

Square Tilings and the Poisson Boundary

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The Riemann mapping theorem

Theorem (Riemann? '1851, Carathéodory 1912)

For every simply connected open set $\Omega \subsetneq \mathbb{C}$, $\Omega \neq \emptyset$, there is a bijective conformal map from Ω onto the open unit disk.

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Theorem (Koebe 1908)

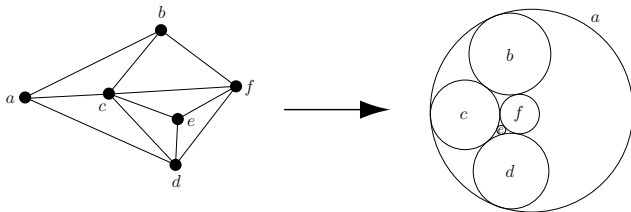
*For every open set $\Omega \subsetneq \mathbb{C}$, $\Omega \neq \emptyset$ with **finitely many boundary components**, there is a bijective conformal map from Ω onto **a circle domain**.*

The circle packing theorem

The Koebe-Andreev-Thurston circle packing theorem

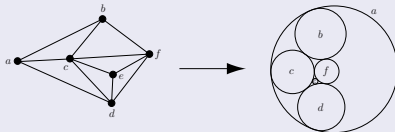
For every finite planar graph G , there is a circle packing in the plane (or S^2) with nerve G .

The packing is unique (up to Möbius transformations) if G is a triangulation of S^2 .



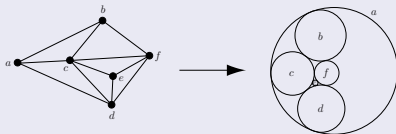
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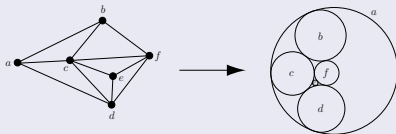
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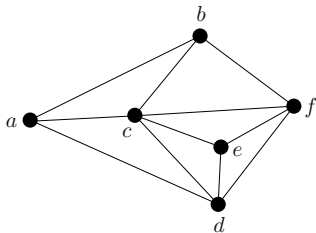
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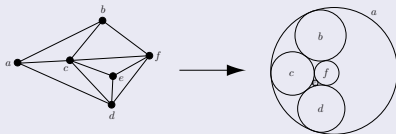


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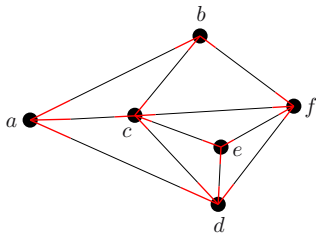


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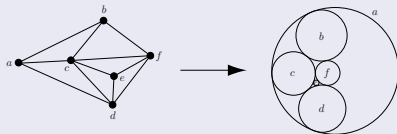


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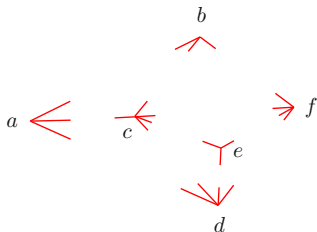


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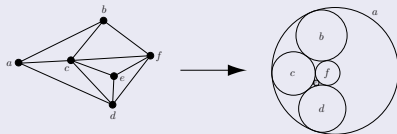


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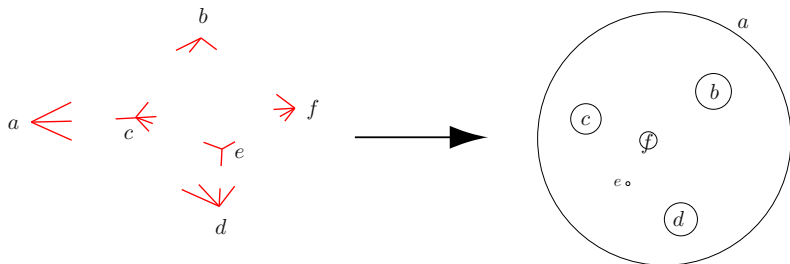


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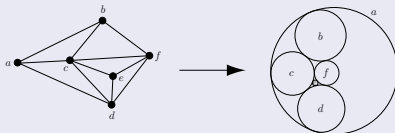


