24 elements.

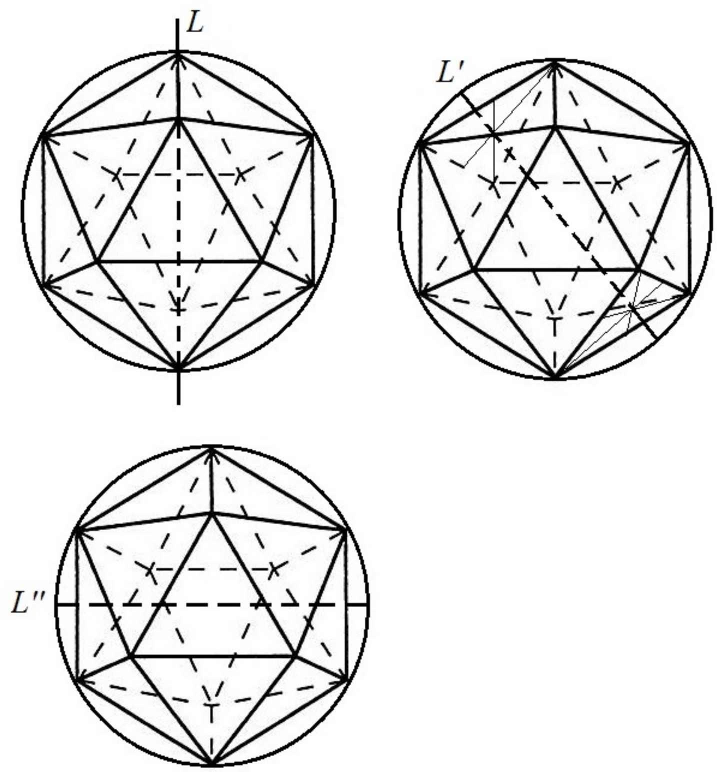
2.6.1 Binary Octahedral group in terms of quaternions

Directly from 3.5.1, replacing $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mapsto i$ by $D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \mapsto \frac{1+i}{\sqrt{2}}$, we have

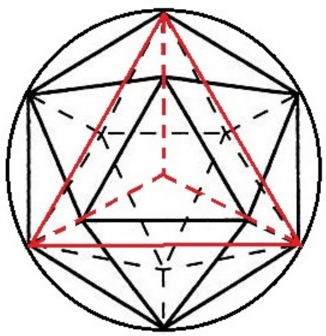
$$\mathrm{BO}_{48} = \left\langle \frac{1+i}{\sqrt{2}}, j, \frac{1}{2}\left(1+i-j+k\right) \right\rangle.$$

2.7 Binary Icosahedral group: BI_{120}

The last type of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ is given by the group of rotations of the icosahedron. This group includes three kinds of axes of rotations: ten axes of rotation $L_1, \ldots, L_{10}$ of index 3 provided by joinning the middle points of opposite faces; six axes of rotation $L_{11}, \ldots, L_{16}$ of index 5 provided by joinning opposite vertices; and fifteen axes of rotation $L_{17}, \ldots, L_{31}$ of index 2 provided by joinning the middle point of each pair of edges:



Now the axes $L_1, \ldots, L_{10}$ can be separeted in five sets of four axes (we count twice every axis) satisfying that in every set the axes are the same as the axes of index 3 in the rotation group of the tetrahedron, that is we get five regular tetrahedrons. Note that no rotation can transform each of the five tetrahedrons into itself, that every axis of $L_{11}, \ldots, L_{31}$ moves one of the tetrahedrons into the other four, thus the group of rotations of the icosahedron can be studied in terms of its five subgroups, corresponding to the groups of rotations of the five tetrahedrons, of order 12, that is the rotation group can be written as a permutation group on 5 letters, and the rotations of index 3 are represented by the cycles on 3 letters.



Recall that the group of rotations of the tetrahedron is isomorphic to the alternating group A_4, so the group of rotations of the icosahedron is isomorphic to the alternating group of degree 5, A_5, of order 60, and we know that A_5 is generated by $\langle (1,2,3,4,5), (1,2,3)\rangle$. Thus the generators of the finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ are the linear transformations corresponding to the permutation (identifying $P_1, \ldots, P_5$ with the five tetrahedrons) $(P_1P_2P_3P_4P_5)$, which are $A = \begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, with ε a fifth primitive root of unity (since now we have five elements); and to the permutation $(P_1P_2P_3)$, C, which has to satisfy the following relations:

$$\begin{cases} C^2 = -I_2 \\ (AC)^3 = (BC)^3 = -I_2 \\ A^5 = I_2 \\ B^4 = I_2 \end{cases},$$

so $C = \frac{1}{\sqrt{5}} \begin{pmatrix} -\varepsilon + \varepsilon^4 & \varepsilon^2 - \varepsilon^3 \\ \varepsilon^2 - \varepsilon^3 & \varepsilon - \varepsilon^4 \end{pmatrix}$, being ε the fifth primitive root of unity given by A.

Therefore G is the group generated by $A = \begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $C = \frac{1}{\sqrt{5}} \begin{pmatrix} -\varepsilon + \varepsilon^4 & \varepsilon^2 - \varepsilon^3 \\ \varepsilon^2 - \varepsilon^3 & \varepsilon - \varepsilon^4 \end{pmatrix}$, with ε a fifth primitive root of unity: $G = \langle A, B, C\rangle$. The order of G is $|G| = 120$ since the relations given above. This group is called the *binary icosahedral group* and is denoted by BI_{120}, BI or $\mathbb{I}$.

2.7.1 Binary Icosahedral group in terms of quaternions

Proceeding as before we have

$$\mathrm{BI}_{120} = \left\langle j, \frac{1}{2}\left(1+i+j+k\right), \frac{1}{2}\left(\frac{1+\sqrt{5}}{2} + \left(\frac{1+\sqrt{5}}{2}\right)^{-1} i + j\right)\right\rangle.$$

2.8 Jordan's Classification

To finish the first section we include the construction scheme used by C. Jordan to list the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$, given in [Jordan, 1870], because in this book there are many results which allow Miller, Blichfeldt and Dickson to list the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(3,\mathbb{C})$, such as important results on permutation groups, the Jordan's theorem on finite linear groups or the mentioned process:

1. If two different Abelian groups in two variables have an element g in common, which is neither I_2 nor $-I_2$, then the elements of the two groups are mutually commutative, so they form a single Abelian group.

2. Let $G \subset \mathrm{SL}(2,\mathbb{C})$ be a finite group and let $K \subset G$ be an Abelian subgroup of G, given in canonical form (in suitable coordinates). If $g \in G$ satisfies that $gKg^{-1} = K$ and g is not commutative with all the elements of K, then g has the form $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$, that is g interchanges the variables of K.

3. Let $G \subset \mathrm{SL}(2,\mathbb{C})$ be a finite group of order $|G|$, by the first step we can separate the elements of G into distinct Abelian subgroups, say $K_1, \ldots, K_m$, of orders $|K_1|, \ldots, |K_m|$, such that no element different than $\pm I_2$ occurs in two distinct groups. Thus we have $|G| = 1 + |K_1| - 1 + \ldots + |K_m| - 1$.

4. The different subgroups $K_1, \ldots, K_m$ can be distributed into conjugate sets and hence if $K_1, \ldots, K_k$ are in the same set their orders are equal. Now let H' be a subgroup of G such that $K_1 \trianglelefteq H$ with order (by the second step) $2|K_1|$ if $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \in G$ or $|K_1|$ in other case. We get then $|G| = 1 + \frac{|G|}{|K_1|}\left(|K_1| - 1\right) + \ldots + \frac{|G|}{|K_p|}\left(|K_p| - 1\right) + \ldots$, and hence $1 = \frac{1}{|G|} + \sum \frac{|K'|-1}{|K'|} + \sum \frac{|K''|-1}{|K''|}$. Jordan last shows that this diophantine equation have a finite number of solutions and he determines the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ through these solutions.

3 Finite Subgroups of SL(3,$\mathbb{C}$)

In this section we are following the works of Miller, Blichfeldt and Dickson in [MBD, 1916] and Yau and Yu in [YY, 1993]. The aim is clear: we want to list the finite subgroups of $\mathrm{SL}(3,\mathbb{C})$, so we need to start by defining $\mathrm{SL}(3,\mathbb{C})$:

Definition 17. The *special linear group* of degree 3 over the field $\mathbb{C}$ is the set of 3×3 matrices with determinant 1 and complex entries, which is a group with respect to the matrix multiplication and matrix inversion, denoted by:

$$\mathrm{SL}(3,\mathbb{C}) = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} : a, \ldots, i \in \mathbb{C}, \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1 \right\}.$$

Remark 18. The special linear group of degree 3 is the group of the linear transformations of $\mathbb{C}^3$ with determinant 1:

$$\mathbb{C}^3 \to \mathbb{C}^3$$

$$(z_1, z_2, z_3) \mapsto (z_1', z_2', z_3') := (az_1 + bz_2 + cz_3, dz_1 + ez_2 + fz_3, gz_1 + hz_2 + iz_3).$$

We aim to list the finite subgroups of $\mathrm{SL}(3,\mathbb{C})$, that is $G \subset \mathrm{SL}(3,\mathbb{C})$ such that the order of G is finite, denoted by$|G| < \infty$.

The classification was done by Miller, Blichfeldt and Dickson in [MBD, 1916] in 1916, where they obtained ten types of finite subgroups of $\mathrm{SL}(3,\mathbb{C})$:

- (A) diagonal Abelian groups;
- (B) groups coming from finite subgroups of $\mathrm{GL}(2,\mathbb{C})$;
- (C) groups generated by (A) and T;
- (D) groups generated by (C) and Q;
- (E) group of order 108 generated by S, T, V;
- (F) group of order 215 generated by (E) and $P = UVU^{-1}$;
- (G) Hessian group of order 648 generated by (E) and U;
- (H) group of order 60 isomorphic to the alternating group of degree five, A_5;
- (I) group of order 168 isomorphic to the permutation group generated by $(1234567), (142)(356), (12)(35)$;
- (L) group G of order 1080 whose quotient G/F is isomorphic to alternating group A_6;

where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, V = \frac{1}{i\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

$$U = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon\omega \end{pmatrix}, P = \frac{1}{i\sqrt{3}} \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix}, Q = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix},$$

$$F = \left\{ I_3, \omega I_3, \omega^2 I_3 \right\},$$

and $abc = -1, \omega = \exp \frac{2\pi i}{3}, \varepsilon = \left(\exp \frac{2\pi i}{9}\right)^2 = \exp \frac{4\pi i}{9}$.

But they missed two types of subgroups which were given by Yau and Yu in [YY, 1993] in 1993, completing the classification (up to conjugacy):

- (J) Group of order 180 generated by (H) and F;
- (K) Group of order 504 generated by (I) and F.

We have then four infinite series of finite subgroups (A) - (D); and eight sporadic finite subgroups (E) - (L). In the case $n = 2$ we had a nice way to list the finite subgroups: list the finite subgroups of SO(3). For $n = 3$ we can not find such nice way, but we can use the following classification of the finite subgroups of $\mathrm{SL}(3, \mathbb{C})$:

1. intransitive groups;
2. transitive groups:
    (a) imprimitive groups;
    (b) primitive groups:
        i. groups with normal intransitive subgroups;
        ii. groups with normal imprimitive subgroups;
        iii. groups with normal primitive subgroups;
        iv. simple groups.

We have to define all these new concepts and show that they give a complete classification:

Definition 19. Let G be a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$, that is G is a linear group in $n \in \mathbb{N}$ variables. Then G is said to be *intransitive* if we can separate these n variables into two or more *sets of intransitivity* after a suitable change of variables, where a set of intransitivity is a set satisfying that their variables are transformed by the elements of G into linear functions of themselves. We say that G is *transitive* if such a division is not possible.

From this definition we get that a finite subgroup of $\mathrm{SL}(3, \mathbb{C})$ is either intransitive or transitive.

Example 20. 1. The symmetric group in three letters, S_3, is transitive since it is isomorphic to the group formed by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^2 \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon^2 \\ \varepsilon & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & \varepsilon \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon \\ \varepsilon^2 & 0 \end{pmatrix},$$

where ε is a primitive cube root of unity. But it contains an intransitive subgroup of order three and three intransitive subgroups of order two.

2. If $n = 4$, an example of G intransitive is when all its elements are of the form

$$A = \begin{pmatrix} a & b & e & f \\ b & a & f & e \\ g & h & c & d \\ h & g & d & c \end{pmatrix},$$

and with the change of variables (assume that x_1, x_2, x_3, x_4 are the old variables)

$$\begin{cases} y_1 = x_1 + x_2 \\ y_2 = x_3 + x_4 \\ z_1 = x_1 - x_2 \\ z_2 = x_3 - x_4 \end{cases},$$

we have

$$A = \begin{pmatrix} i & j & 0 & 0 \\ k & l & 0 & 0 \\ 0 & 0 & m & n \\ 0 & 0 & o & p \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where A_1 and A_2 are the set of intransitivity.

Definition 21. Let G be a transitive group, then G is said to be *imprimitive* if we can separate its n variables into two or more *sets of imprimitivity* after a suitable change of variables, where a set of imprimitivity is a set satisfying that their variables are transformed by the elements of G into linear functions of either themselves or the variables of another set. We say that G is *primitive* if such a division is not possible.

From this definition we get that a transitive subgroup of $\mathrm{SL}(3, \mathbb{C})$ is either imprivitive or primitive.

Example 22. 1. The symmetric group in three letters, S_3, is imprivitive.

