Discrete Riemann maps and the Poisson Boundary

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THE UNIVERSITY OF WARWICK

Southampton, 1/11/13

The Riemann mapping theorem

Theorem (Riemann ? '1851, Carathéodory 1912)

For every simply connected open set $\Omega \subsetneq \mathbb{C}, \Omega \neq \emptyset$, there is a bijective conformal map from Ω onto the open unit disk.

The Riemann mapping theorem

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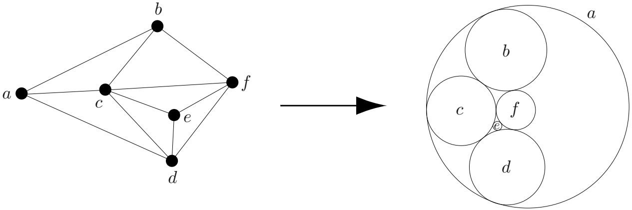
Theorem (Koebe 1908)

For every open set $\Omega \subsetneq \mathbb{C}, \Omega \neq \emptyset$ with finitely many boundary components, there is a bijective conformal map from Ω onto a circle domain.

The circle packing theorem

The Koebe-Andreev-Thurston circle packing theorem

For every finite planar graph G, there is a circle packing in the plane (or S^2) with nerve G.
The packing is unique (up to Möbius transformations) if G is a triangulation of S^2.



Circle Packing <=> Conformal map

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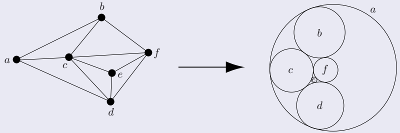

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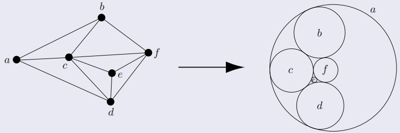

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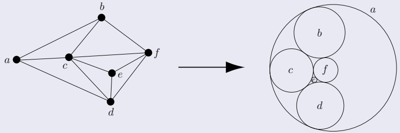

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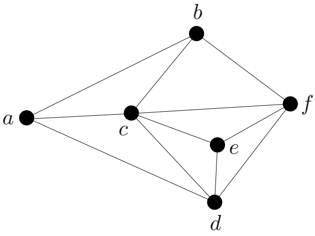

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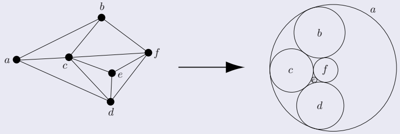

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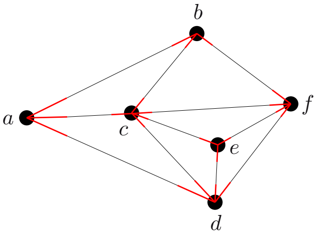

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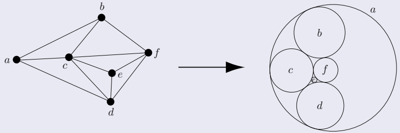

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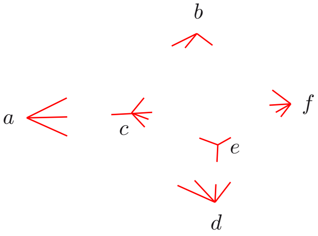

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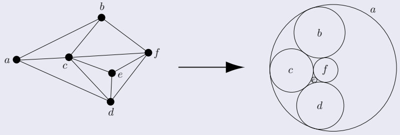

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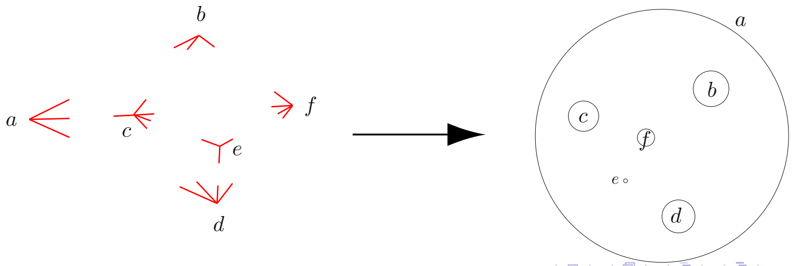