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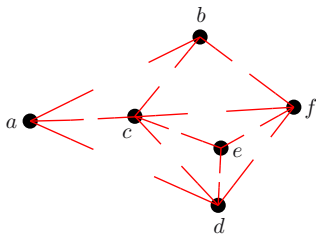


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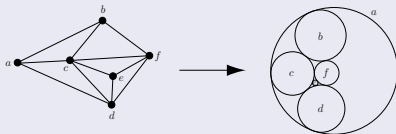


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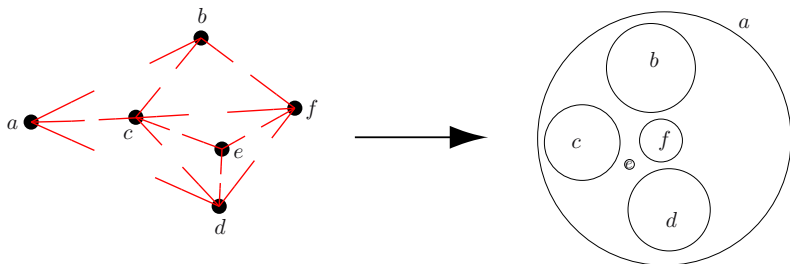


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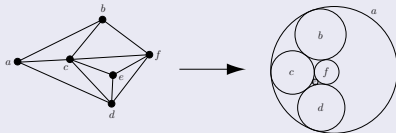


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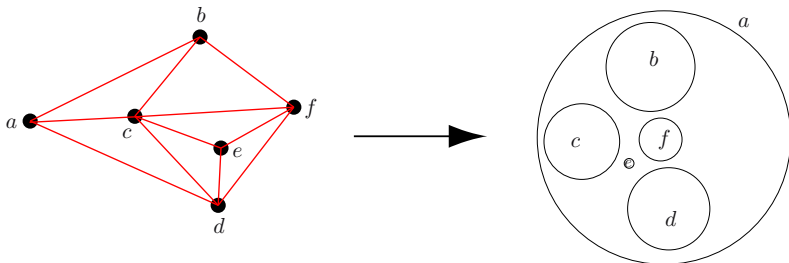


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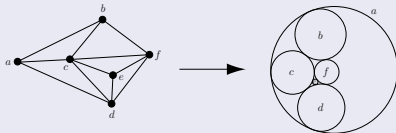


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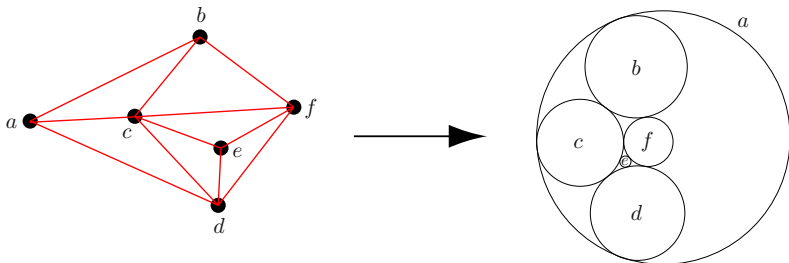


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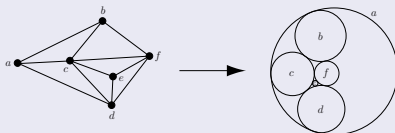


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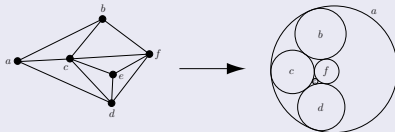
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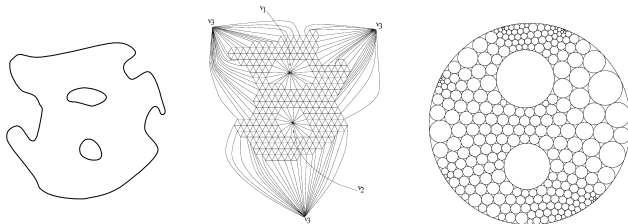
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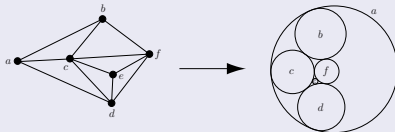