2. Following the Example 20.2, an example of G imprimitive is when G also contains elements of the form

$$B = \begin{pmatrix} a & b & e & f \\ -b & -a & -f & -e \\ g & h & c & d \\ -h & -g & -d & -c \end{pmatrix},$$

and with the change of variables (assume that x_1, x_2, x_3, x_4 are the old variables)

$$\begin{cases} y_1 = x_1 + x_2 \\ y_2 = x_3 + x_4 \\ z_1 = x_1 - x_2 \\ z_2 = x_3 - x_4 \end{cases},$$

we have

$$B = \begin{pmatrix} 0 & 0 & a-b & e-f \\ 0 & 0 & g-h & c-d \\ a+b & e+f & 0 & 0 \\ g+h & c+d & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix},$$

where B_1 and B_2 are the sets of imprimitivity.

Now we have the following weak classification:

1. intransitive groups;
2. transitive groups:
   (a) imprimitive groups;
   (b) primitive groups:

We are going to list the types of the primitive groups in terms of their normal subgroups:

Definition 23. A *normal subgroup* N of a group G is a subgroup $N \subset G$ such that $Ng = gN$ for every element $g \in G$, denoted by $N \trianglelefteq G$.

A group G is *simple* if it only has 1_G and G as normal subgroups, where 1_G denotes the identity element of G.

Therefore we can separate the primitive groups into four types: primitive groups with normal intransitive subgroups; primitive groups with normal imprimitive subgroups; primitive groups with normal primitive subgroups; and primitive groups wich are simple. So we get the classification given above:

1. intransitive groups;
2. transitive groups:
   (a) imprimitive groups;
   (b) primitive groups:
      i. groups with normal intransitive subgroups;
      ii. groups with normal imprimitive subgroups;
      iii. groups with normal primitive subgroups;
      iv. simple groups.

But we want to give the generators of the finite subgroups of $\mathrm{SL}(3,\mathbb{C})$, so we are going to study all of these types and try to give their generators. The separation of $\mathrm{SL}(3,\mathbb{C})$ into intransitive/transitive groups, and imprimitive/primitive groups uses outdated terminology: the concepts intransitive, transitive, imprimitive and primitive are strange for us, but we know by Representation Theory something near: the reducible and irreducible groups. For example, Gomi, Nakamura and Shinoda give in [GNS, 2002] a review of the classification by using a review of these strange concepts. The definitions 25 and 26 are the review definitions given in [GNS, 2002] (this part is to help the lector to understand the concepts intransitive, transitive, imprimitive and primitive, but I prefer to follow, in 3.1−3.6, the construcions given in the classical works [MBD, 1916] and [YY, 1993] because of their importance):

Definition 24. Let G be a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$, that is G is a linear group in $n \in \mathbb{N}$ variables, say $x_1, \ldots, x_n$. Then G is said to be *reducible* if m of these variables are transformed by the elements of G into linear functions of themselves after a suitable change of variables, say $x'_1, \ldots, x'_m$. Then the m variables $x'_1, \ldots, x'_m$ form a *reduced set* of G. We say that G is *irreducible* if such change of variables is not possible. In the current terminology this is the same as $\mathbb{C}^n$ being a reducible G-module.

Definition 25. A subgroup G of $\mathrm{GL}(3, \mathbb{C})$ is called *intransitive* (resp. *transitive*) if $\mathbb{C}^3$ is a completely reducible G-module (resp. $\mathbb{C}^3$ is an irreducible G-module) in the current terminology.

Definition 26. A subgroup G of $\mathrm{GL}(3, \mathbb{C})$ is called *imprimitive* if $\mathbb{C}^3$ is an irreducible G-module and if there is a direct sum decomposition $\mathbb{C}^3 = \oplus_{k=1}^{3} V_k$ such that for any i there is j such that $\sigma(V_i) \subset V_j$. And G is called *primitive* if $\mathbb{C}^3$ is an irreducible G-module and if there is no nontrivial such decomposition of $\mathbb{C}^3$.

3.1 Intransitive groups

The first case is when $G \subset \mathrm{SL}(3, \mathbb{C})$ is an intransitive group. There are two types: the three variables can be separated in two or three sets of intransitivity: let g_1, g_2 be elements of two intransitive groups of $\mathrm{SL}(3, \mathbb{C})$ such that g_1 can be separated in three sets of intransitivity and g_2 can be separated in two sets of intransitivity. Then g_1, g_2 are of the form (up to conjugacy):

$$g_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, g_2 = \begin{pmatrix} e & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix},$$

satisfying the conditions to be an element of a finite subgroup of $\mathrm{SL}(3, \mathbb{C})$, that is they have determinant unity. This gives us two types of finite subgroups of $\mathrm{SL}(3, \mathbb{C})$:

- (A) diagonal Abelian groups;
- (B) groups coming from finite subgroups of $\mathrm{GL}(2, \mathbb{C})$.

3.1.1 (A) Diagonal Abelian groups

A diagonal Abelian subgroup of $\mathrm{SL}(3, \mathbb{C})$ is a finite subgroup G such that every element $g \in G$ has the form

$$g = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix},$$

with $abc = 1$. Clearly (A) is an infinite serie of finite subgroups of $\mathrm{SL}(3,\mathbb{C})$.

3.1.2 (B) Groups coming from finite subgroups of $\mathrm{GL}(2,\mathbb{C})$

When the variables of an intransitive group G can be separated into two sets of intransitivity we have the type mentioned in the Introduction: we get finite subgroups of $\mathrm{SL}(3,\mathbb{C})$ coming from finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ via the group monomorphism:

$$\mathrm{GL}(2,\mathbb{C}) \hookrightarrow \mathrm{SL}(3,\mathbb{C})$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{\det(g)} & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} \frac{1}{ad-bc} & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}.$$

The finite subgroups of $\mathrm{GL}(2,\mathbb{C})$ were described by Behnke and Riemenschneider in [BR, 1995] in an easy way: they come from a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ and a finite cyclic extension (one can also see [DuVal, 1964], [Prill, 1967] or [Riemenschneider, 1977] for the proof). We list their work omitting the case when we have a cyclic group $C_{n,q} = \left\langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^q \end{pmatrix} : 0 < q < n, (n,q) = 1 \right\rangle$, with ε being a n-primitive root of unity and $(n,q) = 1$ denoting the greatest common divisor, because it is included in (A) diagonal Abelian groups. Note that nowadays $C_{n,q}$ is denoted by $\frac{1}{n}(1,q)$ (actually by $\frac{1}{r}(1,a)$), but I prefer to use the notation given in [YY, 1993] to be coherent with the notation used in the rest of the paper. Recall the generators of the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$:

$$\mathrm{BD}_{4n} = \left\langle \begin{pmatrix} \varepsilon_{2n} & 0 \\ 0 & \varepsilon_{2n}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle,$$

$$\mathrm{BT}_{24} = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix} \right\rangle,$$

$$\mathrm{BO}_{48} = \left\langle \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix} \right\rangle,$$

$$\mathrm{BI}_{120} = \left\langle \begin{pmatrix} \varepsilon_5^3 & 0 \\ 0 & \varepsilon_5^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} -\varepsilon_5 + \varepsilon_5^4 & \varepsilon_5^2 - \varepsilon_5^3 \\ \varepsilon_5^2 - \varepsilon_5^3 & \varepsilon_5 - \varepsilon_5^4 \end{pmatrix} \right\rangle,$$

where ε_k denotes the kth primitive root of unity. Then we have four cases:

1. The dihedral groups

$$\mathrm{D}_{n,q} = \begin{cases} \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_{2q} & 0 \\ 0 & 0 & \varepsilon_{2q}^{-1} \end{pmatrix}, \begin{pmatrix} \varepsilon_{4m}^{-2} & 0 & 0 \\ 0 & 0 & \varepsilon_{4m} \\ 0 & -\varepsilon_{4m} & 0 \end{pmatrix} \right\rangle, m := n - q \equiv 0(2) \\ \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_{2q} & 0 \\ 0 & 0 & \varepsilon_{2q}^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{2m}^{-2} & 0 & 0 \\ 0 & \varepsilon_{2m} & 0 \\ 0 & 0 & \varepsilon_{2m} \end{pmatrix} \right\rangle, m \equiv 1(2) \end{cases},$$

where $1 < q < n, (n,q) = 1$, hence $(n-q,q) = 1$ and $|\mathrm{D}_{n,q}| = 4\,(n-q)\,q$.

2. The tetrahedral groups

$$\mathrm{T}_m = \begin{cases} \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1+i & -1+i \\ 0 & 1+i & 1-i \end{pmatrix}, \begin{pmatrix} \varepsilon_{2m}^{-2} & 0 & 0 \\ 0 & \varepsilon_{2m} & 0 \\ 0 & 0 & \varepsilon_{2m} \end{pmatrix} \right\rangle, m \equiv 1,5(6) \\ \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1+i & -1+i \\ 0 & 1+i & 1-i \end{pmatrix} \circ \begin{pmatrix} \varepsilon_{6m}^{-2} & 0 & 0 \\ 0 & \varepsilon_{6m} & 0 \\ 0 & 0 & \varepsilon_{6m} \end{pmatrix} \right\rangle, m \equiv 3(6) \end{cases},$$

hence $|\mathrm{T}_m| = 24m$.

3. The octahedral groups

$$\mathrm{O}_m = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_8 & 0 \\ 0 & 0 & \varepsilon_8^7 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1+i & -1+i \\ 0 & 1+i & 1-i \end{pmatrix}, \begin{pmatrix} \varepsilon_{2m}^{-2} & 0 & 0 \\ 0 & \varepsilon_{2m} & 0 \\ 0 & 0 & \varepsilon_{2m} \end{pmatrix} \right\rangle,$$

where $(m,6) = 1$ and hence $|\mathrm{O}_m| = 48m$.

4. The icosahedral groups

$$\mathrm{I}_m = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_5^3 & 0 \\ 0 & 0 & \varepsilon_5^2 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} \sqrt{5} & 0 & 0 \\ 0 & -\varepsilon_5 + \varepsilon_5^4 & \varepsilon_5^2 - \varepsilon_5^3 \\ 0 & \varepsilon_5^2 - \varepsilon_5^3 & \varepsilon_5 - \varepsilon_5^4 \end{pmatrix}, \begin{pmatrix} \varepsilon_{2m}^{-2} & 0 & 0 \\ 0 & \varepsilon_{2m} & 0 \\ 0 & 0 & \varepsilon_{2m} \end{pmatrix} \right\rangle,$$

where $(m,30) = 1$ and hence $|\mathrm{I}_m| = 120m$.

Clearly (B) is an infinite serie (actually four infinite series) of finite subgroups of $\mathrm{SL}(3,\mathbb{C})$.

3.2 Imprimitive groups

The second case is when $G \subset \mathrm{SL}(3,\mathbb{C})$ is an imprimitive group. There are two types: the three variables can be separated in two or three sets of imprimitivity, so one thinks in elements of the form

$$g_1 = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}, g_2 = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix},$$

with determinant unity. But in this case we need the following theorem, given in [MBD, 1916, p. 229-230], to characterize all the imprimitive groups:

Theorem 27. *Let G be an imprimitive linear group in n variables, then we can separate its n variables into k sets of imprimitivity, say $Y_1, \dots, Y_k$, of m variables each, that is $n = km$, permuted according to a transitive permutation group K on k letters, isomorphic with G. The subgroup of G which corresponds to the subgroup of K leaving one letter unaltered, say Y_1, is primitve as far as the m variables of Y_1 are concerned.*

Proof. Let the n variables of the group G separate into k' sets, say $Y_1, \dots, Y_{k'}$, permuted among themselves according to a permutation group K' on k' letters. Clearly K' is a transitive group as is a permutation group. This is possible since otherwise G would not be a transitive group. Now K' contains $k' - 1$ permutations, say $S_2, \dots, S_{k'}$, which replace Y_1 to $Y_2, Y_3, \dots, Y_{k'}$ respectively. We choose the corresponding $k' - 1$ elements of G, say $g_2, \dots, g_{k'}$, whose determinants do not vanish, so $Y_1, \dots, Y_{k'}$ contain the same number of variables, that is $m' := n/k'$.

In K' we have a subgroup K'_1 whose permutations leave Y_1 inaltered, and this subgroup together with $S_2, \dots, S_{k'}$ generate K'. Thus G is generated by the elementes $g_2, \dots, g_{k'}$ corresponding with $S_2, \dots, S_{k'}$ and the subgroup of G corresponding with K'_1, say G_1, which transforms the variables of the set Y_1 into linear functions of themselves. We want to study how the elements of G transform the m variables of the set Y_1, say $y^1_1, \dots, y^1_m$. These elements form the group G_1 in the m variales of Y_1, and we claim that if G_1 is not primitive, then new variables may be introduced into G such that the number of new sets of imprimitivity is greater than k':

The variables of G_1 can be separated into at least two subsets of intransitivity or imprimitivity, say $Y^1_1, \dots, Y^p_1$. The new variables will be introduced into the sets $Y_2, \dots, Y_{k'}$ such that g_t will replace $Y^1_1, \dots, Y^p_1$ by distinct subsets $Y^1_t, \dots, Y^p_t$, for every $t = 2, \dots, k'$. Hence the variables of G will be separated into pk' subsets and thus we have to prove that any element $g \in G$ will permute the pk' subsets among themselves: g will transform the variables of any subset into linear functions of the variables of one of these pk' subsets. Let g be such that replace Y_a by Y_b, so $g' := g_a g g_b^{-1}$ transforms Y_1 into itself and hence $g' \in G_1$ and then $g = g_a^{-1} g g_b$ transforms any subset of Y_a into some subset of Y_b, so our claim is proved.

Now, using the claim we can change the variables increasing the number of sets of imprimitivity until either the sets contain one variable each, or the group G_1 is primitive. □

Definition 28. Let G be an imprimitive linear group in n variables, we say that G has the *monomial form* (or that G is a *monomial group*) if $m = 1, k = n$, where m, k are as in the previous theorem.