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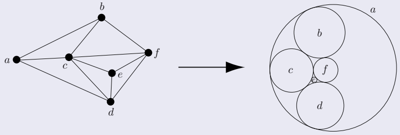

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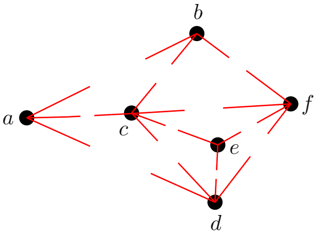

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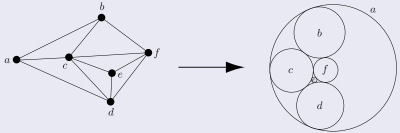

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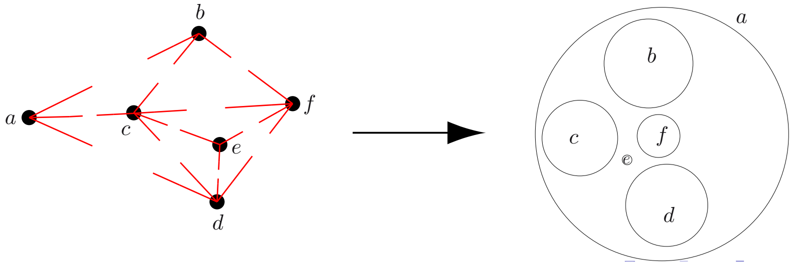

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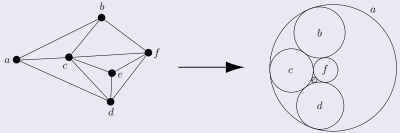

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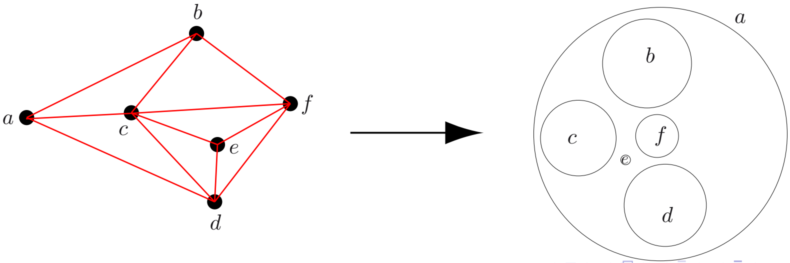

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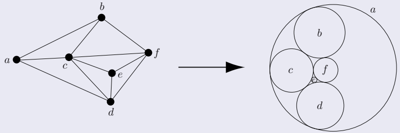

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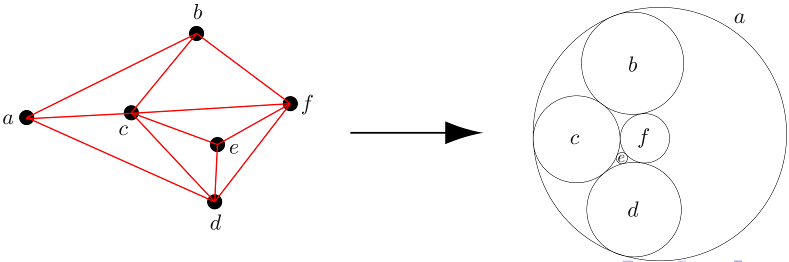

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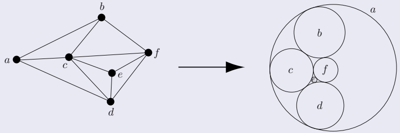

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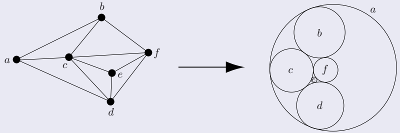

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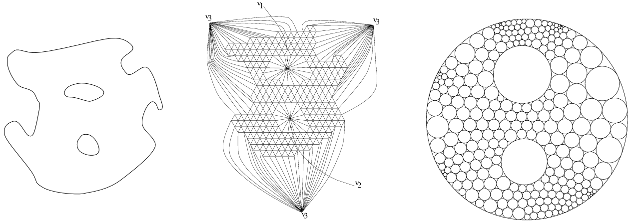

