The planar cubic Cayley graphs

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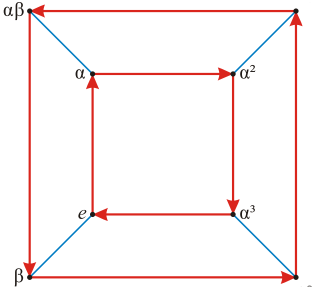
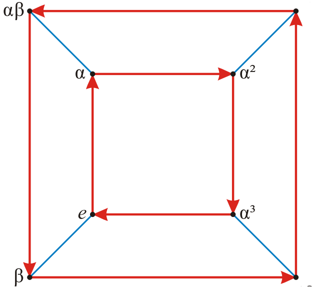
Paris, 17.02.11

Cayley graphs



$$\langle \alpha, \beta \mid, \beta^2, \alpha^4, (\alpha\beta)^2 \rangle$$



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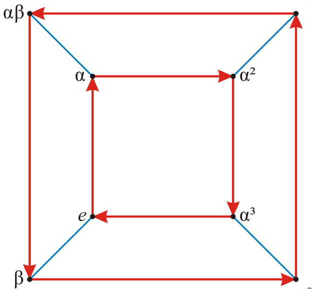

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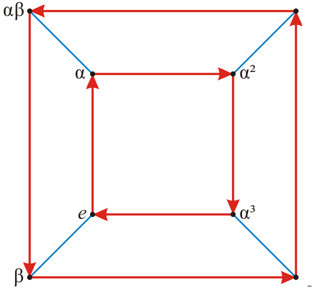

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- for every $g \in \Gamma$ and $s \in \{a, b, c, \ldots\}$, put in an edge:

$$g \xrightarrow{s} gs$$

Theorem (Sabidussi's Theorem)

A properly *edge-coloured digraph is a Cayley graph iff for every $x, y \in V(G)$ there is a colour-preserving automorphism mapping x to y.*

properly edge-coloured := no vertex has two incoming or two outgoing edges with the same colour

---

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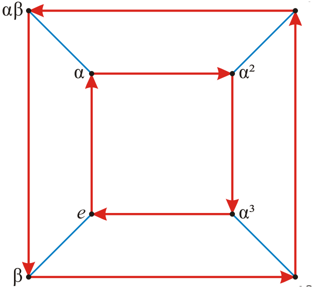
Charactisation of the finite planar groups

Theorem (Maschke 1886)

Every finite planar group is a group of isometries of S^2.



planar group := a group having at least 1 planar Cayley graph.

The Cayley complex

Let $\Gamma = \langle a, b, c, \ldots \mid R_1, R_2 \ldots \rangle$ be a group presentation. Define the corresponding simplified Cayley complex $CC \langle a, b, c, \ldots \mid R_1, R_2 \ldots \rangle$ by:

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Given a planar Cayley graph, can you find a presentation in which the relators induce precisely the face boundaries?

Yes!

Proving Maschke's Theorem

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Theorem (Whitney '32)

Let G be a 3-connected plane graph. Then every automorphism of G extends to a homeomorphism of the sphere.

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Let X be the corresponding simplified Cayley complex.

X is homeomorphic to S^2

Since $\Gamma(G)$ acts on X, we have:

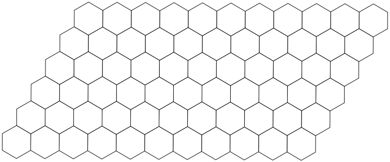
Theorem (Maschke 1886)

Every finite planar group is a group of homeomorphisms of S^2.

The 1-ended planar groups

Theorem ((classic) Macbeath, Wilkie, ...)

Every 1-ended planar Cayley graph corresponds to a group of isometries of $\mathbb{R}^2$ or $\mathbb{H}^2$.



