Hyperbolic graphs, fractal boundaries, and graph limits

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Technische Universität Graz

Oberwolfach, 25.2.2010

Applications of infinite graphs

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- Finite graph theorists?

Ends

end: equivalence class of rays

two rays are equivalent if no finite vertex set separates them

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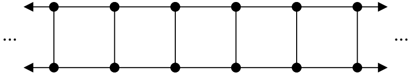
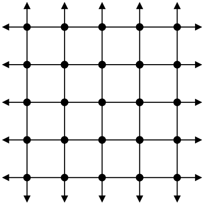
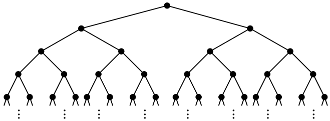


two ends



one end

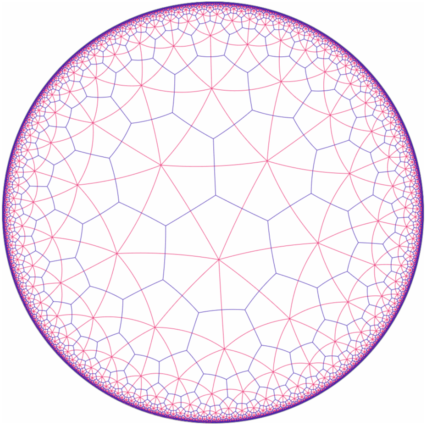


uncountably many ends

An interesting boundary

end: equivalence class of rays

two rays are equivalent if no finite vertex set separates them



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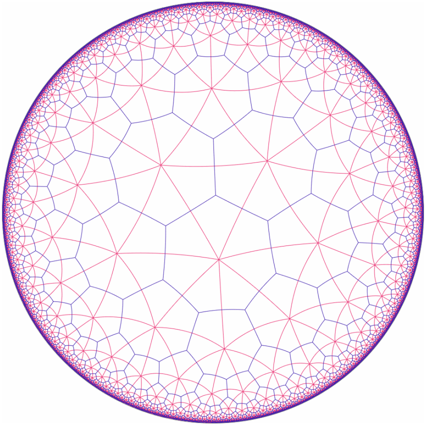

An 1-ended graph...

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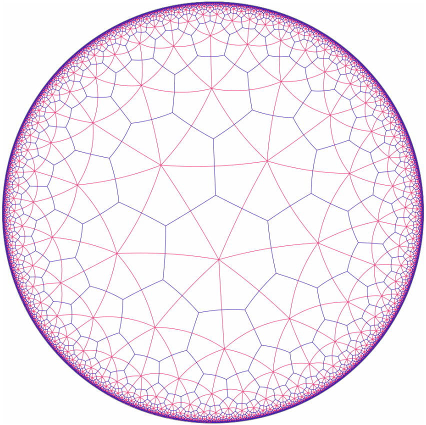

An 1-ended graph...

...with a large "hyperbolic boundary"

Gromov's hyperbolic graphs and groups

Definition *(Gromov '87)*: A graph is hyperbolic if all its geodetic triangles are δ-thin for some fixed $\delta \in \mathbb{N}$.

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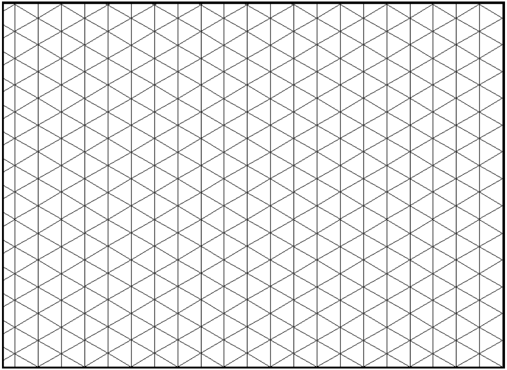

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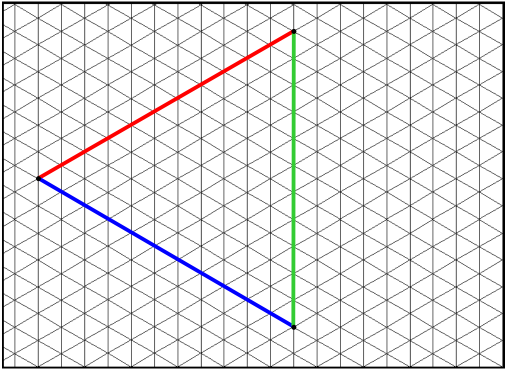

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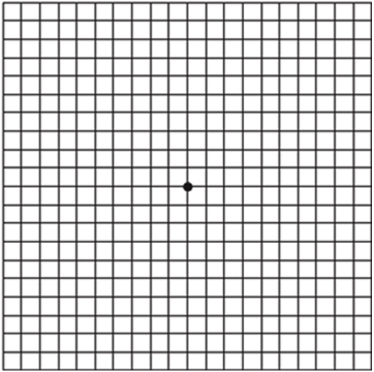

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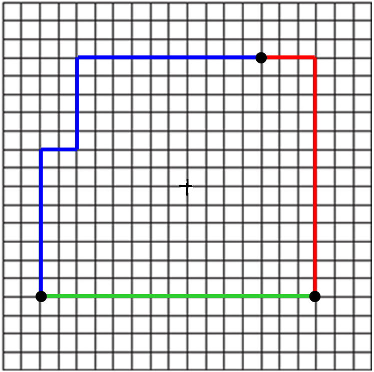