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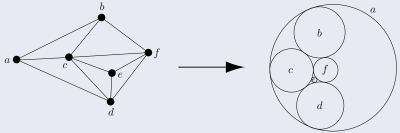

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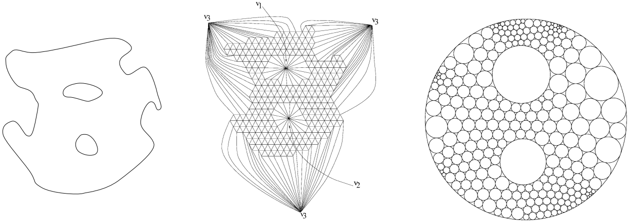
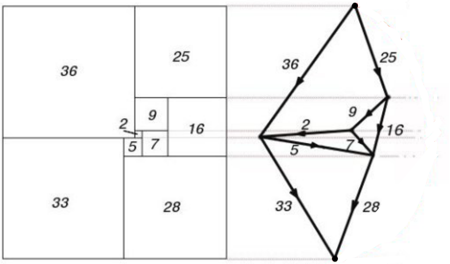


[*S. Rohde: "Oded Schramm: From Circle Packing to SLE", '10*]

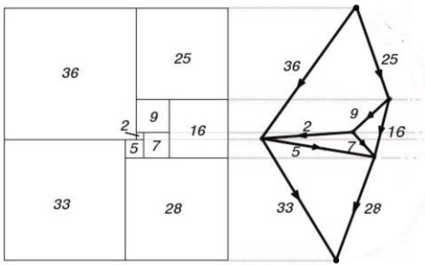
Square Tilings

Theorem (Brooks, Smith, Stone & Tutte '40)

... for every finite planar graph G, there is a square tiling with incidence graph G ...



Properties of square tilings



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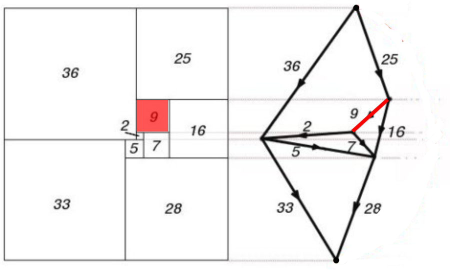

- every edge is mapped to a square;

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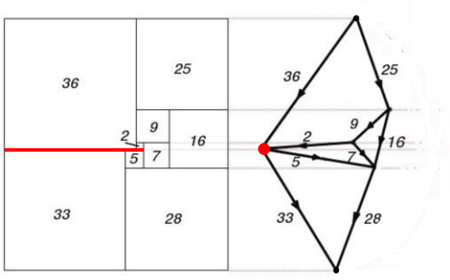

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- vertices correspond to horizontal segments tangent with their edges;

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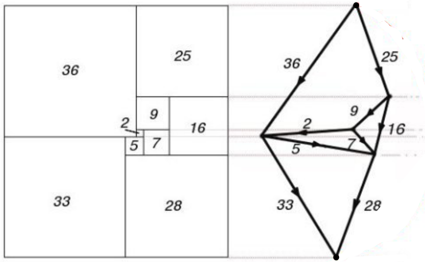

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- there is no overlap of squares, and no ‘empty’ space left;

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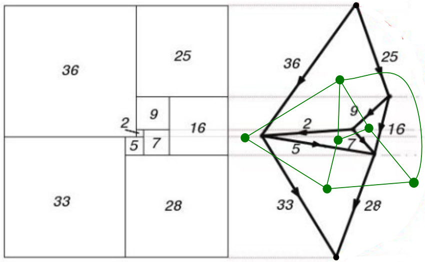

- every edge is mapped to a square;
- vertices correspond to horizontal segments tangent with their edges;
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- the square tiling of the dual of G can be obtained from that of G by a 90° rotation.