From this definition and the previous theorem we get that the imprimitive groups of $\mathrm{SL}(3, \mathbb{C})$ are all monomial: $n = 3$, so the only possibility to separate the three variables into k sets of imprimitivity of m variables each is that $m = 1, k = n = 3$, and we have two possibilities of sets of intransitivity: either the three sets have variables that are transformed into linear functions of the variables of another set; or one set has a variable which is transformed into linear funcionts of itself and the other two have variables that are transformed into linear functions of the variables of another set. Then we have two types coming from the type (A):

- (C) groups generated by (A) and T;

- (D) groups generated by (C) and Q,

where

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}.$$

Note that we have again two infinite series of finite subgroups of $\mathrm{SL}(3, \mathbb{C})$:

3.2.1 (C) Groups generated by (A) and a permutation

We have groups generated by the groups given in (A) and the permutation of the variables $(x_1x_2x_3)$ (we assume that the variables of the group are x_1, x_2, x_3) as the one given by

$$g_1 = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix},$$

where $abc = 1$. We can replace the generator g_1 by

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

since they are conjugate:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \sqrt[3]{bc^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{bc} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix} \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix} \frac{1}{\sqrt[3]{bc^2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & bc & 0 \\ 0 & 0 & c \end{pmatrix}.$$

3.2.2 (D) Groups generated by (C) and a permutation

We also have groups generated by the groups given in (A), the permutation of the variables $(x_1x_2x_3)$, that is the generator T, and the permutation of the variables (x_2x_3), given by the generator

$$Q = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix},$$

where $abc = -1$. Note that if we have one of the following groups coming from (A)

$$G_1 = \langle I_3, T, Q \rangle, G_2 = \langle \varepsilon_3 I_3, T, Q \rangle,$$

where ε_3 denotes the primitive cube root of unity and satisfying that $a = b = c$, then G_1 and G_2 would be intransitive groups.

3.3 Primitive groups with normal imprimitive subgroups

The rest of cases are when $G \subset \mathrm{SL}(3, \mathbb{C})$ is a primitive group, and we study them in terms of their normal subgroups. We leave the case when we have primitive groups with normal intransitive subgroups for later since they give the two "lost types" and we need the primitive groups which are simple, so chronologically they were the last to be discovered. Then the first case of G being a primitive group to be studied is when G contains a normal imprimitive subgroup, that is a group of type (C) or (D). Before do it, we need to know more about (C) and (D), and some properties about normal subgroups given in [MBD, 1916] and in [Blichfeldt, 1917]:

Definition 29. A homogeneous polynomial f of the variables $x_1, \ldots, x_n$ of a group G is called an *invariant polynomial* of G when

$$f(g(x_1, \ldots, x_n)) = \alpha f(x_1, \ldots, x_n), \alpha \in \mathbb{C} \setminus \{0\},$$

for every element g of G. Then we say that G *leaves* f *invariant.*

Remark 30. Let (x_1, x_2, x_3) be the three variables of a group G of the type (C) or (D), we can see x_1, x_2, x_3 as homogeneous coordinates of the projective plane. Then the triangle whose sides are given by the equations $x_1 = 0, x_2 = 0, x_3 = 0$ is transformed into itself by the action of the elements of G, that is $x_1x_2x_3$ is an invariant polynomial of the groups (C) and (D). Are there other invariant triangles? In that case, they would be of the form

$$(a_1x_1 + a_2x_2 + a_3x_3)(b_1x_1 + b_2x_2 + b_3x_3)(c_1x_1 + c_2x_2 + c_3x_3) = 0.$$

We want the triangles to be invariant, so we study the action of the elements of (C) and (D) and what conditions they impose. In the case of (C), we find that the triangle described above could not be distinct from $x_1x_2x_3 = 0$ unless the group of type (A) is the group generated by the elements

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_3 & 0 \\ 0 & 0 & \varepsilon_3^2 \end{pmatrix}, g_2 = \begin{pmatrix} \varepsilon_3 & 0 & 0 \\ 0 & \varepsilon_3 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} = \varepsilon_3 I_3,$$

where ε_3 is the primitive cube root of unity. Then we have four invariant trinagles for (C):

$$\begin{gathered} x_1x_2x_3 = 0, \\ (x_1 + x_2 + x_3)\left(x_1 + \varepsilon_3x_2 + \varepsilon_3^2x_3\right)\left(x_1 + \varepsilon_3^2x_2 + \varepsilon_3x_3\right) = 0, \\ (x_1 + x_2 + \varepsilon_3x_3)(x_1 + \varepsilon_3x_2 + x_3)\left(x_1 + \varepsilon_3^2x_2 + \varepsilon_3^2x_3\right) = 0, \\ \left(x_1 + x_2 + \varepsilon_3^2x_3\right)(x_1 + \varepsilon_3x_2 + \varepsilon_3x_3)\left(x_1 + \varepsilon_3^2x_2 + x_3\right) = 0. \end{gathered}$$

In the case of (D), the same triangles are invariant if the group is generated by (C) as before, that is generated by g_1, g_2 and T from (C), and Q of the form

$$Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Lemma 31. *Let $G \in \mathrm{GL}(n, \mathbb{C}), n \in \mathbb{N}$ be a linear group with a normal Abelian subgroup N such that the elements of N are not all of the form $\alpha I_n, \alpha \in \mathbb{C}$ (the matrices of the form $\alpha I_n, \alpha \in \mathbb{C}$ are called scalar matrices or scalar). Then G is either intransitive or imprimitive.*

Proof. We can write by conjugacy the normal subgroup N in canonical form (diagonal matrix), and then we can separate the variables into sets such that every element of N sends the variables of every set to a constant factor of themselves. Now, since N is a normal subgroup of G, the elements of G permute these sets of N among themselves: let $g \in G, h \in N$, the variables of a given set of N, say Y_1, are transformed by g into linear functions of the variables of G, forming a new set of variables, say X_1. But N is invariant under G, so ghg^{-1} is an element of N, hence the variables of a given set of N are transformed by ghg^{-1} into a constant factor of themselves, hence $ghg^{-1}(Y_1) = \alpha Y_1, \alpha \neq 0$ and then $gh(Y_1) = \left(ghg^{-1}\right) g(Y_1)$, that is $h(X_1) = \alpha X_1, \alpha \neq 0$. Therefore the linear functions of X_1 contain variables from only one set of N, so $g \in G$ transforms the variables of Y_1 into linear functions of some set of N. Consequently, the sets $Y_1, Y_2, \ldots$ of N are permuted among themselves by the elements of G. □

Theorem 32. *Let $G \in \mathrm{GL}(n, \mathbb{C}), n \in \mathbb{N}$ be a linear group whose order is the power of a prime number. Then G can be written as a monomial group by a suitable change of coordinates $x_1, \ldots, x_n$. Thus the elements of G are of the form:*

$$x_i = a_{ij} x'_j,$$

where x'_j are the old coordinates and $i, j = 1, \ldots, n$

Proof. It follows from the fact (see for example [MBD, 1916, pp. 118-119] or a handbook of Group Theory) that a group G whose order is the power of a prime numbre, say p^k, is either Abelian or it contains a normal Abelian subgroup N whose elements are not separately normal in G. Now if G is Abelian, then we can introduce new variables such that the elements of G will have the canonical form (diagonal matrix); and if G contains a normal Abelian subgroup N whose elements are not separately normal in G, then the elementes of N can be written in canonical form, so using Lemma 31 G is intransive or imprimitive. We only study the imprimitive case because if the theorem is true for transitive groups, crearly it holds if G is intransitive. So assume G is imprimitive, we apply Theorem 27 and hence G is monomial because if we had a subgroup of G, say G_1, such that G_1 is primitive in the m variables of a set, say Y_1, then the order or G_1 is a power of a prime number, and using the shown before, G_1 has to be intransive or imprimitive (using Lemma 31). □

Corollary 33. *Let $G \in \mathrm{GL}(n, \mathbb{C}), n \in \mathbb{N}$ be a linear group whose order is the power of a prime number, say p^k, greater than n. Then G is Abelian.*

Proof. By the previous theorem we can write G in monomial form. Then any element $g \in G$ which does not have the canonical form permutes the variables, say $x_1, \ldots, x_n$, by a permutation on these letters. Let q be the order of the permutation, then the order of g is q or a multiple of q. But the order of g is a power of p since it is an element of a group whose order is the power of a prime number, p^k, so q is a power of p. This is a contradiction since q equals n or a product of numbers all less than n, and no one of the prime factors involved can be p. Thus every element g of G has canonical form and hence G is Abelian. $\square$

Theorem 34. *Let $G \in \mathrm{SL}(n, \mathbb{C}), n \in \mathbb{N}$ be a linear group which is reducible. The G is intransitive, and a reduced set of G is one of the sets of intransitivity of G.*

Proof. G can be written as a reducible group as

$$\begin{pmatrix} G_1 & 0 \\ G_2 & G_3 \end{pmatrix},$$

so we see the variables of G as

$$y_i = a_{i1}y_1' + \ldots + a_{im}y_m',$$

$$y_j = a_{j1}y_1' + \ldots + a_{jn}y_n',$$

where $i = 1, \ldots, m, j = m+1, \ldots, n$, and using Remark 11 we have that every group of $\mathrm{SL}(n, \mathbb{C})$ is conjugate to a group of $\mathrm{SU}(n)$, so it imposes the following condition for the elements of G_2:

$$\sum a_{ij}\overline{a_{ij}} = 0,$$

but $a_{ij}\overline{a_{ij}}$ are real and non-negative, so they vanish and hence the elements of G_2 are all zero, and we can write G as

$$\begin{pmatrix} G_1 & 0 \\ 0 & G_3 \end{pmatrix}.$$

$\square$

Now we can list the types when G contains a normal imprimitive subgroup, that is a group of type (C) or (D). So let G be a group containing a normal subgroup of type (C) or (D). By the Remark 30 we know that the types (C) and (D) leave invariant either one or four triangles, but if the triangle $x_1x_2x_{3=0}$ is the only one, then we see easily that G would also leave invariant that triangle and hence G would not be primitive. Thus we assume that there are four invariant triangles and that the elements of G permute the four triangles among themselves. We denote the triangles by t_1, t_2, t_3, t_4:

$$t_1 : x_1x_2x_3 = 0,$$

$$t_2 : (x_1 + x_2 + x_3)\left(x_1 + \varepsilon_3 x_2 + \varepsilon_3^2 x_3\right)\left(x_1 + \varepsilon_3^2 x_2 + \varepsilon_3 x_3\right) = 0,$$

$$t_3 : (x_1 + x_2 + \varepsilon_3 x_3)(x_1 + \varepsilon_3 x_2 + x_3)\left(x_1 + \varepsilon_3^2 x_2 + \varepsilon_3^2 x_3\right) = 0,$$

$$t_4 : \left(x_1 + x_2 + \varepsilon_3^2 x_3\right)\left(x_1 + \varepsilon_3 x_2 + \varepsilon_3 x_3\right)\left(x_1 + \varepsilon_3^2 x_2 + x_3\right) = 0.$$

Then we get a permutation group K on four letters isomorphic to G since we associate the elements of G with permutations on the letters t_1, t_2, t_3, t_4, where the permutations on the letters indicate the way in which the elements of G permute the four triangles, and clearly the normal subgroup of G, (C) or (D), corresponds to the identity of K. We claim that no one of the four letters can be left unaltered by the permutations of K: in such that case, the corresponding triangle would be an invariant triangle of G, and arguing in the same way as when we have discarded that $x_1x_2x_{3=0}$ is the only invariant triangle, we would have that G is not primitive. Moreover, we see directly that the elements of G can not interchange two triangles and leave the other two unaltered. With these conditions we have the following cases for K (up to conjugacy):

- (E) identity, $(t_1t_2)(t_3t_4)$;
- (F) identity, $(t_1t_2)(t_3t_4), (t_1t_3)(t_2t_4), (t_1t_4)(t_2t_3)$ (actually it is generated by identity, $(t_1t_2)(t_3t_4), (t_1t_4)(t_2t_3)$ since $(t_1t_3)(t_2t_4)$ is the composition of the other two permutations);
- (G) the alternating group on four letters, A_4, generated by $(t_1t_2)(t_3t_4)$, $(t_2t_3t_4)$.

Notice that if we have the group (D) as in the Remark 30, that is (D) generated by

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_3 & 0 \\ 0 & 0 & \varepsilon_3^2 \end{pmatrix}, g_2 = \begin{pmatrix} \varepsilon_3 & 0 & 0 \\ 0 & \varepsilon_3 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} = \varepsilon_3 I_3,$$

where ε_3 is the primitive cube root of unity, and T from (C) and Q of the form

$$Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

then this group (D) contain all the elements which leave invartiant the triangles, and if we have an element $g \in G$, which permutes the triangles, then any element g' of G which permutes the triangles in the same way as g can be written as $g' = hg$, where h is an element coming from (D), so $g'g^{-1} = h$ leaves unaltered each triangle. Now we construct the generators of the groups corresponding with the permutations on the letters given in (E) - (G) using the equations of the four invariant triangles, that is $(t_2t_3t_4), (t_1t_2)(t_3t_4), (t_1t_4)(t_2t_3)$ and we get the generators U, V, UVU^{-1}, respectively:

$$U = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon\omega \end{pmatrix}, V = \frac{1}{i\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, UVU^{-1} = \frac{1}{i\sqrt{3}} \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix},$$

where $\omega = \exp \frac{2\pi i}{3}, \varepsilon = \left(\exp \frac{2\pi i}{9}\right)^2 = \exp \frac{4\pi i}{9}$.