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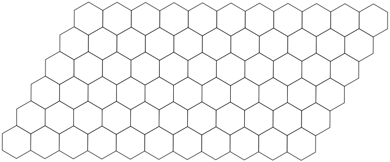

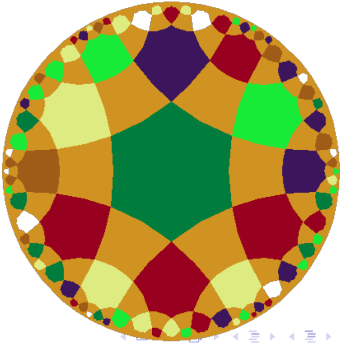

Planar groups and fundamental groups of surfaces

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Theorem (G '10)

A group has a planar simplified Cayley complex if and only if it has a VAP-free Cayley graph.

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Classification of the cubic planar Cayley graphs

Theorem (G '10)

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Let G be a planar cubic Cayley graph. Then G is colour-isomorphic to precisely one element of **the list**.
Conversely, for every element of the list and any choice of parameters, the corresponding Cayley graph is planar.

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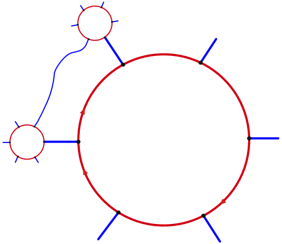

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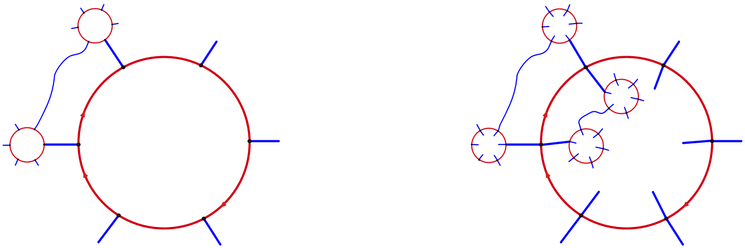

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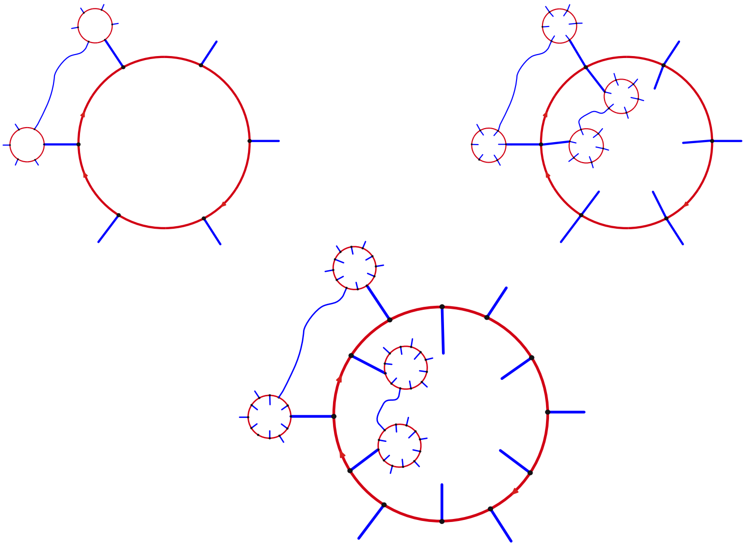

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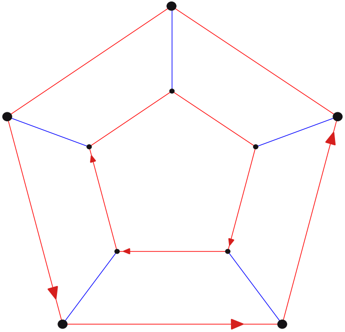

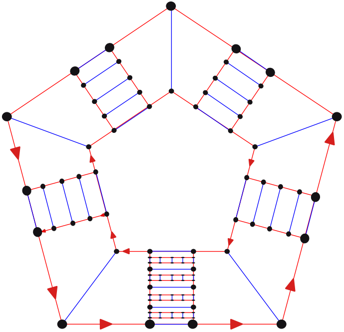

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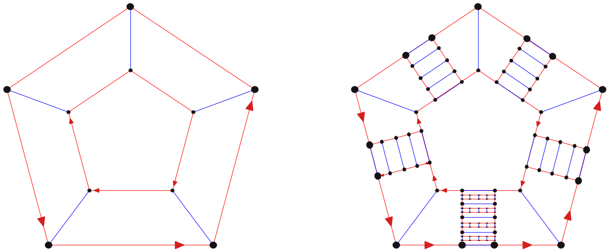

Corollary (G '10)

Every planar cubic Cayley graph has an almost planar Cayley complex.