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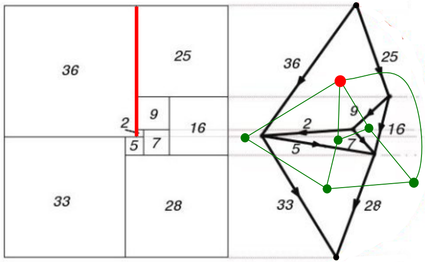

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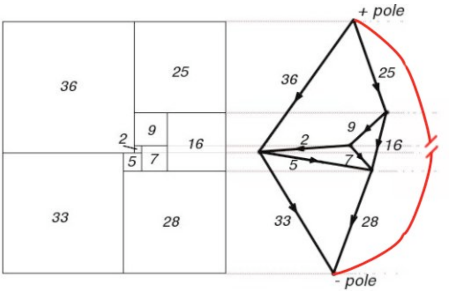
The construction of square tilings



- Think of the graph as an electrical network;

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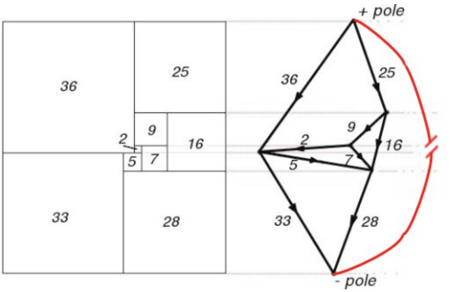

- Think of the graph as an electrical network;
- impose an electrical current from p to q;

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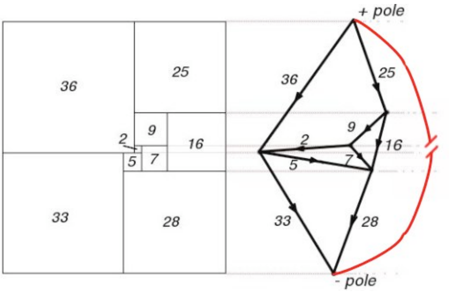

- Think of the graph as an electrical network;
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- let the square corresponding to edge e have side length the flow $i(e)$;

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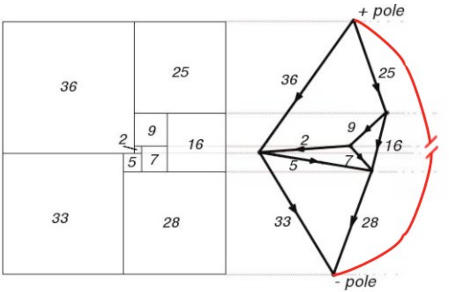

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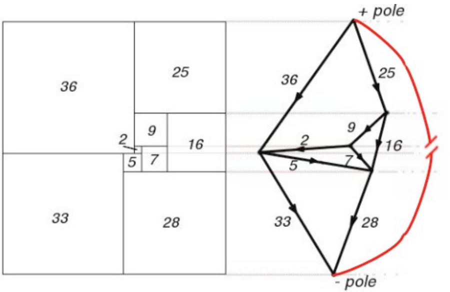

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- use a duality argument to determine the width coordinates.

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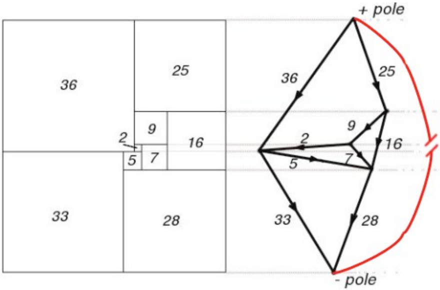

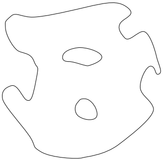

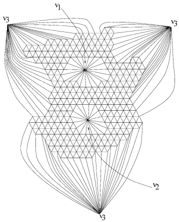

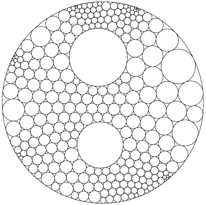

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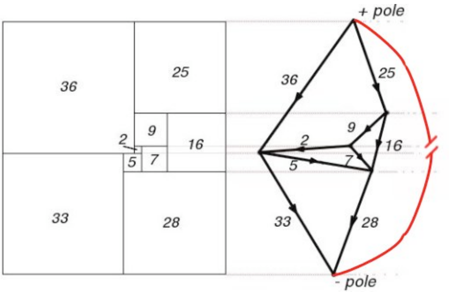

