ℓ-TOP: graph topologies induced by edge-lengths

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Mathematisches Seminar
Universität Hamburg

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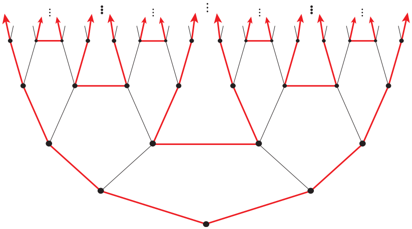
Kirchhoff's second law



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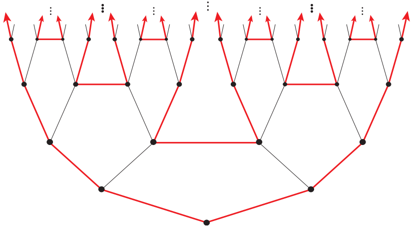

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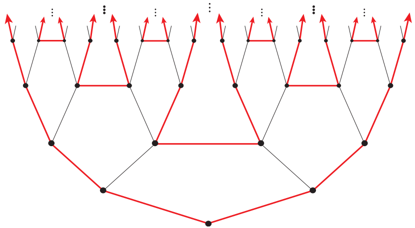

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What about:

Kirchhoff's second law: in an electrical network, the net potential drop along a cycle is 0.

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Kirchhoff's 2nd law: $\sum_{\vec{e} \in \vec{C}} i(\vec{e}) r(e) = 0$ for every cycle $\vec{C}$.

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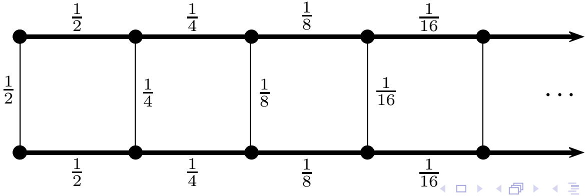

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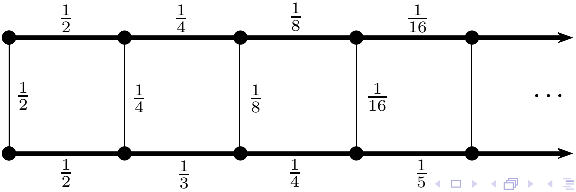

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Theorem (G '06 (easy))

If $\sum_{e \in E(\Gamma)} \ell(e) < \infty$ *then* ℓ-$TOP(\Gamma) \approx |\Gamma|$.

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A circle is *proper* if its edges: – have finite total resistance, and
– form a dense subset.

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Problem

Are there other important spaces that are a special case of ℓ-TOP?

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(*separable*: has a countable dense subset)

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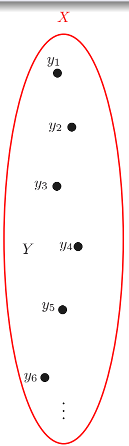

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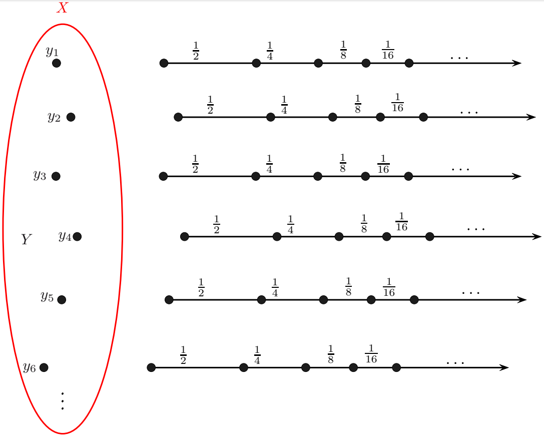

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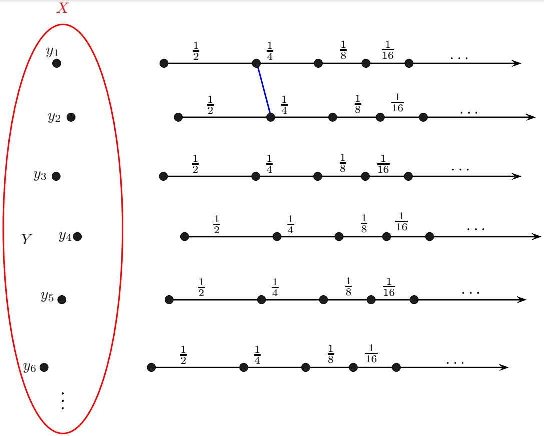

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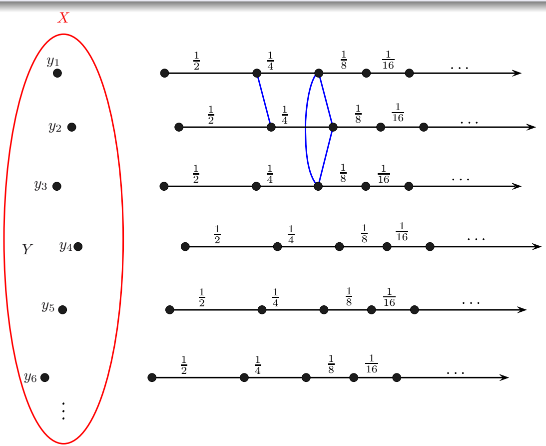