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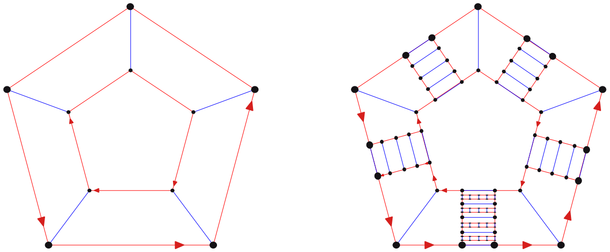

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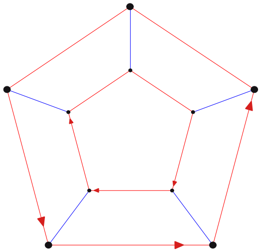

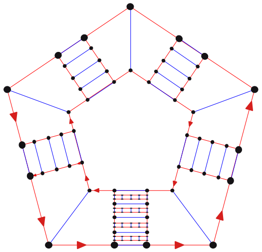

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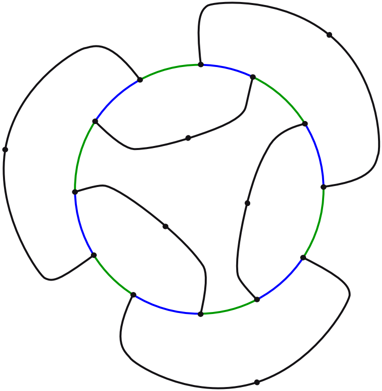

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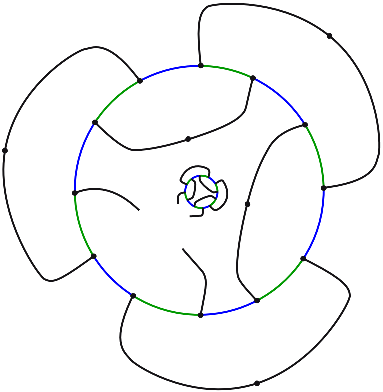

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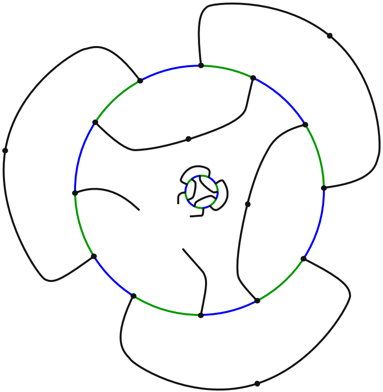
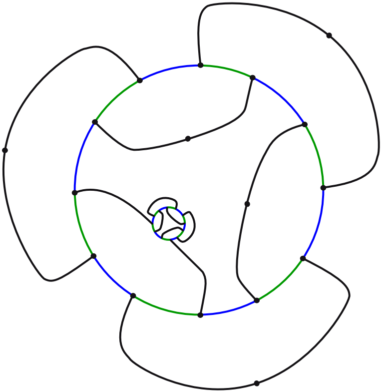
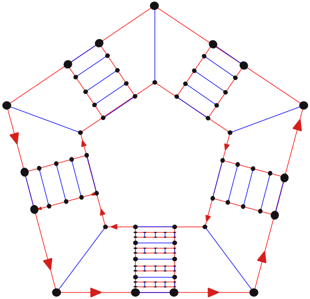




Spot the societies!