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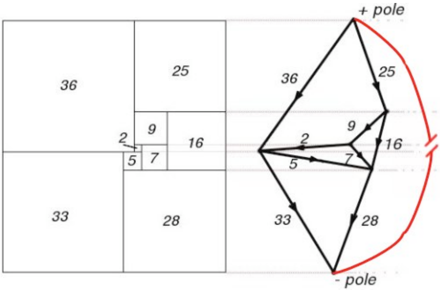
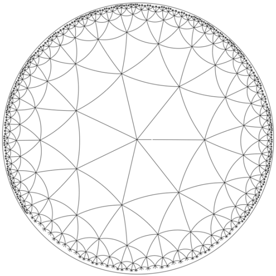


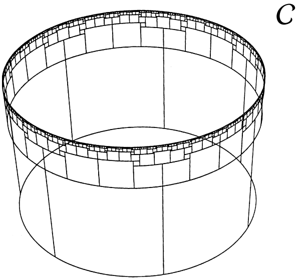
[*J. W. Cannon, W. J. Floyd, and W. R. Parry: "Squaring rectangles: The finite Riemann mapping theorem."*]

The square tilings of Benjamini & Schramm

Theorem (Benjamini & Schramm '96)

Every transient (infinite) graph G of bounded degree that has a uniquely absorbing embedding in the plane admits a square tiling.





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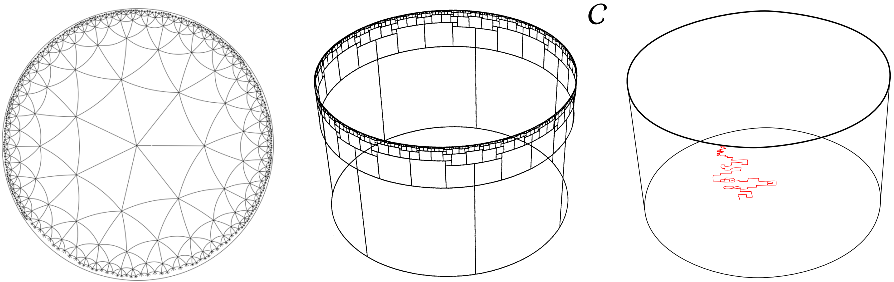

The Poisson integral representation formula

The classical Poisson formula

$$h(z) = \int_0^1 \hat{h}(\theta) P(z, \theta) d\theta$$

where $P(z, \theta) := \frac{1 - |z|^2}{|e^{2\pi i \theta} - z|^2}$,

recovers every continuous harmonic function h on $\mathbb{D}$ from its boundary values $\hat{h}$ on the circle $\partial\mathbb{D}$.

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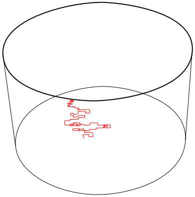
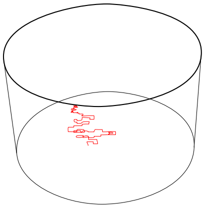


The boundary of the square tiling coincides with the Poisson boundary

Can the bounded harmonic functions on a plane graph G be expressed as a Poisson-like integral using $\mathcal{C}$?



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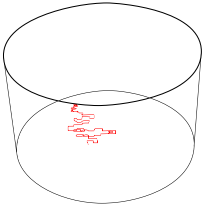
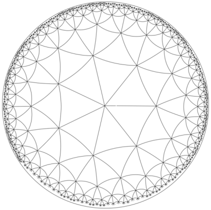


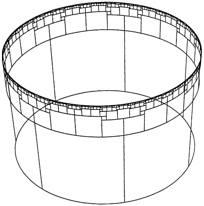
A function $h : V(G) \to \mathbb{R}$,
is **harmonic**, if $h(x) = \sum_{y \sim x} h(y)/d(x)$.

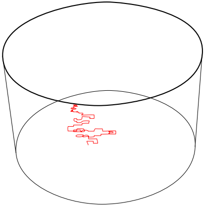
The boundary of the square tiling coincides with the Poisson boundary

Question (Benjamini & Schramm '96)

Does the Poisson boundary of every graph as above coincide with the boundary of its square tiling?