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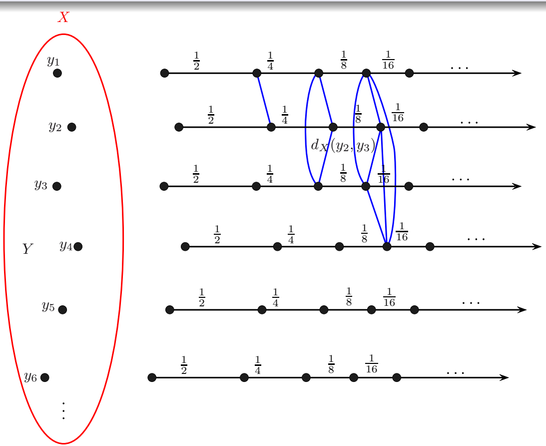

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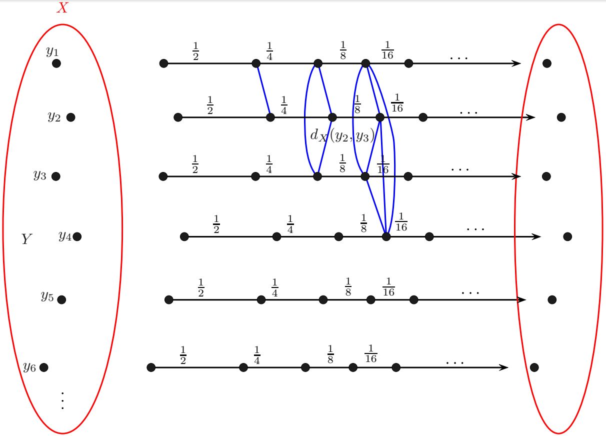

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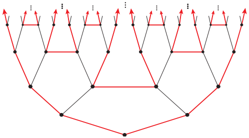

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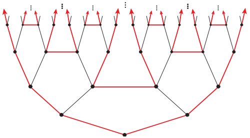

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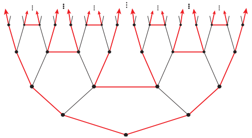

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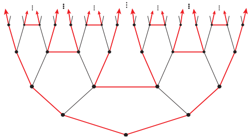

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Problem

Define brownian motion on ℓ-TOP of a (Cayley) graph.

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easy: The fundamental circuits of a spanning tree generate $\mathcal{C}(\Gamma)$	Diestel & Kühn

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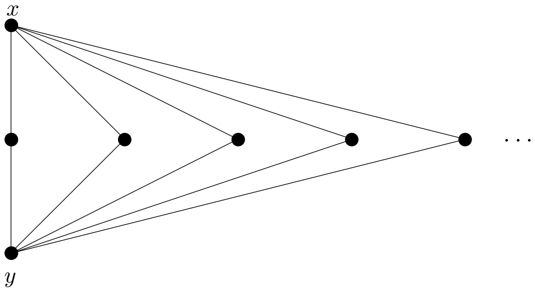

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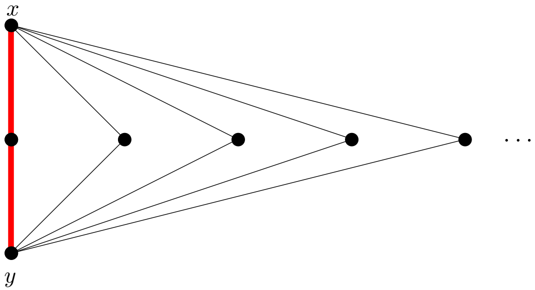

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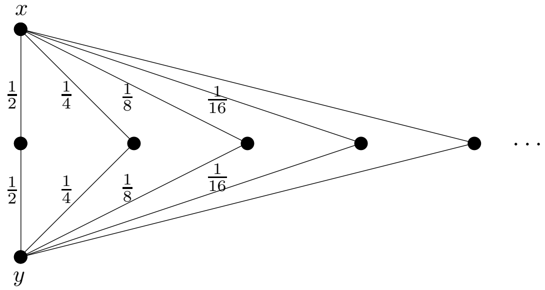

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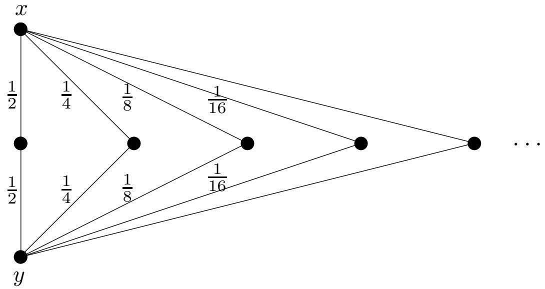


Idea: allow only sums of families of circuits of finite total length

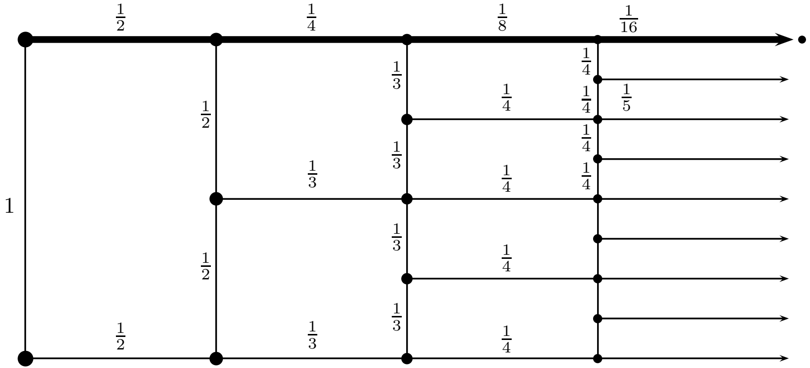
A new Homology

Work in progress:

We aim at a homology that

- generalises the topological cycle space
- is defined for any metric space
- allows generalisations of theorems from graphs to other spaces

A monster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