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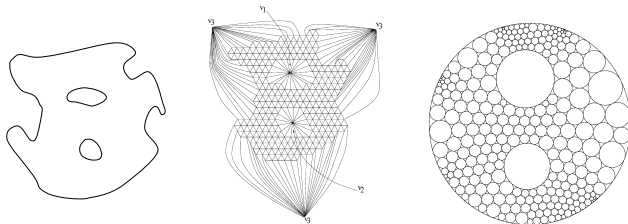


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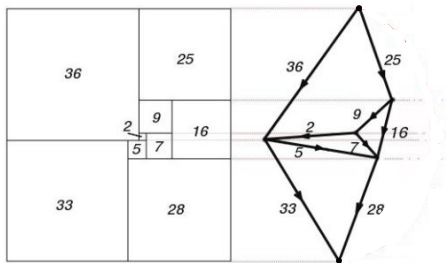


[S. Rohde: "Oded Schramm: From Circle Packing to SLE", '10]

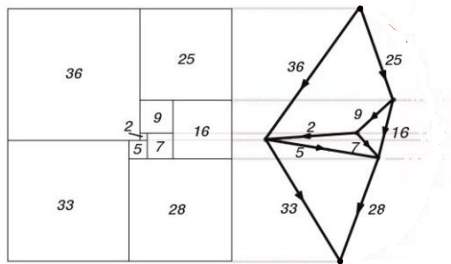
Square Tilings

Theorem (Brooks, Smith, Stone & Tutte '40)

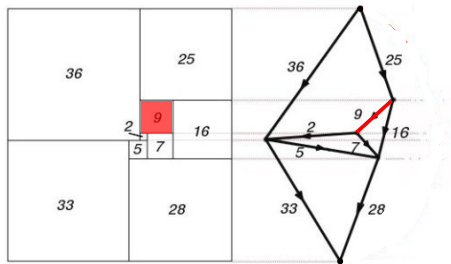
... for every finite planar graph G , there is a square tiling with incidence graph G ...



Properties of square tilings

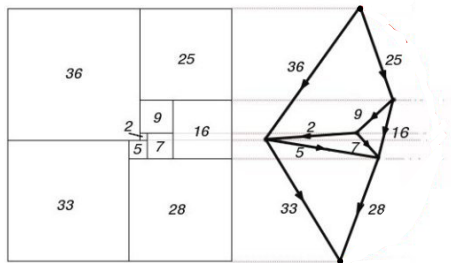


Properties of square tilings



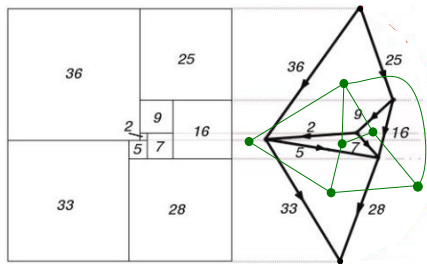
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Properties of square tilings



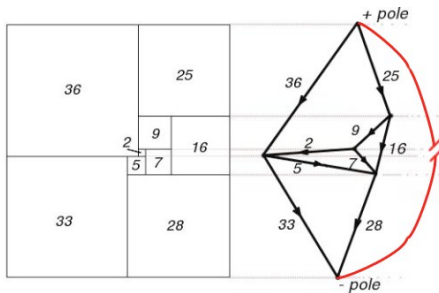
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- vertices correspond to horizontal segments tangent with their edges;
- there is no overlap of squares, and no 'empty' space left;

Properties of square tilings



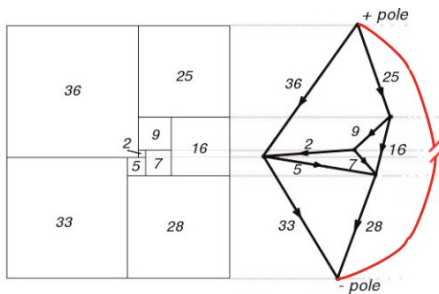
- every edge is mapped to a square;
- vertices correspond to horizontal segments tangent with their edges;
- there is no overlap of squares, and no 'empty' space left;
- the square tiling of the dual of G can be obtained from that of G by a 90° rotation.