3.3.1 (E) Group of order 108

We have then (E) generated by the identity and the permutation $(t_1t_2)\,(t_3t_4)$, that is generated by

$$V=\frac{1}{i\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{pmatrix},$$

where the identity corresponds to the normal subgroup (C) or (D). We have to determine which is the normal subgroup:

Since all the groups contain the generator V corresponding to the permutation $(t_1t_2)\,(t_3t_4)$, then the three groups contain the element hV, where h is an element of (D). Thus if G contains (D) as the normal subgroup, it also contains V. Else, if G contains (C) as the normal subgroup, but not (D), then either G contains V or hV, where h is an element of (D) but not of (C). If G contains hV, we can write $h=h_1Q$, where h_1 is an element of (C) and

$$Q=\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}.$$

Hence either V or QV is an element of G, but $V^2=(QV)^2=Q$, so V and Q are contained in G in any case, and it is enough to consider when the normal subgroup is (C).

Therefore we have

- (E) group of order 108 generated by S,T,V,

where

$$S=\begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2\end{pmatrix},T=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix},V=\frac{1}{i\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{pmatrix},$$

with $\omega=\exp\frac{2\pi i}{3}$.

3.3.2 (F) Group of order 216

We have (F) generated by the identity and the permutations $(t_1t_2)\,(t_3t_4)\,,(t_1t_4)\,(t_2t_3)\,$, that is generated by

$$V=\frac{1}{i\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{pmatrix},UVU^{-1}=\frac{1}{i\sqrt{3}}\begin{pmatrix}1 & 1 & \omega^2\\ 1 & \omega & \omega\\ \omega & 1 & \omega\end{pmatrix},$$

where the identity corresponds to the normal subgroup (C) or (D). Again we have to determine which is the normal subgroup:

If G contains the generator UVU^{-1} corresponding to the permutation (t_1t_4) (t_2t_3), then the group contain the element $hUVU^{-1}$, where h is an element of (D). Arguing as before, G contains (D) and hence UVU^{-1}. Therefore we have

- (F) group of order 216 generated by S, T, V, UVU^{-1},

where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, V = \frac{1}{i\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

$$UVU^{-1} = \frac{1}{i\sqrt{3}} \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix},$$

with $\omega = \exp \frac{2\pi i}{3}$.

3.3.3 (G) Hessian group of order 648

Finally, we have (G) generated by the identity and A_4, that is generated by

$$U = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon\omega \end{pmatrix}, V = \frac{1}{i\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, UVU^{-1} = \frac{1}{i\sqrt{3}} \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix},$$

where the identity corresponds to the normal subgroup (C) or (D). Again we have to determine which is the normal subgroup, and doing the same as in 4.3.1, 4.3.2 with hU, we get

- (G) group of order 648 generated by S, T, V, U,

where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, V = \frac{1}{i\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

$$U = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon\omega \end{pmatrix},$$

with $\omega = \exp \frac{2\pi i}{3}, \varepsilon = \left(\exp \frac{2\pi i}{9}\right)^2 = \exp \frac{4\pi i}{9}$.

In the literature, this group is called the *Hessian group*: it was introduced by Camille Jordan in [Jordan, 1878, p. 209], who named it for Otto Hesse.

Note that we have the following chain of normal subgroups:

$$(C') \trianglelefteq (D') \trianglelefteq (E) \trianglelefteq (F) \trianglelefteq (G),$$

where (C') and (D') are the groups of type (C) and (D) satisfying the conditions imposed by Remark 30.

3.4 Primitive groups which are simple groups

The primitive subgroups of $\mathrm{SL}(3, \mathbb{C})$ which are simple may be the most important types of finite subgroups of $\mathrm{SL}(3, \mathbb{C})$ since the type (L) and the "two lost types" (J) and (K) come from them, but they are the more tedious to get: we need so many results to restrict their order and get them, which are not difficult to be proved but need a lot of Linear Algebra computations and previous results given in [MBD, 1916], so we sketch or refer the proofs of some of these results, which are given in [Blichfeldt, 1917], [MBD, 1916] and [YY, 1993]:

Theorem 35. *If $p > 7$ is a prime number and $G \subset \mathrm{SL}(3, \mathbb{C})$ is primitive, then p can not divide the order of G.*

Proof. We sketch the proof, for the complete proof we refer to Theorem 13 of [MBD, 1916, pp. 241-246] or [YY, 1993, pp. 20-26].

We want to show that if a group G has order $|G|$ with a prime factor $p > 7$, then G is intransitive or imprimitive, in contradiction with the assumtion. The sketch is divided into three parts:

- it is proved the existence of an equation $f = 0$, where f is a certain sum of roots of unity;
- it is given and used a method for transforming the equation $f = 0$ into a congruence modulo $p, p > 7$;
- it follows that G has a Abelian normal subgroup of order p^k and hence by Lemma 31, G is either intransitive or imprimitive.

$\square$

Notation. We are going to denote the order of a group G by $|G| = \phi\,|G'|$, where if $F = \left\{I_3, \omega I_3, \omega^2 I_3\right\} \subset G$ then $G' := G/F$ and $\phi := |F| = 3$, and $G' := G, \phi := 1$ otherwise.

Theorem 36. *Let $p > 2$ be a prime number and let $G \subset \mathrm{SL}(3, \mathbb{C})$ be a group containing an element g of order $p^2\phi$, then G has a normal subgroup N_p (it is possible that $N_p = G$) such that $g^p \in N_p$. If $p = 2$ then G has a normal subgroup N_p if the order of g is p^3, in which case $g^{p^2} \in N_p$; also if $g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}$, whose order is p^2, in which case $g^2 \in N_p$. Moreover, the order of N_p is a power of p.*

Proof. We refer to Theorem 14 of [MBD, 1916, pp. 246-247]. $\square$

The normal subgroup N_p If the assumptions of the previous theorem are satisfied then G has a normal subgroup N_p whose order is a power of p, so N_p is monomial by Theorem 32 and hence if $N_p = G$ we have that G is either intransitive or imprimitive by Lemma 31, or equivalently that if G is primitive then $N_p \neq G$.

Corollary 37. *Let $G \subset \mathrm{SL}(3,\mathbb{C})$ be a primitive simple group, then it can not contain an element g of order*

$$\begin{cases} p^2, & p > 3 \\ p^2\phi, & p = 3 \\ p^3, & p = 2 \end{cases}.$$

Proof. This Corollary is a direct consequence of the previous theorem and that if G is a primitive simple group then $N_p \neq G$ since G is primitive. □

Theorem 38. *Let $G \subset \mathrm{SL}(3,\mathbb{C})$ be a primitive simple group, then it can not contain an element g of prime order $p > 3$, which has at most two distinct multipliers (the elements of a diagonal matrix are called multipliers).*

Proof. By Theorem 35 we have that $p \not> 7$, so we have to check only when $p = 5, 7$. Let $g = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}, \alpha_1 \neq \alpha_2$, then g leaves invariant the point $x_1 = x_2 = 0$ and the straight line through it, of the form $L = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : ax_1 + bx_2 = 0\}$. This holds (for another point and straight line) if we have an element g' conjugate to g since they have the same eigenvalues, and hence the line joining the two invariant points is invariant for g and g'. Now

- assume $p = 7$. In suitable coordinates, we can write the common invariant line as $y_1 = 0$, so the group generated by g and g' is reducible and by Theorem 34 it is also intransitive, with sets of intransitiviy given by y_1 and (y_2, y_3). But there is no intransitive groups in two variables generated by two transformations of order 7, as g and g', thus g and g' are commutative. g' is an arbitrary conjugate to g, so all the conjugates to g are commutative and then they generate an Abelian group, normal in G, so by Lemma 31 G would be either imprimitive or intransitive, in contradiction with the assumption that G is a primitive simple group.

- assume $p = 5$. If g and g' are commutative we are done. If they are not commutative, then they generate the binary icosahedral group in the variables (y_2, y_3). We saw that it contains the element $-I_2$ and an element of order 3, $\begin{pmatrix} \varepsilon_3 & 0 \\ 0 & \varepsilon_3^2 \end{pmatrix}$, whose product, see as an element in three variables, is $h := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\varepsilon_3 & 0 \\ 0 & 0 & -\varepsilon_3^2 \end{pmatrix}$, but it is impossible to have this element by the following theomem (putting $g_1 = h^2$ and $g_2 = h^3$) :

□

Theorem 39. *Let $G \subset \mathrm{SL}(3,\mathbb{C})$ be a primitive simple group, then it can not contain an element g of order pq, where p and q are different prime numbers, and $g_1 = g^p$has three distinct multipliers and $g_2 = g^q$ has at least two.*

Proof. We refer to Theorem 16 of [MBD, 1916, pp. 248-249]. □

Corollary 40. *Let $G \subset \mathrm{SL}(3, \mathbb{C})$ be a primitive simple group, then it can not contain an element g of order $35, 15\phi, 21\phi, 10$ or 14. Moreover, if $G \subset \mathrm{SL}(3, \mathbb{C})$ is a primitive group, with an element g of order $35, 15\phi, 21\phi, 10$ or 14, then G contains an imprimitive normal subgroup, so G is not simple.*

Proof. This Corollary is a direct consequence of the Theorem 38 and 39. □

Now we need to talk about Sylows subgroups and state the well known Sylows theorems, whose proof is in every handbook of Group Theory:

The Sylow p-subgroups Let p be a prime number, a Sylow p-subgroup, p-Sylow subgroup or Sylow subgroup of a group G is a maximal p-subgroup of G, that is a subgroup of G that is a p-group (the order of any group element is a power of p), and which is not a proper subgroup of other p-subgroup of G.

The Sylows theorems

1. *For any prime factor p with multiplicity n of the order of a finite group G, there exists a Sylow p-subgroup of G, of order p^n.*
2. *Given a finite group G and a prime number p, all Sylow p-subgroups of G are conjugate to each other.*
3. *Let p be a prime factor with multiplicity n of the order of a finite group G, such that the order of G can be written as $p^n m$, where $n > 0$ and p does not divide m. Let n_p be the number of Sylow p-subgroups of G. Then:*
   - *n_p divides m, which is the index of the Sylow p-subgroup in G;*
   - *$n_p \equiv 1(p)$;*
   - *$n_p = |G : N_G(P)|$, where P is any Sylow p-subgroup of G and $N_G(P) := \{g \in G : gP = Pg\}$ denotes the normalizer.*

The following three theorems are consequence of the Sylows theorems and allow us to get the types (H) and (I) of primitive groups which are simple by restricting more the order of G:

Theorem 41. *Let $G \subset \mathrm{SL}(3, \mathbb{C})$ be a linear group, then if it has an element g_1 of order 5 and an element g_2 of order 7, that $|G|$ is divisible by 35, then G has an element of order 35.*

Theorem 42. *Let $G \subset \mathrm{SL}(3, \mathbb{C})$ be a linear group, then*

- *if $|G| = 3^4\phi$, then G contains an element of order $3^2\phi$;*
- *if $|G| = 2^4$, then G contains an element of order 2^3 or the element $g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}$, of order 2^2.*

Theorem 43. *Let $G \subset \mathrm{SL}(3,\mathbb{C})$ be a linear group, then if it contains an element g of order 5 (or 7) and the element* $U = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon\omega \end{pmatrix}$, *where* $\omega = \exp\frac{2\pi i}{3}, \varepsilon = \left(\exp\frac{2\pi i}{9}\right)^2 = \exp\frac{4\pi i}{9}$, *of order 9, then G has an element of order $45 = 9{\cdot}5$ (respectively G has an element of order $63 = 9{\cdot}7$).*

Proof. For these three theorems we refer to Theorem 20 and Corollary 1, 2 and 3 of [MBD, 1916, pp. 267-268]. □

Now let $G \subset \mathrm{SL}(3,\mathbb{C})$ be a primitive simple group of order $|G| = \phi\,|G'|$ and let H be a subgroup of G, then

- if H has order 5^2 (or 7^2) then by Corollary 33, H is Abelian. If we try to construct such a group H, we violate Corollary 37 and Theorem 38: if H is cyclic then it has an element of order 5^2 (or 7^2), in contradiction with Corollary 37; and if H is not cyclic then it has an element of order 5 (or 7), which has at most two distinct multipliers, in contradiction with Theorem 38;
- if $|G|$ is divisible by $35 = 5{\cdot}7$ then by Theorem 41 G contains an element of order 35, in contradiction with Corollary 40;
- if H has order $3^2\phi\,(\phi = 1)$ then H is Abelian by Corollary 33. If H is cyclic, it contradicts Corollary 37; and if H is not cyclic, then it contains eight elements of order 3 and when we try to construct such a group it contains $F = \left\{I_3, \omega I_3, \omega^2 I_3\right\}$ in contradiction with $\phi = 1$;
- if H has order 2^4 then by Theomer 42 it has an element of order 2^3, in contradiction with Corollary 37.

Therefore $|G|$ is divisible by 5 or 7, but not by $5{\cdot}7 = 35$; $|G|$ is divisible by 3; and $|G|$ is divisible by 2^2 or 2^3. Thus $|G|$ is a factor of one of

$$2^3{\cdot}3{\cdot}5 = 120, 2^3{\cdot}3{\cdot}7 = 168,$$

and the simple groups whose orders do not exceed $2^3{\cdot}3{\cdot}7 = 168$ have been listed, so we have two possibilities $|G| = 60, 168$.

3.4.1 (H) Simple group of order 60

The first type of simple group has order 60 and is isomorphic to the alternating group A_5, generated by the permutations

$$(12345)\,, (14)\,(23)\,, (12)\,(34)\,,$$

which correspond with the generators

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_5^4 & 0 \\ 0 & 0 & \varepsilon_5 \end{pmatrix}, H_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, H_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix},$$

where ε_5 is the primitive fifth root of unity and $s = \varepsilon_5^2 + \varepsilon_5^3 = \frac{-1-\sqrt{5}}{2}, t = \varepsilon_5 + \varepsilon_5^4 = \frac{-1+\sqrt{5}}{2}$.