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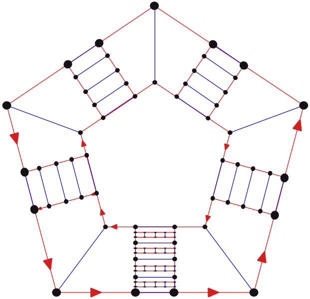
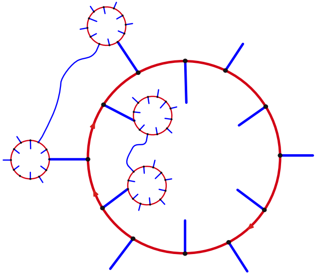
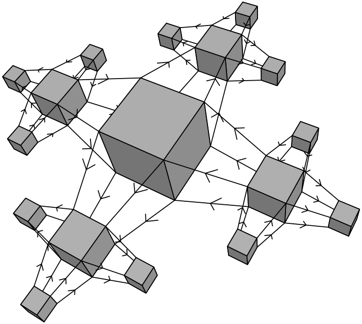




Stallings' Theorem

Theorem (Stallings '71)

Every group with >1 ends can be written as an HNN-extension or an amalgamation product over a finite subgroup.



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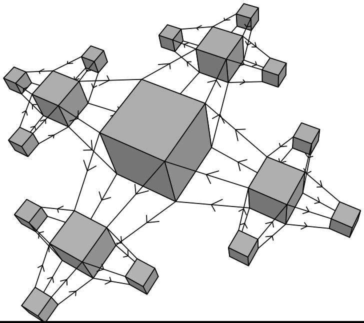

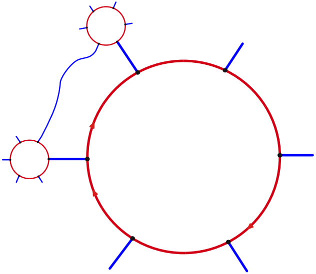

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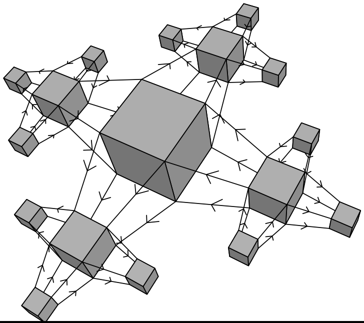
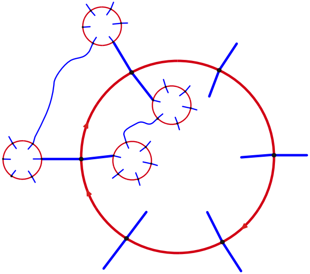
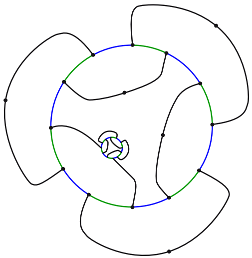




Group splittings by topological minors



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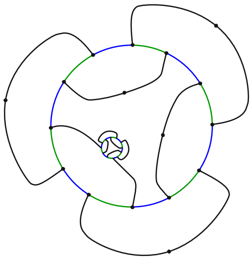

Conjecture

Let $G = Cay(\Gamma, S)$ be a Cayley graph with > 1 ends. Then there is a non-trivial splitting of G as a union of subdivisions of Cayley graphs.

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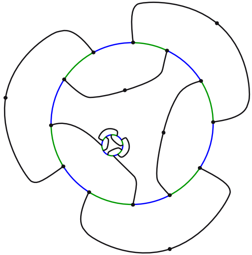

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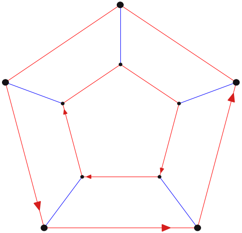

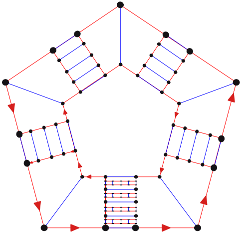

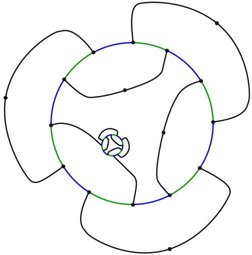

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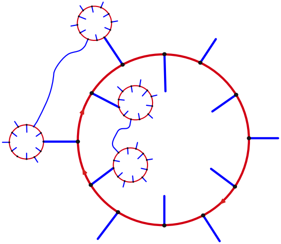

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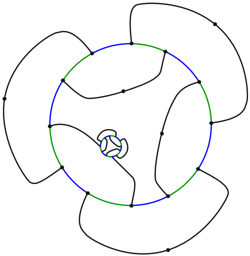