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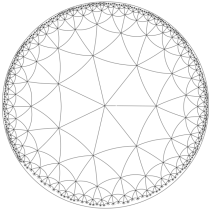

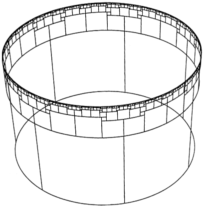

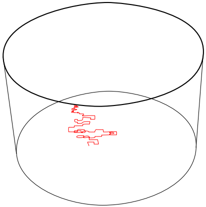

Theorem (G '12)

Yes!

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such that

- every bounded harmonic function h can be obtained by

$$h(z) = \int_{\mathcal{P}_G} \hat{h}(\eta) d\nu_z(\eta)$$

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- every bounded harmonic function h can be obtained by

$$h(z) = \int_{\mathcal{P}_G} \hat{h}(\eta) d\nu_z(\eta)$$

- this $\hat{h} \in L^\infty(\mathcal{P}_G)$ is unique up to modification on a null-set;

The Poisson-Furstenberg boundary

The Poisson boundary of an (infinite) graph G consists of

- a measurable space $(\mathcal{P}_G, \Sigma)$, and
- a family of probability measures $\{\nu_z, z \in V_G\}$,

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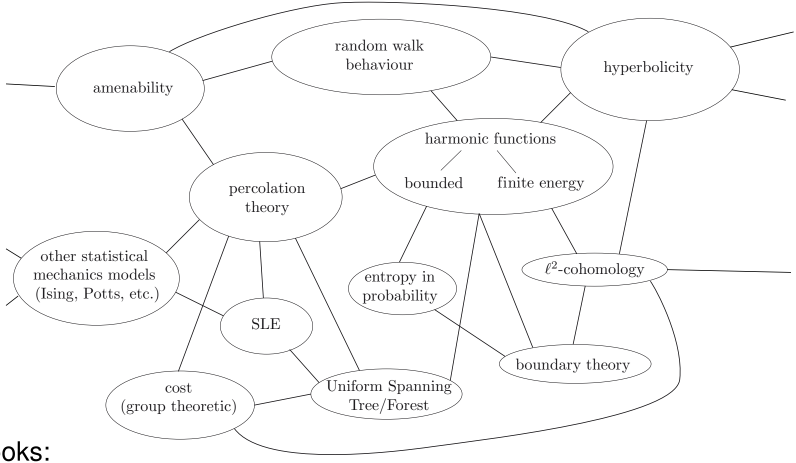
- this $\hat{h} \in L^\infty(\mathcal{P}_G)$ is unique up to modification on a null-set;
- conversely, for every $\hat{h} \in L^\infty(\mathcal{P}_G)$ the function $z \mapsto \int_{\mathcal{P}_G} \hat{h}(\eta) d\nu_z(\eta)$ is bounded and harmonic.

i.e. there is Poisson-like formula establishing an isometry between the Banach spaces $H^\infty(G)$ and $L^\infty(\mathcal{P}_G)$.

Selected work on the Poisson boundary

- Introduced by Furstenberg to study semi-simple Lie groups [*Annals of Math. '63*]
- Kaimanovich & Vershik give a general criterion using the entropy of random walk [*Annals of Probability '83*]
- Kaimanovich identifies the Poisson boundary of hyperbolic groups, and gives general criteria [*Annals of Math. '00*]

The context



Textbooks:

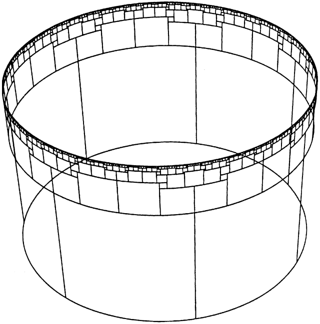
[*Woess: Random Walks on Infinite Graphs and Groups*]

[*Lyons & Peres: Probability on Trees and Networks*]

The theorem

Theorem (G '12)

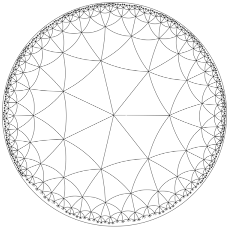
For every bounded degree graph admitting a square tiling, the Poisson boundary coincides with C.