The construction of square tilings



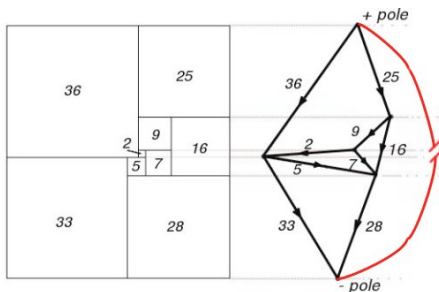
- Think of the graph as an electrical network;

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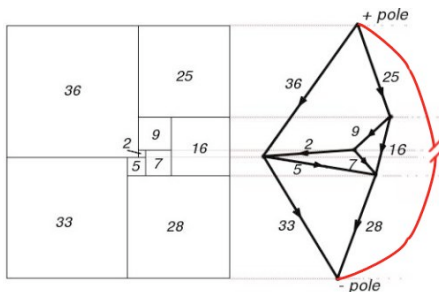
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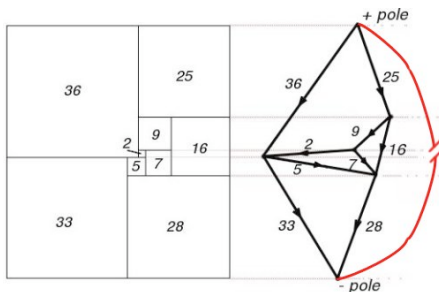
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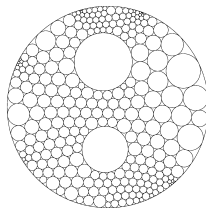
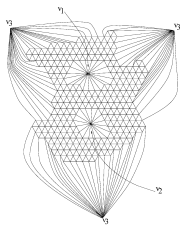
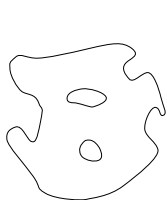
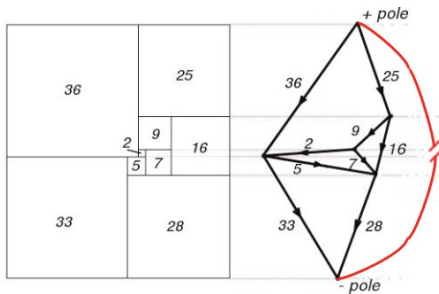
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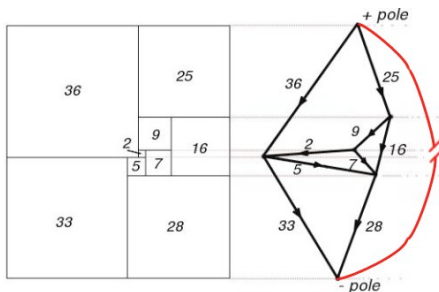


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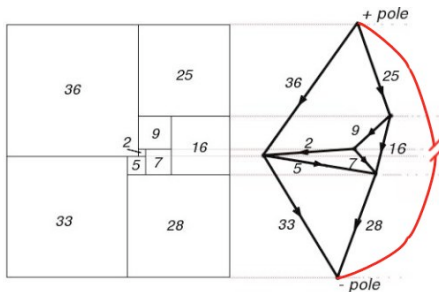


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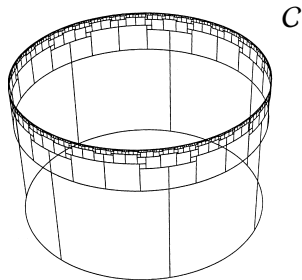
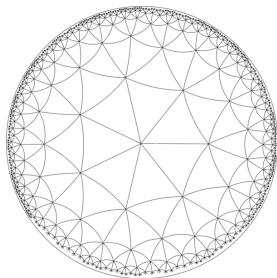


[J. W. Cannon, W. J. Floyd, and W. R. Parry: "Squaring rectangles: The finite Riemann mapping theorem."]

The square tilings of Benjamini & Schramm

Theorem (Benjamini & Schramm '96)

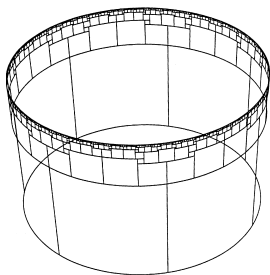
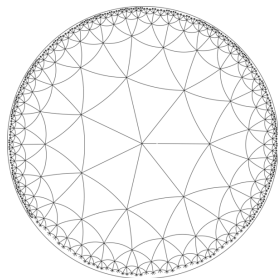
Every transient (infinite) graph G of bounded degree that has a uniquely absorbing embedding in the plane admits a square tiling.