3.4.2 (I) Simple group of order 168

The simple group with order 168 is isomorphic to the permutation group generated by the permutations $(1234567), (142)(356), (12)(35)$, which correspond with the generators

$$I_1 = \begin{pmatrix} \varepsilon_7 & 0 & 0 \\ 0 & \varepsilon_7^2 & 0 \\ 0 & 0 & \varepsilon_7^4 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, I_2 = \frac{1}{i\sqrt{7}} \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix},$$

where ε_7 is the primitive seventh root of unity and $a = \varepsilon_7^4 - \varepsilon_7^3, b = \varepsilon_7^2 - \varepsilon_7^5, c = \varepsilon_7 - \varepsilon_7^6$.

3.5 Primitive groups with normal intransitive subgroups

We want to study the case when $G \subset \mathrm{SL}(3, \mathbb{C})$ is a primitive group with normal intransitive subgroups. Recall that we have two types of intransitive subgroups: type (A) and (B). The two "lost types" are going to be included in the primitive groups with normal intransitive subgroups and we explain the mistake made by Miller, Blichfeldt and Dickson in *Theory and applications of finite groups* [MBD, 1916]: they claimed the following (note that they called a normal subgroup by *invariant subgroup* and that their lemma is our Lemma 31):

> "All such groups are intransitive or imprimitive. This follows from the fact that the type (B) has a single linear invariant x_1, which is therefore also an invariant of a group containing (B) invariantly; and the fact that a group containing (A) invariantly cannot be primitive by the lemma."

Most of this is true, but they forgot to study the case when the elements of (A) are all scalar matrices, that is when we have the group $F = \{I_3, \omega I_3, \omega^2 I_3\}$ of type (A), where ω is the primitive cube root of unity.

Indeed, if G has a normal subgroup of type (A) different than F, then G is intransitive or imprimitive by Lemma 31; and if G has a normal subgroup of type (B), then G is intransitive: let $g \in G$ and let h be a element of its normal subgroup, of type (B), so $ghg^{-1} = h_1$ is an element of its normal subgroup by definition. Now if x_1 denotes the first variable of the group, then $h_1(x_1) = \alpha x_1, \alpha \in \mathbb{C}$ and say $g(x_1) = y$, so

$$h(y) = g^{-1} h_1 g(y) = g(\alpha x_1) = \alpha y,$$

thus $y = 0$ is an invariant straight line of (B), but as Miller, Blichfeldt and Dickson say, $x_1 = 0$ is the only straight line of (B), so $y = \beta x_1, \beta \in \mathbb{C}$, and using the Theorem 34, G would be reducible and hence intransitive.

Now we discuss when the group $G \subset \mathrm{SL}(3,\mathbb{C})$ has the normal subgroup $F = \{I_3, \omega I_3, \omega^2 I_3\}$ of type (A), generated by ωI_3 and with order $|F| = 3$. Clearly the order of G is then $|G| = 3\,|G/F|$, where G/F is the quotient group, which is clearly simple, and allows us to use all the results stated in the case when the normal subgroup is simple. We argue as in the Sylow subgroups, putting $\phi := |F| = 3$, and we get the same restrictions for factors 5, 7 and 2 of $|G/F|$, and using Theorem 42, if G has a subgroup of order $3^4\phi$, then G contains an element of order $3^2\phi$, in contradiction with Corollary 37. Else if G has a subgroup of order $3^3\phi$, then G has a subgroup of order $3^2\phi$, say K, which is Abelian. If K is cyclic then it contradicts Corollary 37, and if it is not cyclic, then Kcontains an element of the form

$$U = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon\omega^2 \end{pmatrix},$$

where $\omega = \exp\frac{2\pi i}{3}, \varepsilon = \left(\exp\frac{2\pi i}{9}\right)^2 = \exp\frac{4\pi i}{9}$, with order 9. But if $|G|$ is divisible by 5 or 7 then $|G/F|$ is divisible by 5 or 7, so G/F, which is simple, and using Theorem 43 it would have an element of order $5{\cdot}9 = 45 = 15{\cdot}3 = 15{\cdot}\phi$ or $7{\cdot}9 = 63 = 21{\cdot}3 = 21{\cdot}\phi$, in contradiction with Corollary 40.

Summing up, we have that $|G/F|$ is a factor of one of

$$2^3{\cdot}3^3 = 216, 2^3{\cdot}3^2{\cdot}5 = 360, 2^3{\cdot}3^2{\cdot}7,$$

and hence four possibilities $|G/F| = 60, 168, 360, 504$.

By the work of Cole about simple groups in [Cole, 1893] we know that we cannot have $G \subset \mathrm{SL}(3,\mathbb{C})$ isomorphic with the simple group of order 504: the simple group of order 504 has an Abelian subgroup of order 8, formed by 7 distinct elements of order 2 and the identity, and it is impossible to write this subgroup in canonical form since G is a group in three variables.

Therefore we have the following three types:

3.5.1 (J) Group of order 180

The first type is when $|G/F| = 60$, that is when G/F is the simple group given in (H): the simple group of order 60 isomorphic to A_5, and hence G is generated by (H) and F:

- (J) group of order 180 generated by (H) and F,

where $F = \{I_3, \omega I_3, \omega^2 I_3\}$ with ω being the primitive cube root of unity.

3.5.2 (K) Group of order 504

The second lost type is when $|G/F| = 168$, that is when G/F is the simple group given in (I): the permutation group generated by $(1234567)\,,(142)\,(356)\,,(12)\,(35)$, and hence G is generated by (I) and F:

- (K) group of order 504 generated by (I) and F,

where $F = \{I_3, \omega I_3, \omega^2 I_3\}$ with ω being the primitive cube root of unity.

3.5.3 (L) Group of order 1080

Paradoxically, this group should be a "lost type" as the other two but it is found by Miller, Blichfeldt and Dickson when they are getting the simple groups, specifically when they are studying the simple group of order 306, which is isomorphic to the alternating group A_6. A_6 is generated by A_5, that is the generators of (I), and the permutation (14) (56), corresponding with the element

$$W = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \lambda_1 & \lambda_1 \\ 2\lambda_2 & s & t \\ 2\lambda_2 & t & s \end{pmatrix},$$

where $\lambda_1 = \frac{1}{4}\left(-1 + i\sqrt{15}\right), \lambda_2 = \overline{\lambda_1}$, and s, t are as in (H).

3.6 Primitive groups with normal primitive subgroups

We start the last case by defining the center, orbit and stabilizer of a group, but actually we have been using these concepts before:

Definition 44. Given a group G, and consider G acting on a set X (note that in our case $G \subset \mathrm{SL}(3, \mathbb{C})$ and $X = \mathbb{C}$), we define

- the *center* of G as the set of the elements of G which commute with all the elements of G. It is denoted by

  $$Z(G) = \{z \in G : zg = gz, g \in G\},$$

  and it is a normal subgroup of G.

- the *orbit* of an element $x \in X$, denoted by O_x, is the set of elements of X such that

  $$O_x = \{g(x) \in X : g \in G\}.$$

- the *stabilizer* of an element $x \in X$, denoted by G_x, is the set the elements of G which fix x:

  $$G_x = \{g \in G : g(x) = x\},$$

  which is a subgroup of G.

Example 45. The center of $\mathrm{SL}(3, \mathbb{C})$ is $F = \left\{I_3, \omega I_3, \omega^2 I_3\right\}$, with ω being the primitive cube root of unity.

Now we want to find primitive subgroups of $\mathrm{SL}(3, \mathbb{C})$ which have normal primitive subgroups, and we can use the Theorem 35 and Theorem 36: we could have a normal subgroup N_p as in the simple case, which is monomial by Theorem 32, but we have determined the primitive groups with a normal subgoup N_p (types (J) - (L)), thus by Theorems 41, 42 and 43 we have to study the groups whose order is factor of

$$2^3{\cdot}3^3\phi, 2^3{\cdot}3^2{\cdot}5\phi, 2^3{\cdot}3^2{\cdot}7\phi,$$

with $\phi = 3$ since the case $\phi = 1$ gives the types (J) - (L). We have to study if our group can contain a group of type (E) - (L):

- the group (L) has order $1080 = 2^3{\cdot}3^2{\cdot}5\phi = 2^3{\cdot}3^3{\cdot}5$, so it has already a maximum order, and if our primitive group had a normal primitive subgroup (J) or (K), then it would contain $Z(\mathrm{SL}(3,\mathbb{C})) = \left\{I_3, \omega I_3, \omega^2 I_3\right\}$ as a normal intransitive subgroup only, and it is reduced to the case when we have primitive groups with normal intransitive subgroups.

- if we had a normal subgroup (E), (F) or (G), then our group, larger than them, would permutes among themselves the four triangles given in Remark 30, but this condition was imposed to get (E), (F) and (G).

- if (H) is a normal subgroup of a primitive group $G \subset \mathrm{SL}(3,\mathbb{C})$, then $|G| = |(H)|\,2^a{\cdot}3^b = 2^{2+a}{\cdot}3^{1+a}{\cdot}5$, with at least one of $a, b > 0$. We are going to show that necessarily $a = b = 0$ and it does not give new groups. (H) has ten subgroups of order 3, say $H_1, \ldots, H_{10}$, so G has to acts on the by conjugation since (H) is a normal subgroup of G. Let $G_i := \left\{g \in G : gH_ig^{-1} = H_i\right\}, i = 1, \ldots, 10$, so G_i acts on H_i by conjugation and $|G/G_i| = |G| / |G_i| = 10$, thus $|G_i| = 2^{1+a}{\cdot}3^{1+b}, i = 1, \ldots, 10$. We choose $H_1 = \{identity, (123), (132)\}$ seeing the elements in A_5 isomorphic to (H). The element $(23)(45) \in G_1$ sends (123) to (132), hence the orbit of G_1 containing (123) is $\{(123), (132)\}$ and the stabilizer of (123), say $G_{1_{(123)}}$ has order $|G_1|/2 = 2^a{\cdot}3^{1+b}$. Assume that $a > 0$, so G contains an element of order 2 commutative with (123) or an element of order 3, and then by Theorem 39 we would have a normal group N_p, thus $a = 0$. To show that $b = 0$ we argue as before but using that (H) has six subgroups of order 5, and again assuming $b > 0$ we would have a normal group N_p by Theorem 39, so $b = 0$.

- if (I) is a normal subgroup of a primitive group $G \subset \mathrm{SL}(3,\mathbb{C})$, then $|G| = |(I)|\,3^b = 2^3{\cdot}3^{1+b}{\cdot}7$, with $b > 0$. We argue as when $(H) \trianglelefteq G$ using that (I) has eight subgroups of order 7, and again by Theorem 39 we would have a normal group N_p, thus $b = 0$.

Therefore we do not get new types of finite subgroups of $\mathrm{SL}(3,\mathbb{C})$ and we have ended the classification. In the last part we show the review classification using the modern terminology presented in the Definition 24, 25 and 26, by enunciating the following results from [GNS, 2002]:

3.7 Review of the classification

Let $G \subset \mathrm{SL}(3,\mathbb{C})$ be a finite subgroup, then

- if $\mathbb{C}^3$ is a reducible G-module, then G is type (A) or (B);

- if $\mathbb{C}^3$ is an irreducible G-module and if there is a direct sum decomposition $\mathbb{C}^3 = \oplus_{k=1}^3 V_k$ such that for any i there is j such that $\sigma(V_i) \subset V_j$, then G is type (C) or (D);

- if $\mathbb{C}^3$ is an irreducible G-module and if there is no nontrivial such decomposition of $\mathbb{C}^3$ and$G$ has a normal subgroup N such that $\mathbb{C}^3$ is an irreducible

N-module and there is a direct sum decomposition $\mathbb{C}^3 = \oplus_{k=1}^3 V_k$ such that for any i there is j such that $\sigma(V_i) \subset V_j$, then G is of type (E), (F) or (G);

- if $\mathbb{C}^3$ is an irreducible G-module and if there is no nontrivial such decomposition of $\mathbb{C}^3$ and$G$ has a normal subgroup N such that $\mathbb{C}^3$ is a reducible N-module, then G is of type (J), (K) or (L);
- if $\mathbb{C}^3$ is an irreducible G-module and if there is no nontrivial such decomposition of $\mathbb{C}^3$ and$G$ is a simple group, then G is of type (H) or (I).

4 Application to Algebraic Geometry

In this section we assume basic definitions and results on Invariant Theory, Representation Theory and Algebraic Geometry which are in the corresponding handbooks (we follow [Dolgachev, 2003], [Kraamer, 2013], [Bartel, 2014], [Reid, 1988], [Reid, 1996], [Reid, 1985] and [Reid, 1997] among others).

Let G be a finite subgroup of $\mathrm{SL}(n, \mathbb{C})$, then G acts on $\mathbb{C}^n$ and we study the quotient variety $\mathbb{C}^n/G$ and its resolutions $f : Y \to \mathbb{C}^n/G$ using G-invariant polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. For $n = 2$, McKay observed that there is a correspondence between the ADE Dynkin diagrams and the McKay graphs of the finite subgroups of $\mathrm{SL}(2, \mathbb{C})$, called the McKay correspondence or McKay's observation. In higher dimension the problem is presented as giving bijections between the irreducible representations of G and the basis of $H^*(Y, \mathbb{Z})$: some important results are the work of González-Sprinberg and Verdier in [G-SV, 1983] using sheaves on Y and the work of Ito and Nakamura in [IN, 1999] using G-Hilbert schemes.

In the first part we get the invariant polynomials and their relations when $n = 2, 3$ and in the second part the works of McKay; González-Sprinberg and Verdier; and Ito and Nakamura are exposed.

4.1 Invariants

We recall the Definition 29: a homogeneous polynomial f of the variables $x_1, \ldots, x_n$ of a group G is called an *invariant polynomial* of G when

$$f(g(x_1, \ldots, x_n)) = \alpha f(x_1, \ldots, x_n), \alpha \in \mathbb{C} \setminus \{0\},$$

for every element g of G. Then we say that G *leaves* f *invariant*. In the literature, these polynomiales are usually called *relative invariants* meanwhile an invariant polynomial is when $f(g(x_1, \ldots, x_n)) = f(x_1, \ldots, x_n)$, which we call *absolute invariant*, following the notation given in [MBD, 1916].