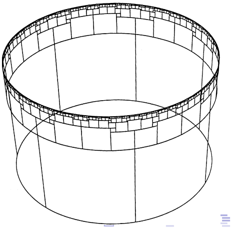


Probabilistic interpretation of the tiling

Lemma (G '12)

Let C be a 'horizontal' circle in the tiling T of G, and let B the set of points of G at which C 'dissects' T. Then the widths of the points of B in T coincide with the probability distribution of the first visit to B by brownian motion on G starting at o.





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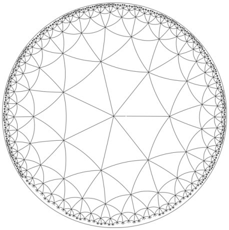

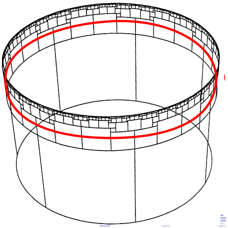

Probabilistic interpretation of the tiling

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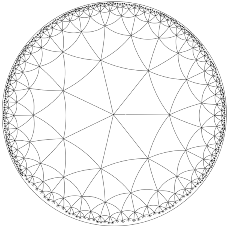
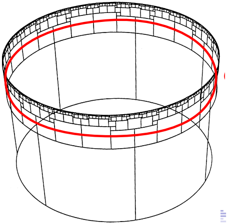
For every 'meridian' M in T, the probability that brownian motion on G starting at o will 'cross' M clockwise equals the probability to cross M counter-clockwise.





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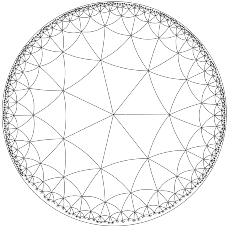
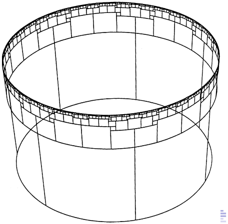




Here come some
'geometric' random graphs

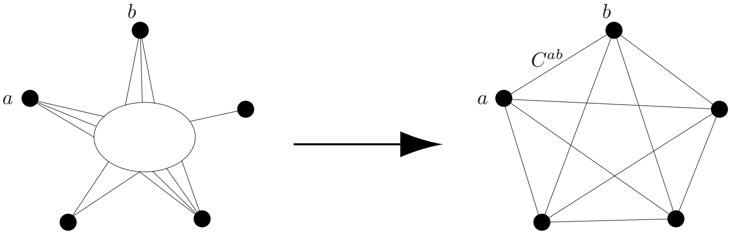
Here come some
‘geometric’ random graphs
... from groups

Electrical Network Reduction

Theorem

Let N be an electrical network and B its set of external nodes. Then there is an equivalent network with vertex set B in which each edge (a, b) has conductance

$$C_{eff}(a, b) = d(a)\mathbb{P}_a(b).$$



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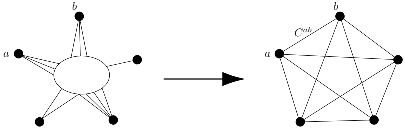

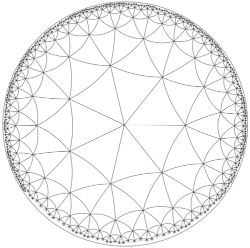

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Energy $E(u)$ $:= \sum_{x \sim y} (u(x) - u(y))^2$

The Effective Conductance Measure

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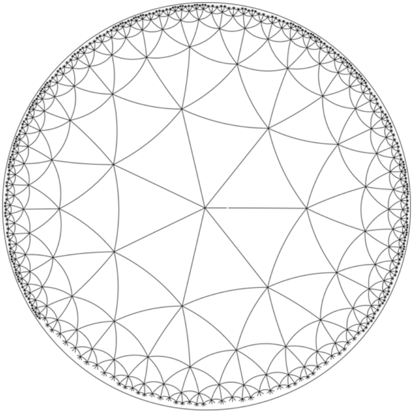

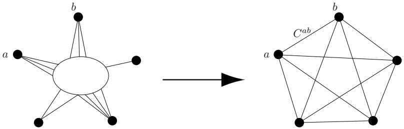

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How much do they depend on the choice of the (finite) generating set?

How are the geometric or algebraic properties of the group reflected in the graph-theoretic or geometric properties of the typical random graph?

Summary