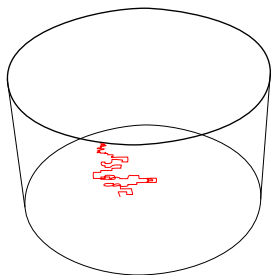
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Every transient (infinite) graph G of bounded degree that has a uniquely absorbing embedding in the plane admits a square tiling. Moreover, random walk on G converges a. s. to a point in C .



C



The Poisson integral representation formula

The classical Poisson formula

$$h(z) = \int_0^1 \hat{h}(\theta) P(z, \theta) d\theta$$

$$\text{where } P(z, \theta) := \frac{1-|z|^2}{|e^{2\pi i\theta} - z|^2},$$

recovers every continuous harmonic function h on \mathbb{D} from its boundary values \hat{h} on the circle $\partial\mathbb{D}$.

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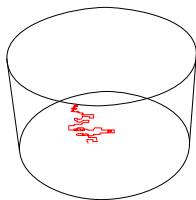
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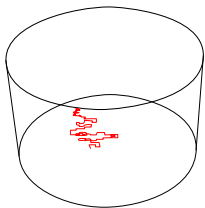
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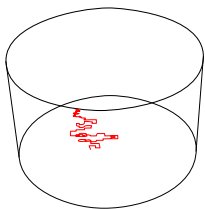
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Can the bounded harmonic functions on a plane graph G be expressed as a Poisson-like integral using C ?



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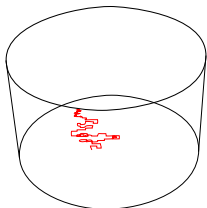
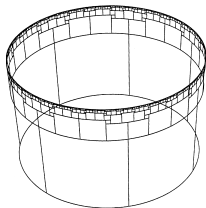
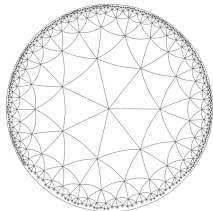


A function $h : V(G) \rightarrow \mathbb{R}$,
is **harmonic**, if $h(x) = \sum_{y \sim x} h(y)/d(x)$.

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Question (Benjamini & Schramm '96)

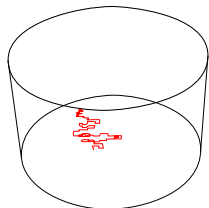
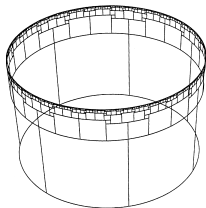
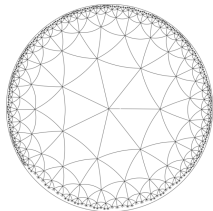
Does the Poisson boundary of every graph as above coincide with the boundary of its square tiling?



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Theorem (G '12)

Yes!



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- this $\hat{h} \in L^\infty(\mathcal{P}_G)$ is unique up to modification on a null-set;
- conversely, for every $\hat{h} \in L^\infty(\mathcal{P}_G)$ the function $z \mapsto \int_{\mathcal{P}_G} \hat{h}(\eta) d\nu_z(\eta)$ is bounded and harmonic.

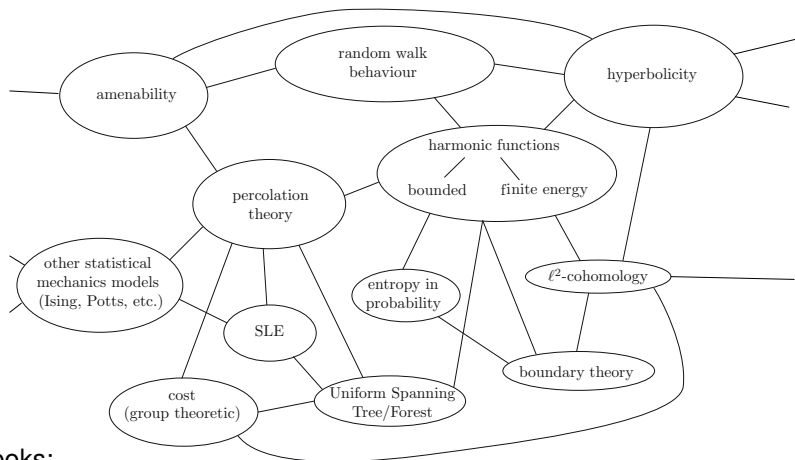
i.e. there is Poisson-like formula establishing an isometry between the Banach spaces $H^\infty(G)$ and $L^\infty(\mathcal{P}_G)$.