Let $\mathbb{C}[x_1, \ldots, x_n]$ then $\mathbb{C}[x_1, \ldots, x_n]^G = \{f \in \mathbb{C}[x_1, \ldots, x_n] : f \text{ invariant of } G\}$ is finitely generated and contains a minimal set of polynomials $f_1, \ldots, f_k$ which generate $\mathbb{C}[x_1, \ldots, x_n]^G$ as a $\mathbb{C}-$algebra and hence are called the *minimal generators* of $\mathbb{C}[x_1, \ldots, x_n]^G$, and it induces a ring homomorphism

$$\varphi : \mathbb{C}[y_1, \ldots, y_k] \to \mathbb{C}[x_1, \ldots, x_n]^G,$$

where $\varphi(F) := F(f_1, \dots, f_k)$ and $\mathbb{C}[x_1, \dots, x_n]^G \cong \mathbb{C}[y_1, \dots, y_k] / \ker(\varphi)$ since $\ker(\varphi)$ is an ideal of $\mathbb{C}[y_1, \dots, y_k]$. The minimal generators of $\ker(\varphi)$ are called the *relations* of $\mathbb{C}[x_1, \dots, x_n]^G$. These relations define the image of the quotient variety $\mathbb{C}^n/G$ in $\mathbb{C}^k$ as affine algebriac variety. Note that for our purpose of get the minimal generators of $\ker(\varphi)$ it is indifferent if in the definition of $\mathbb{C}[x_1, \dots, x_n]^G$ we are using relative or absolute invariants since the minimal generators of $\ker(\varphi)$ are the same modulo suitable coordinates (this will be clearer when we give them), so for $n = 2$ the method gives the relative invariants and for $n = 3$ the absolute ones, but in both cases we will say invariants.

In some cases, one can calculate the invariant polynomials directly, for example the invariant polynomials of the cyclic groups of $\mathrm{SL}(2, \mathbb{C})$ are generated by $f_1(x_1, x_2) = x_1, f_2(x_1, x_2) = x_2$. These are the minimal generators, obviously we are going to omitte all the invartiant polynomials, of the form $\sum a_i f_1^{n_i} f_2^{m_i}$. But in other cases it can be really complicated: about get the invariant polynomials of the type (L) of $\mathrm{SL}(3, \mathbb{C})$ and their relations, Yau and Yu say in [YY, 1993]:

> "The most difficult one is type (L), its invariants take a few pages long to write down. We had a hard time to find their relation. It took us more than 3 months even with the aid of computer. However the final relation is quite simple."

In [Noether, 1916] Noether proves that if $G \subset \mathrm{GL}(n, \mathbb{C})$ is finite, then the generators of $\mathbb{C}[x_1, \dots, x_n]^G$ has not more than $\binom{|G| + n}{n}$ invariants, of degree not exceeding the order of G, $|G|$, and we can get them by taking the average over G of all monomials $x_1^{b_1} x_2^{b_2} \cdot \dots \cdot x_n^{b_n}$, with $\sum b_i \leq |G|$. For the cases $n = 2, 3$ we have the following results of Miller, Blichfeldt and Dickson in [MBD, 1916] and Yau and Yu in [YY, 1993] (starting from Noether's result and using the computation programs Caley and Reduce) respectively:

- $n = 2$: they write an invariant into linear factors, say $f = f_1 \cdot \dots \cdot f_k$ and reduce the problem to determine the different conjugate sets in G of Abelian subgroups which are not subgroups in larger Abelian subgroups: let G_i be the subgroup of G such that G_i levaes f_i invariant, then G_i forms an Abelian group and the G_i are conjugate. The invariant f will be made up of one factor for each of the subgroups of the set if there is no element in G which transforms one of the linear invariants of G_1 into the other, that is if G_1 can be writen as $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and there is no element of the form $\begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}$ in G, then we get two invariants $f_1(x_1, x_2) = x_1$ and $f_1(x_1, x_2) = x_2$; in other case we get $f_1(x_1, x_2) = x_1 x_2$.
- $n = 3$: actually this is a method to find minimial generators of the ring of invariant polynomials as well as their relations when $G \subset \mathrm{GL}(n, \mathbb{C})$: they prove the Theorem 46 and it follows that $\mathbb{C}[x_1, \dots, x_n]^G$ can be written as

$\mathbb{C}[f_1,\dots,f_n] \oplus \mathbb{C}[f_1,\dots,f_n]\,h_1 \oplus \dots \oplus \mathbb{C}[f_1,\dots,f_n]\,h_k$, called the *basic decomposition of* $\mathbb{C}[x_1,\dots,x_n]^G$, where $f_1,\dots,f_n,h_1,\dots,h_k$ are invariants of G, called *basic invariants of* $\mathbb{C}[x_1,\dots,x_n]^G$, and $f_1,\dots,f_n$ are algebraically independent. Then any $p \in \mathbb{C}[x_1,\dots,x_n]^G$ can be written as $B(p) := p_0(f_1,\dots,f_n) + p_1(f_1,\dots,f_n)\,h_1 + \dots + p_k(f_1,\dots,f_n)\,h_k$, called the *basic form of* p, where $p_i(f_1,\dots,f_n)$ are polynomials in $f_1,\dots,f_n$. Let $\{f_1,\dots,f_n,h_1,\dots,h_k\}$ be a set of basic invariants of $\mathbb{C}[x_1,\dots,x_n]^G$ such that $\{f_1,\dots,f_n,h_1,\dots,h_t\}$ is a set of minimal generators of $\mathbb{C}[x_1,\dots,x_n]^G$, $t \le k$, and let $P := \{h_ih_j : 1 \le i \le t, i \le j \le k\} \setminus \{h_1,\dots,h_k\}$, then $\{p - B(p) : p \in P\}$ generates $\ker(\varphi)$ for h_i being a polynomial in $h_1,\dots,h_t$, $t+1 \le i \le k$, and if $Q := \{p \in P : p$ has no a factor $p' \in P \setminus \{p\}\}$, then $\{p - B(p) : p \in Q\}$ are the relations of $\mathbb{C}[x_1,\dots,x_n]^G$.

Theorem 46. *Let* H *be a subgroup of* $G \subset \mathrm{GL}(n,\mathbb{C})$ *and let* $\{f_1,\dots,f_k\}$ *be a set of minimal generators of* $\mathbb{C}[x_1,\dots,x_n]^H$*. Let* $G = Ha_1 \bigcup Ha_2 \bigcup \dots \bigcup Ha_r$*, where* $r = |G|/|H|$ *and* $a_i \in G, i = 1,\dots,r$*. Then the set of generators of* $\mathbb{C}[x_1,\dots,x_n]^G$ *is given by* $\left(f_1^{d_1} \cdot \dots \cdot f_k^{d_k}\right) a_1 + \dots + \left(f_1^{d_1} \cdot \dots \cdot f_k^{d_k}\right) a_r$*, where* $\sum d_i \deg f_i \le |G|$.

Obviously the calculation of the minimal generator of $\mathbb{C}[x_1,\dots,x_n]^G, n = 2,3$ is not included since it is beyond the scope of this paper. In the mentioned book, Yau and Yu extend a known result when $G \subset \mathrm{SL}(2,\mathbb{C})$ (if G is a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ then $\mathbb{C}^2/G$ has only isolated singularity and it must be a rational double point) to $G \subset \mathrm{SL}(3,\mathbb{C})$: $\mathbb{C}^3/G$ has isolated singularities if and only if G is Abelian and 1 is not an eigenvalue of the nontrivial elements of G.

4.1.1 Relative invariants of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$

Assuming that (x,y) are the variables of $G \subset \mathrm{SL}(2,\mathbb{C})$ finite, then we have the following minimal generators of the relative invariants:

- $C_n, n \in \mathbb{N}$:
$$f_1(x,y) = x,$$
$$f_2(x,y) = y.$$
If we fix $n_0 \in \mathbb{N}$, then we have three absolute invariants which are the minimal generators:
$$F_1(x,y) = x^{n_0},$$
$$F_2(x,y) = y^{n_0},$$
$$F_3(x,y) = xy,$$
which satisfy the relation
$$F_1F_2 - F_3^{n_0} = 0.$$

Thus $\mathbb{C}[x,y]^G = \mathbb{C}[F_1, F_2, F_3] \cong \mathbb{C}[u,v,w] / \ker\varphi$, where $\ker\varphi$ is the ideal $(uv - w^{n_0})$, which in suitable coordinates is

$$\mathbb{C}[x,y]^G \cong \mathbb{C}[x,y,z] / \left(x^2 + y^2 + z^{n_0}\right);$$

- $\mathrm{BD}_{4n}, n \in \mathbb{N}$:

$$f_3(x,y) = xy.$$

Note that in the cases C_n and BD_{4n} we are giving relative invariants valid for all the cyclic and binary dihedral groups. If we fix $n_0 \in \mathbb{N}$, for a particular BD_{4n_0} we have two relative invariants more:

$$f_{12}(x,y) = x^{2n_0} + y^{2n_0},$$

$$f_{13}(x,y) = x^{2n_0} - y^{2n_0},$$

which satisfy the relation

$$f_3^{2n_0} - f_{12}^2 + f_{13}^2 = 0.$$

Thus $\mathbb{C}[x,y]^G = \mathbb{C}[f_3, f_{12}, f_{13}] \cong \mathbb{C}[u,v,w] / \ker\varphi$, where $\ker\varphi$ is the ideal $\left(u^{2n} - v^3 + w^3\right)$, which in suitable coordinates is (if we choose the invariants $F_1 = f_3 f_{13}, F_2 = f_{12}, F_3 = f_3^2$, then we have the relation $F_1^2 - F_3 F_2^2 + 4F_3^{n_0+1} = 0$)

$$\mathbb{C}[x,y]^G \cong \mathbb{C}[x,y,z] / \left(x^2 + zy^2 + z^{n_0+1}\right);$$

- BT_{24} :

$$f_4(x,y) = xy(x^4 - y^4),$$

$$f_5(x,y) = x^4 + 2i\sqrt{3}x^2y^2 + y^4,$$

$$f_6(x,y) = x^4 - 2i\sqrt{3}x^2y^2 + y^4,$$

which satisfy the relation

$$12i\sqrt{3}f_4^2 - f_5^3 + f_6^3 = 0.$$

Thus $\mathbb{C}[x,y]^G = \mathbb{C}[f_4, f_5, f_6] \cong \mathbb{C}[u,v,w] / \ker\varphi$, where $\ker\varphi$ is the ideal $\left(12i\sqrt{3}u^2 - v^3 + w^3\right)$, which in suitable coordinates is (if we choose the invariants $F_1 = f_4, F_2 = f_5 f_6, F_3 = f_5^3 + f_6^3$, then we have the relation $F_3^2 - F_1^4 - 4F_2^3 = 0$)

$$\mathbb{C}[x,y]^G \cong \mathbb{C}[x,y,z] / \left(z^2 + x^4 + y^3\right);$$

- BO_{48} :

$$f_4(x,y),$$

$$f_7(x,y) = f_5(x,y)\, f_6(x,y) = x^8 + 14x^4y^4 + y^8,$$

$$f_8\left(x,y\right)=x^{12}-33x^8y^4-33x^4y^8+y^{12},$$

which satisfy the relation

$$108f_4^{12}-f_7^3+f_8^2=0.$$

Thus $\mathbb{C}\left[x,y\right]^G=\mathbb{C}\left[f_4,f_7,f_8\right]\cong\mathbb{C}\left[u,v,w\right]/\ker\varphi$, where $\ker\varphi$ is the ideal $\left(108u^{12}-v^3+w^2\right)$, which in suitable coordinates is (if we choose the invariants $F_1=f_4^2, F_2=f_5f_6=f_7, F_3=f_4f_8$, then we have the relation $F_3^2-F_1\left(F_2^3-108F_1^2\right)=0$)

$$\mathbb{C}\left[x,y\right]^G\cong\mathbb{C}\left[x,y,z\right]/\left(z^2+xy^3+x^3\right);$$

- BI_{120} :

$$f_9\left(x,y\right)=x^{30}+y^{30}+522\left(x^{25}y^5-x^5y^{25}\right)-100005\left(x^{20}y^{10}+x^{10}y^{20}\right),$$
$$f_{10}\left(x,y\right)=-x^{20}-y^{20}+228\left(x^{15}y^5-x^5y^{15}\right)-494x^{10}y^{10},$$
$$f_{11}\left(x,y\right)=xy\left(x^{10}+11x^5y^5-y^{10}\right),$$

which satisfy the relation

$$f_9^2+f_{10}^3-1728f_{11}^5=0.$$

Thus $\mathbb{C}\left[x,y\right]^G=\mathbb{C}\left[f_9,f_{10},f_{11}\right]\cong\mathbb{C}\left[u,v,w\right]/\ker\varphi$, where $\ker\varphi$ is the ideal $\left(u^2+v^3-1728w^5\right)$, which in suitable coordinates is

$$\mathbb{C}\left[x,y\right]^G\cong\mathbb{C}\left[x,y,z\right]/\left(x^2+y^3+z^5\right).$$

Note that for all the types of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ the number of minimal generators of $\mathbb{C}\left[x,y\right]^G$ is three and hence the quotient varieties of these groups are hypersufaces in $\mathbb{C}^3$.