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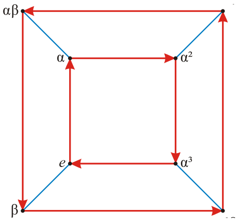
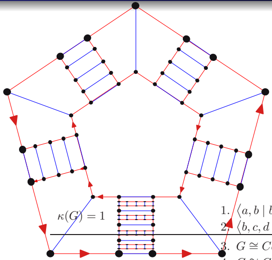
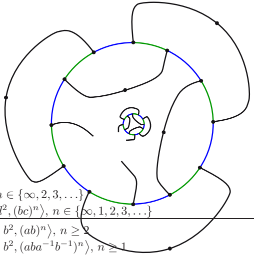
Corollary (G '10)

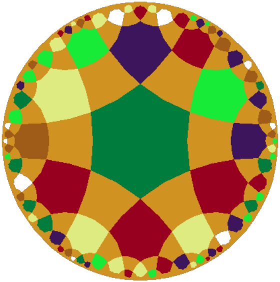
True for planar cubic Cayley graphs.

Summary









$\kappa(G) = 1$	1. $\langle a, b \mid b^2, a^n \rangle$, $n \in \{\infty, 2, 3, \ldots\}$ 2. $\langle b, c, d \mid b^2, c^2, d^2, (bc)^n \rangle$, $n \in \{\infty, 1, 2, 3, \ldots\}$
$\kappa(G) = 2$	3. $G \cong Cay\langle a, b \mid b^2, (ab)^n \rangle$, $n \geq 2$ 4. $G \cong Cay\langle a, b \mid b^2, (aba^{-1}b^{-1})^n \rangle$, $n \geq 1$ 5. $G \cong Cay\langle a, b \mid b^2, a^4, (a^2b)^2 \rangle$ 6. $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (bc)^2, (bcd)^m \rangle$, $m \geq 2$ 7. $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (bc)^{2n}, (cbcd)^m \rangle$, $n, m \geq 2$ 8. $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (bd)^m \rangle$, $n, m \geq 2$ 9. $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (b(cb)^n d)^m \rangle$, $n \geq 1, m \geq 2$ 10. $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (bcbd)^m \rangle$, $m \geq 1$ 11. $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, cd \rangle$, $n \geq 1$
$\kappa(G) = 3$, G is 1-ended or finite, with two generators	12. $G \cong Cay\langle a, b \mid b^2, a^n, (ab)^m \rangle$, $n \geq 3$, $m \geq 2$ 13. $G \cong Cay\langle a, b \mid b^2, a^n, (aba^{-1}b)^m \rangle$, $n \geq 3, m \geq 1$ 14. $G \cong Cay\langle a, b \mid b^2, (a^2b)^m \rangle$, $m \geq 1$ 15. $G \cong Cay\langle a, b \mid b^2, (a^2ba^{-2}b)^m \rangle$, $m \geq 1$
$\kappa(G) = 3$, G is 1-ended or finite, with three generators	16. $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (bcd)^n \rangle$, $n \geq 1$ 17. $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (cbcdbd)^n \rangle$, $n \geq 1$ 18. $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (bdcd)^m \rangle$, $n \geq 2, m \geq 1$ 19. $G \cong Cay\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (cd)^m, (db)^p \rangle$, $n, m, p \geq 2$
$\kappa(G) = 3$, G is multi-ended, with two generators	20. $G \cong Cay\langle a, b \mid b^2, (a^2b)^m; a^{2n} \rangle$, $n \geq 3, m \geq 2$ 21. $G \cong Cay\langle a, b \mid b^2, (a^2ba^{-2}b)^m; a^{2n} \rangle$, $n \geq 3, m \geq 1$ 22. $G \cong Cay\langle a, b \mid b^2, a^2ba^{-2}b; (baba^{-1})^n \rangle$, $n \geq 2$ 23. $G \cong Cay\langle a, b \mid b^2, (a^2ba^{-2}b)^m; (baba^{-1})^n \rangle$, $n, m, p \geq 2$ 24. $G \cong Cay\langle a, b \mid b^2, (a^2b)^2; (ab)^{2m} \rangle$, $m \geq 2$ 25.