The Poisson-Furstenberg boundary

Selected work on the Poisson boundary

- Introduced by Furstenberg to study semi-simple Lie groups [Annals of Math. '63]
- Kaimanovich & Vershik give a general criterion using the entropy of random walk [Annals of Probability '83]
- Kaimanovich identifies the Poisson boundary of hyperbolic groups, and gives general criteria [Annals of Math. '00]

The context



Textbooks:

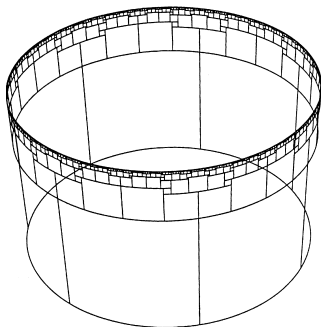
[Woess: *Random Walks on Infinite Graphs and Groups*]

[Lyons & Peres: *Probability on Trees and Networks*]

The theorem

Theorem (G '12)

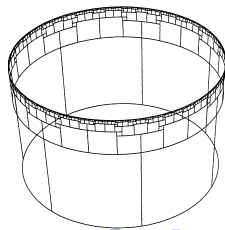
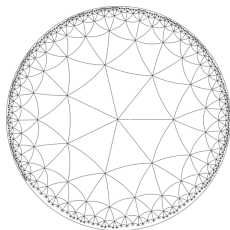
For every bounded degree graph admitting a square tiling, the Poisson boundary coincides with C .



Probabilistic interpretation of the tiling

Lemma (G '12)

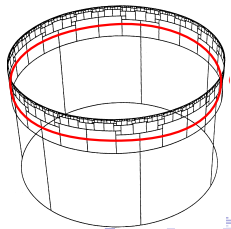
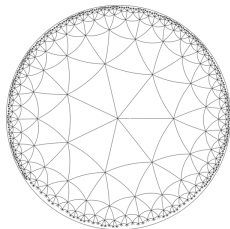
Let C be a 'horizontal' circle in the tiling T of G , and let B the set of points of G at which C 'dissects' T . Then the widths of the points of B in T coincide with the probability distribution of the first visit to B by brownian motion on G starting at o .



Probabilistic interpretation of the tiling

Lemma (G '12)

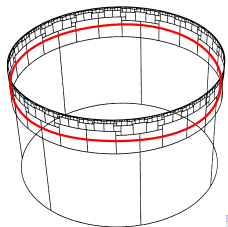
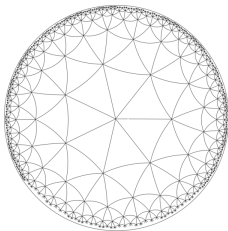
Let C be a 'horizontal' circle in the tiling T of G , and let B the set of points of G at which C 'dissects' T . Then the widths of the points of B in T coincide with the probability distribution of the first visit to B by brownian motion on G starting at o .



Probabilistic interpretation of the tiling

Lemma

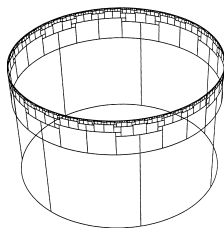
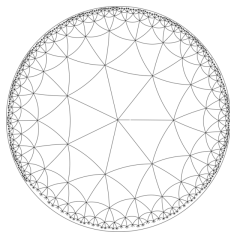
For every 'meridian' M in T , the probability that brownian motion on G starting at o will 'cross' M clockwise equals the probability to cross M counter-clockwise.



Probabilistic interpretation of the tiling

Lemma

For every 'meridian' M in T , the probability that brownian motion on G starting at o will 'cross' M clockwise equals the probability to cross M counter-clockwise.



Here come some
'geometric' random graphs