4.1.2 Absolute invariants of finite subgroups of $\mathrm{SL}(3,\mathbb{C})$

Assuming that (x,y,z) are the variables of $G\subset\mathrm{SL}(3,\mathbb{C})$ finite, then we have the following minimal generators of the absolute invariants (recall that we have four infinite series of finite subgroups, (A) - (D), and eight sporadic types (E) - (L). For (E) - (L) we can give the minimal generators and the relations, but for (A) - (D) it is not possible to give the minimal generators implicitly and the relations unless we have a particular case of them. The minimal generators and an example of (A) are shown, and the cases (B) - (D) are omitted, see [YY, 1993]) :

- (A): let $G\subset\mathrm{SL}(3,\mathbb{C}$ be a diagonal Abelian group, then it can be writen as product of cyclic groups, say $G=G_1\times\ldots\times G_m$, where every cyclic group is generated by an element of the form

$$\begin{pmatrix}\varepsilon_{|G_i|}^{a_i} & 0 & 0\\ 0 & \varepsilon_{|G_i|}^{b_i} & 0\\ 0 & 0 & \varepsilon_{|G_i|}^{c_i}\end{pmatrix},$$

where $\varepsilon_{|G_i|}$ is a primitive $|G_i|$th root of unity and $a_i+b_i+c_i \equiv 0\,(|G_i|)$. We have then that the invariant are monomials of the form $x^r y^s z^t$ satisfying $a_i r + b_i s + c_i t \equiv 0\,(|G_i|)$ and the minimal generators are contained in the basic invariants, which are generated by $x^{n_1}, y^{n_2}, z^{n_3}$, where n_1, n_2, n_3 are the smallest positive integers such that $x^{n_1}, y^{n_2}, z^{n_3}$ are invariantes, and by every solution of $a_i r + b_i s + c_i t \equiv 0\,(|G_i|)$ with $r+s+t \leq |G_i|$.

Example 47. Let G be a diagonal Abelian subgroup of $\mathrm{SL}(3,\mathbb{C})$ generated by

$$\begin{pmatrix} \varepsilon_4^2 & 0 & 0 \\ 0 & \varepsilon_4 & 0 \\ 0 & 0 & \varepsilon_4 \end{pmatrix}, \begin{pmatrix} \varepsilon_6 & 0 & 0 \\ 0 & \varepsilon_6^3 & 0 \\ 0 & 0 & \varepsilon_6^2 \end{pmatrix}.$$

We have the minimal generators

$$f_1(x,y,z) = x^6,$$
$$f_2(x,y,z) = y^4,$$
$$f_3(x,y,z) = z^{12},$$
$$f_4(x,y,z) = xyz,$$
$$f_5(x,y,z) = y^2 z^6,$$
$$f_6(x,y,z) = x^4 z^4,$$
$$f_7(x,y,z) = x^2 z^8,$$

which satisfy the relations

$$f_4^4 - f_2 f_6 = 0,$$
$$f_4^2 f_5 - f_2 f_7 = 0,$$
$$f_4^2 f_6 - f_1 f_5 = 0,$$
$$f_5^2 - f_2 f_3 = 0,$$
$$f_5 f_6 - f_4^2 f_7 = 0,$$
$$f_5 f_7 - f_3 f_4^2 = 0,$$
$$f_6^2 - f_1 f_7 = 0,$$
$$f_6 f_7 - f_1 f_3 = 0,$$
$$f_7^2 - f_3 f_6 = 0.$$

Thus $\mathbb{C}[x,y,z]^G = \mathbb{C}[f_1,\ldots,f_7] \cong \mathbb{C}[q,r,s,t,u,v,w] / \ker\varphi$, where $\ker\varphi$ is the ideal generated by the relations given above in the variables q,r,s,t,u,v,w;

- (E):

$$f_1\left(x,y,z\right)=18x^2y^2z^2+6xyz\left(x^3+y^3+z^3\right)-\left(x^3+y^3+z^3\right)^2,$$

$$f_2(x,y,z)=\left(x^3+y^3+z^3\right)^2-12\left(x^3y^3+y^3z^3+x^3z^3\right),$$

$$f_3\left(x,y,z\right)=27x^4y^4z^4-xyz\left(x^3+y^3+z^3\right)^3,$$

$$f_4\left(x,y,z\right)=18x^3y^3z^3\left(x^3+y^3+z^3\right)-3x^2y^2z^2\left(x^3+y^3+z^3\right)^2$$
$$-xyz\left(x^3+y^3+z^3\right)^3,$$

$$f_5\left(x,y,z\right)=\left(x^3-y^3\right)\left(y^3-z^3\right)\left(z^3-x^3\right),$$

which satisfy the relations

$$9f_4^2-12f_3^2-f_1^2f_3+f_1^2f_4=0,$$

$$432f_5^2-f_3^2+2f_1^2-36f_1f_4+3f_1^2f_3-36f_2f_3=0.$$

Thus $\mathbb{C}\left[x,y,z\right]^G=\mathbb{C}\left[f_1,\ldots,f_5\right]\cong\mathbb{C}\left[r,s,t,u,v\right]/\ker\varphi$, where $\ker\varphi$ is the ideal $\left(9u^2-12t^2-r^2t+r^2u,432v^2-t^2+2r^2-36ru+3r^2t-36st\right)$;

- (F):

$$f_1\left(x,y,z\right)=\left(x^3-y^3\right)\left(y^3-z^3\right)\left(z^3-x^3\right),$$

$$f_2\left(x,y,z\right)=\left(x^3+y^3+z^3\right)^2-12\left(x^3y^3+y^3z^3+x^3z^3\right),$$

$$f_3\left(x,y,z\right)=27x^4y^4z^4-xyz\left(x^3+y^3+z^3\right)^3,$$

$$f_4\left(x,y,z\right)=(18x^3y^3z^3\left(x^3+y^3+z^3\right)-3x^2y^2z^2\left(x^3+y^3+z^3\right)^2$$
$$-xyz\left(x^3+y^3+z^3\right)^3)^2,$$

which satisfy the relation

$$4f_4^4-114f_3f_4^2+1728f_3^2f_4-\left(f_2^3-432f_1^2-3f_2f_4+36f_2f_3\right)^2=0.$$

Thus $\mathbb{C}\left[x,y,z\right]^G=\mathbb{C}\left[f_1,\ldots,f_4\right]\cong\mathbb{C}\left[s,t,u,v\right]/\ker\varphi$, where $\ker\varphi$ is the ideal $\left(4v^4-114uv^2+1728u^2v-\left(t^3-432s^2-3tv+36tu\right)^2\right)$;

- (G):

$$f_1\left(x,y,z\right)=\left(\left(x^3+y^3+z^3\right)^2-12\left(x^3y^3+y^3z^3+x^3z^3\right)\right)^3,$$

$$f_2\left(x,y,z\right)=\left(\left(x^3+y^3+z^3\right)^2-12\left(x^3y^3+y^3z^3+x^3z^3\right)\right)$$
$$\left(18x^3y^3z^3\left(x^3+y^3+z^3\right)-3x^2y^2z^2\left(x^3+y^3+z^3\right)^2-xyz\left(x^3+y^3+z^3\right)^3\right)^2,$$

$$f_3\left(x,y,z\right)=27x^4y^4z^4-xyz\left(x^3+y^3+z^3\right)^3,$$

$$f_4(x,y,z) = \left(x^3-y^3\right)\left(y^3-z^3\right)\left(z^3-x^3\right),$$

which satisfy the relation

$$4f_2^3-9f_1f_2^2+6f_1^2f_2+2592f_1f_2f_4^2-f_1^3+864f_1^2f_4^2+6912f_1f_3^3-186624f_1f_4^4=0.$$

Thus $\mathbb{C}[x,y,z]^G = \mathbb{C}[f_1,\ldots,f_4] \cong \mathbb{C}[s,t,u,v]/\ker\varphi$, where $\ker\varphi$ is the ideal $\left(4t^3-9st^2+6s^2t+2592stv^2-s^3+864s^2v^2+6912su^3-186624sv^4\right)$;

- (H):

$$\begin{aligned} f_1(x,y,z) &= x^2+yz,\\ f_2(x,y,z) &= 8x^4yz-2x^2y^2z^2-x\left(y^5+z^5\right)+y^3z^3,\\ f_3(x,y,z) &= 320x^6y^2z^2-160x^4y^3z^3+20x^2y^4z^4+6y^5z^5\\ &\quad -4x\left(y^5+z^5\right)\left(32x^4-20x^2yz+5y^2z^2\right)+y^{10}+z^{10},\\ f_4(x,y,z) &= \left(y^5-z^5\right)(-1024x^{10}+3840x^8yz-3840x^6y^2z^2+1200x^4\\ &\quad -100x^2y^4z^4+y^{10}+z^{10}+2y^5z^5+x\left(y^5+z^5\right)\left(352x^4-160x^2yz+10y^2z^2\right)), \end{aligned}$$

which satisfy the relation

$$f_4^2+1728f_2^5-f_3^3-720f_1f_2^3f_3+80f_1^2f_2f_3^2-64f_1^3\left(5f_2^2-f_1f_3\right)^2=0.$$

Thus $\mathbb{C}[x,y,z]^G = \mathbb{C}[f_1,\ldots,f_4] \cong \mathbb{C}[s,t,u,v]/\ker\varphi$, where $\ker\varphi$ is the ideal $\left(v^2+1728t^5-u^3-720st^3u+80s^2tu^2-64s^3\left(5t^2-su\right)^2\right)$;

- (I):

$$\begin{aligned} f_1(x,y,z) &= xy^3+yz^3+zx^3,\\ f_2(x,y,z) &= 5x^2y^2z^2-x^5y-y^5z-z^5x,\\ f_3(x,y,z) &= x^{14}+y^{14}+z^{14}-34\left(x^{11}yz^2-x^2y^{11}z-xy^2z^{11}\right)\\ &\quad -250\left(x^9y^4z-xy^9z^4-x^4yz^9\right)+375\left(x^8y^2z^4-x^4y^8z^2-x^2y^4z^8\right)\\ &\quad +18\left(x^7y^7+y^7z^7+x^7z^7\right)-126\left(x^6y^5z^3-x^3y^6z^5-x^5y^3z^6\right),\\ f_4(x,y,z) &= x^{21}+y^{21}+z^{21}-7\left(x^{18}yz^2-x^2y^{18}z-xy^2z^{18}\right)\\ &\quad +217\left(x^{16}y^4z-xy^{16}z^4-x^4yz^{16}\right)-308\left(x^{15}y^2z^4-x^4y^{15}z^2-x^2y^4z^{15}\right)\\ &\quad -57\left(x^7y^{14}+y^7z^{14}+x^7z^{14}\right)-289\left(x^{14}y^7+y^{14}z^7+x^{14}z^7\right)\\ &\quad +4018\left(x^{13}y^5z^3-x^3y^{13}z^5-x^5y^3z^{13}\right)+637\left(x^{12}y^3z^6-x^6y^{12}z^3-x^3y^6z^{12}\right)\\ &\quad +1638\left(x^{11}yz^9-x^9y^{11}z-xy^9z^{11}\right)-6279\left(x^{11}y^8z^2-x^2y^{11}z^8-x^8y^2z^{11}\right)\\ &\quad +7007\left(x^{10}y^6z^5-x^5y^{10}z^6-x^6y^5z^{10}\right)-10010\left(x^9y^4z^8-x^8y^9z^4-x^4y^8z^9\right)\\ &\quad +10296x^7y^7z^7, \end{aligned}$$

which satisfy the relation

$$f_4^2 - f_3^3 + 88 f_1^2 f_2 f_3^2 - 1008 f_1 f_2^4 f_3 - 1088 f_1^4 f_2^2 f_3 + 256 f_1^7 f_3$$
$$-1728 f_2^7 + 60032 f_1^3 f_2^5 - 22016 f_1^6 f_2^3 + 2048 f_1^9 f_2 = 0.$$

Thus $\mathbb{C}[x,y,z]^G = \mathbb{C}[f_1,\ldots,f_4] \cong \mathbb{C}[s,t,u,v] / \ker\varphi$, where $\ker\varphi$ is the ideal $(v^2 - u^3 + 88s^2tu^2 - 1008st^4u - 1088s^4t^2u + 256s^7u - 1728t^7 + 60032s^3t^5 - 22016s^6t^3 + 2048s^9t)$;

- (J):

$$f_1(x,y,z) = \left(x^2+yz\right)^3,$$
$$f_2(x,y,z) = \left(x^2+yz\right)(320x^6y^2z^2 - 160x^4y^3z^3 + 20x^2y^4z^4$$
$$+6y^5z^5 - 4x\left(y^5+z^5\right)\left(32x^4 - 20x^2yz + 5y^2z^2\right) + y^{10} + z^{10},$$
$$f_3(x,y,z) = 8x^4yz - 2x^2y^2z^2 - x\left(y^5+z^5\right) + y^3z^3$$
$$f_4(x,y,z) = \left(y^5-z^5\right)(-1024x^{10} + 3840x^8yz - 3840x^6y^2z^2 + 1200x^4$$
$$-100x^2y^4z^4 + y^{10} + z^{10} + 2y^5z^5 + x\left(y^5+z^5\right)\left(352x^4 - 160x^2yz + 10y^2z^2\right)),$$

which satisfy the relation

$$f_2^3 - f_1\left(f_4^2 + 1728 f_3^5 - 720 f_3^3 f_2 + 80 f_3 f_2^2 - 64 f_1\left(5f_3^2 - f_2\right)^2\right) = 0.$$

Thus $\mathbb{C}[x,y,z]^G = \mathbb{C}[f_1,\ldots,f_4] \cong \mathbb{C}[s,t,u,v] / \ker\varphi$, where $\ker\varphi$ is the ideal $\left(t^3 - s\left(v^2 + 1728u^5 - 720u^3t + 80ut^2 - 64s\left(5u^2-t\right)^2\right)\right)$;

- (K):

$$f_1(x,y,z) = \left(xy^3+yz^3+zx^3\right)^3,$$
$$f_2(x,y,z) = \left(xy^3+yz^3+zx^3\right)(x^{14}+y^{14}+z^{14} - 34\left(x^{11}yz^2 - x^2y^{11}z - xy^2z^{11}\right)$$
$$-250\left(x^9y^4z - xy^9z^4 - x^4yz^9\right) + 375\left(x^8y^2z^4 - x^4y^8z^2 - x^2y^4z^8\right)$$
$$+18\left(x^7y^7 + y^7z^7 + x^7z^7\right) - 126\left(x^6y^5z^3 - x^3y^6z^5 - x^5y^3z^6\right)),$$
$$f_3(x,y,z) = 5x^2y^2z^2 - x^5y - y^5z - z^5x,$$
$$f_4(x,y,z) = x^{21} + y^{21} + z^{21} - 7\left(x^{18}yz^2 - x^2y^{18}z - xy^2z^{18}\right)$$
$$+217\left(x^{16}y^4z - xy^{16}z^4 - x^4yz^{16}\right) - 308\left(x^{15}y^2z^4 - x^4y^{15}z^2 - x^2y^4z^{15}\right)$$
$$-57\left(x^7y^{14} + y^7z^{14} + x^7z^{14}\right) - 289\left(x^{14}y^7 + y^{14}z^7 + x^{14}z^7\right)$$
$$+4018\left(x^{13}y^5z^3 - x^3y^{13}z^5 - x^5y^3z^{13}\right) + 637\left(x^{12}y^3z^6 - x^6y^{12}z^3 - x^3y^6z^{12}\right)$$
$$+1638\left(x^{11}yz^9 - x^9y^{11}z - xy^9z^{11}\right) - 6279\left(x^{11}y^8z^2 - x^2y^{11}z^8 - x^8y^2z^{11}\right)$$
$$+7007\left(x^{10}y^6z^5 - x^5y^{10}z^6 - x^6y^5z^{10}\right) - 10010\left(x^9y^4z^8 - x^8y^9z^4 - x^4y^8z^9\right)$$
$$+10296x^7y^7z^7,$$

which satisfy the relation

$$f_2^3 - f_1(f_4^2 + 88 f_1 f_2 - 1008 f_3^4 f_2 - 1088 f_3^2 f_1 f_2 + 256 f_1^2 f_2$$
$$-1728 f_3^7 + 60032 f_3^5 f_1 - 22016 f_3^3 f_1^2 + 2048 f_3 f_1^3 = 0.$$

Thus $\mathbb{C}[x,y,z]^G = \mathbb{C}[f_1,\ldots,f_4] \cong \mathbb{C}[s,t,u,v] / \ker\varphi$, where $\ker\varphi$ is the ideal $(t^3 - s(v^2 + 88st - 1008u^4t - 1088u^2st + 256s^2t - 1728u^7 + 60032u^5s - 22016u^3s^2 + 2048us^3)$;

- (L): (give the invariants implicitly take a few pages long to write down)

$$f_1(x,y,z) = \left(-3 + 5i\sqrt{15}\right)x^6 + \left(135 + 15i\sqrt{15}\right)x^4yz - 18xy^5$$
$$-18xz^5 - \left(45 - 15i\sqrt{15}\right)x^2y^2z^2 + \left(15 + 5i\sqrt{15}\right)x^3y^3,$$
$$f_2(x,y,z) = \frac{1}{81000}H\left(f_1(x,y,z)\right),$$
$$f_3(x,y,z) = \frac{1}{145800}BH\left(f_1(x,y,z), f_2(x,y,z)\right),$$
$$f_4(x,y,z) = \frac{1}{9720}J\left(f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)\right),$$

where H denotes the Hessian matrix, BH the bordered Hessian matrix and J the Jacobian matrix. The minimal generators satisfy the relation

$$459165024 f_4^2 - 25509168 f_3^3 - \left(236196 + 26244i\sqrt{15}\right) f_3^2 f_1^5$$
$$+1889568\left(1 + i\sqrt{15}\right) f_3^2 f_1^3 f_2 + \left(8503056 - 2834352i\sqrt{15}\right) f_3^2 f_1 f_2^2$$
$$-\left(891 + 243i\sqrt{15}\right) f_3 f_1^{10} - \left(5346 - 8910i\sqrt{15}\right) f_3 f_1^8 f_2$$
$$+\left(36012 - 51516i\sqrt{15}\right) f_3 f_1^6 f_2 + \left(192456 + 21384i\sqrt{15}\right) f_3 f_1^4 f_2^3$$
$$-3569184\left(1 + i\sqrt{15}\right) f_3 f_1^2 f_2^4 - \left(7558272 - 2519424i\sqrt{15}\right) f_3 f_2^5$$
$$-2426112\left(1 + i\sqrt{15}\right) f_2^7 f_1 + \left(7978176 + 886464i\sqrt{15}\right) f_2^6 f_1^3$$
$$-\left(3297168 - 471024i\sqrt{15}\right) f_2^5 f_1^5 + \left(78768 - 131280i\sqrt{15}\right) f_2^4 f_1^7$$
$$+\left(26928 + 7344i\sqrt{15}\right) f_2^3 f_1^9 - \left(1560 - 40i\sqrt{15}\right) f_2^2 f_1^{11}$$
$$+17\left(1 - i\sqrt{15}\right) f_2 f_1^{13} = 0.$$

Thus $\mathbb{C}[x,y,z]^G = \mathbb{C}[f_1,\ldots,f_4] \cong \mathbb{C}[s,t,u,v] / \ker\varphi$, where $\ker\varphi$ is the ideal generated by the relation given above in the variables s,t,u,v.

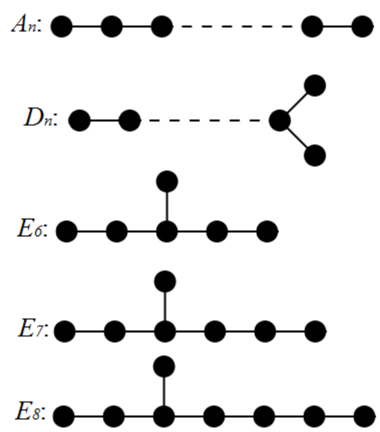
Note that for types (F) - (L) of finite subgroups of $\mathrm{SL}(3,\mathbb{C})$ the number of minimal generators of $\mathbb{C}[x,y,z]^G$ is four and hence the quotient varieties of these groups are hypersufaces in $\mathbb{C}^4$.

4.2 Du Val singularities, Dynkin diagrams and McKay correspondence

As we said, for all the types of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ the number of minimal generators of $\mathbb{C}[x, y]^G$ is three and hence the quotient varieties of these groups are hypersufaces in $\mathbb{C}^3$. The equations of these surfaces coincide with the equations of the Du Val singularities:

- $A_n : x^2 + y^2 + z^{n+1}$;
- $D_n : x^2 + y^2 z + z^{n-1}, n \geq 4$;
- $E_6 : x^4 + y^3 + z^2$;
- $E_7 : x^3 + xy^3 + z^2$;
- $E_8 : x^2 + y^3 + z^5$,

classified by the simply laced A-D-E Dynkin diagrmas:



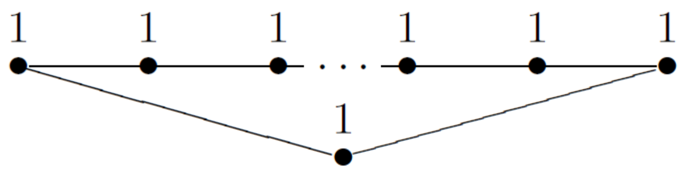
The McKay correspondence answers the question of whether there are any connections between the representation theory of $G \subset \mathrm{SL}(2, \mathbb{C})$ and the geometry of the minimal resolution of a singularity: there is a one-to-one correspondence between the McKay graphs of $G \subset \mathrm{SL}(2, \mathbb{C})$ finite and the extended Dynkin diagrams. We show the definition of McKay quiver (or McKay graph) and the McKay's observation:

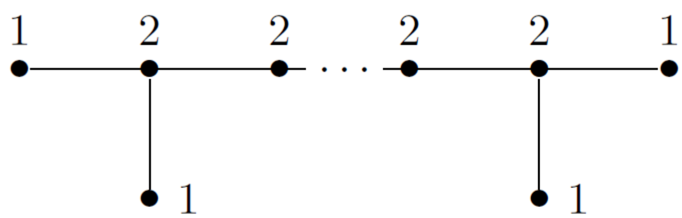
Definition 48. Let G be a finite group, ρ a representation of G, $\rho_1, \ldots, \rho_m$ its irreducible representations and χ its character. We define the *McKay graph* of G, as follow:

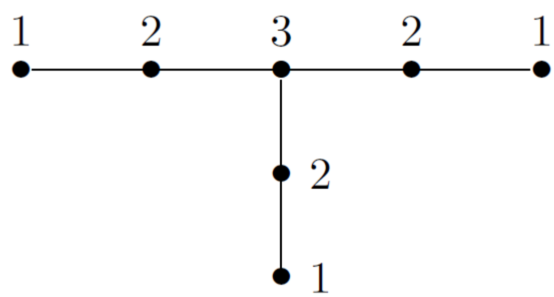
- the vertices/nodes of the graph correspond to the irreducible representations of G, $\rho_1, \ldots, \rho_m$, and we label every vertice with the representation;
- there is an arrow from ρ_i to ρ_j if and only if $n_{ij} > 0$, and there is an edge when $n_{ij} = n_{ji}$ instead of a double arrow, where $n_{ij} := \langle \rho \otimes \rho_i, \rho_j \rangle = \frac{1}{|G|} \sum_{g \in G} \rho(g) \rho_i(g) \overline{\rho_j(g)}$. Note that if $G \subset \mathrm{SL}(2, \mathbb{C})$ then $n_{ij} = n_{ji}$.

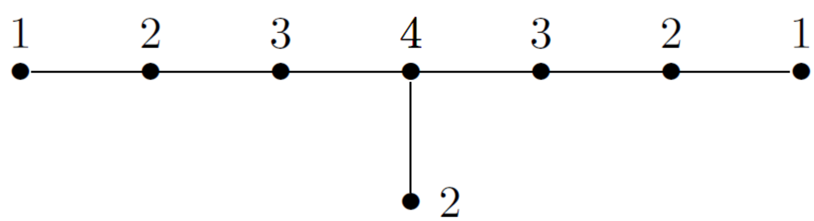
Theorem 49. *Let $G \subset \mathrm{SL}(2, \mathbb{C})$ be finite, then the McKay graph of G is an affine simply laced Dynkin diagram (extended Dynkin diagram).*

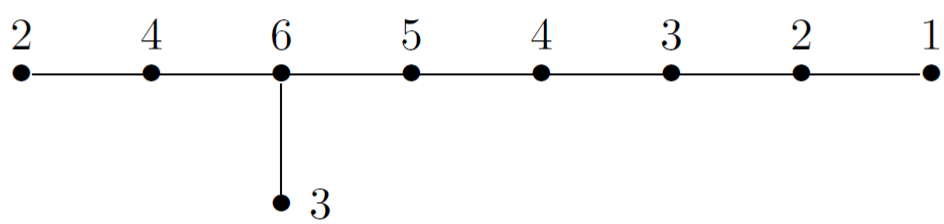
And hence if $G = C_n, \mathrm{BD}_{4n}, \mathrm{BT}_{24}, \mathrm{BO}_{48}, \mathrm{BI}_{120}$ and ρ is its natural 2−dimensional representation given by the inclusion $G \hookrightarrow \mathrm{SL}(2, \mathbb{C}) \subset \mathrm{GL}(2, \mathbb{C})$, then we have the following McKay graphs (respectively):











4.3 Further work

Naturally, it arises a question: how can we extend this correspondence? We have the following conjeture (since 1992) due to Miles Reid, which suggests a correspondence between a basis for the cohomology of a resolution of a quotient singularity and the irreducible representations of the group, and that conjugacy classes correspond to a basis of homology:

Conjecture 50. *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be finite and assume that the quotient $X = \mathbb{C}^n/G$ has a crepant resolution $f : Y \to X$ (that is $K_Y = 0$), then there exist "natural" bijections*

$$\{irreducible\, representations\, of\, G\} \to basis\, of\, H^* (Y, \mathbb{Z}) ,$$

$$\{conjugacy\, classes\, of\, G\} \to basis\, of\, H_* (Y, \mathbb{Z}) .$$

For $n = 2$ we have the McKay correspondence and the work of Gonzalez-Sprinberg and Verdier in [G-SV, 1983] where they give an isomorphism on K-theory: for $G \subset \mathrm{SL}(2, \mathbb{C})$ they construct *sheaves* $\mathcal{F}_\rho$ on Y, corresponding to each irreducible representation ρ of G and it follows the first part of the conjecture for $G \subset \mathrm{SL}(2, \mathbb{C})$. For $n = 3$ we have a weak version given by Reid and Ito in [IR, 1994]: crepant exceptional divisors of Y correspond one-to-one with *junior* conjugacy classes of G, which gives a basis of $H^* (Y, \mathbb{Q})$. In [IN, 1999] Ito and Nakamura discuss the McKay Correspondence from the new point of view of Hilbert schemes: in many cases the G-Hilber scheme is a preferred resolution Y of X. It follows another conjeture due to Reid, which has been proved when $G \subset \mathrm{SL}(3, \mathbb{C})$ finite is Abelian:

Conjecture 51. *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be finite and assume that $Y = G\text{-Hilb}\mathbb{C}^n$ is a crepant resolution of the quotient $X{=}\mathbb{C}^n/G$, then*

- *the González-Sprinberg and Verdier sheaves $\mathcal{F}_\rho$ on Y are locally free and form a $\mathbb{Z}$-basis of the K-theory of Y;*
- *a certain cookery with the Chern classes of the sheaves $\mathcal{F}_\rho$ leads to a $\mathbb{Z}$-basis of the of the cohomology $H^* (Y, \mathbb{Z})$ for which the bijection*

  $$\{irreducible\, representations\, of\, G\} \leftrightarrow basis\, of\, H^* (Y, \mathbb{Z})$$

  holds. This is the McKay correspondence for $Y = G\text{-Hilb}\mathbb{C}^n$.

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