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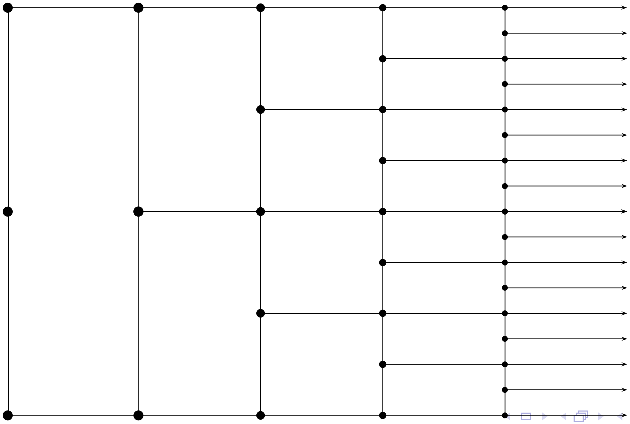

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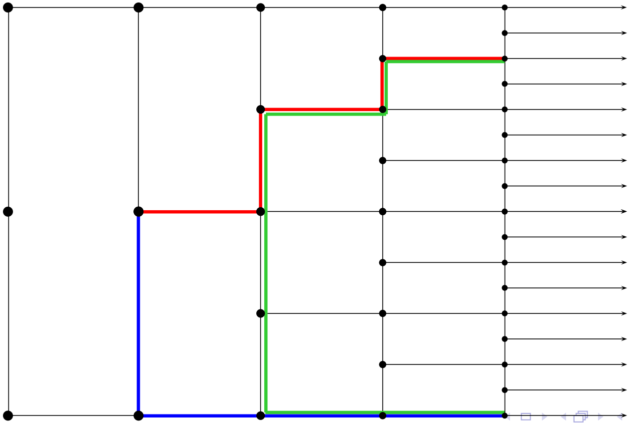

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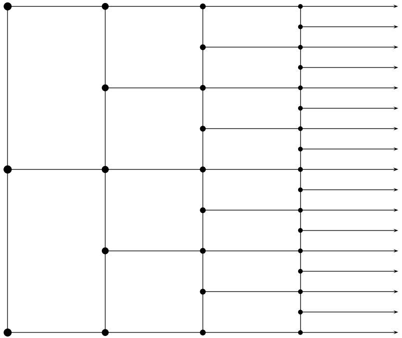

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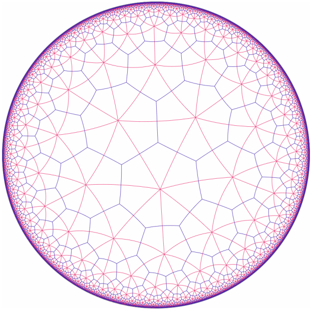

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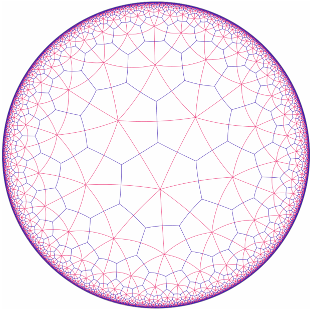

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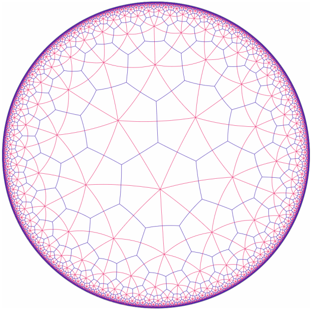

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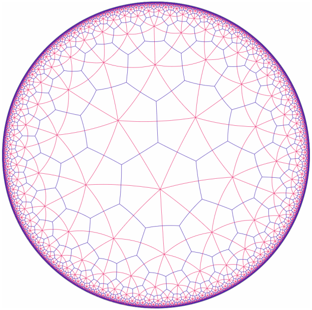

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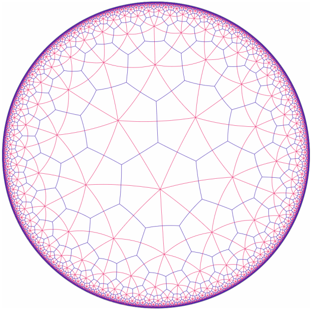

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Metric on $\partial^h G$:
$d_v([\sigma], [\tau]) := exp(-|\text{max common subpath of } \sigma, \tau|)$ *(roughly)*

hyperbolic boundary and topology

Theorem (Gromov '87)

Every compact metric space is isometric to the hyperbolic boundary of some hyperbolic graph

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... but can it be the limit of a sequence of finite graphs?

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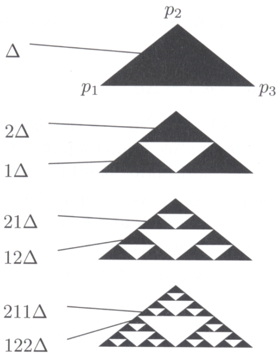
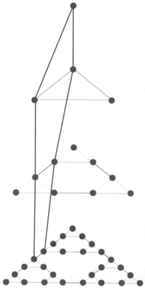
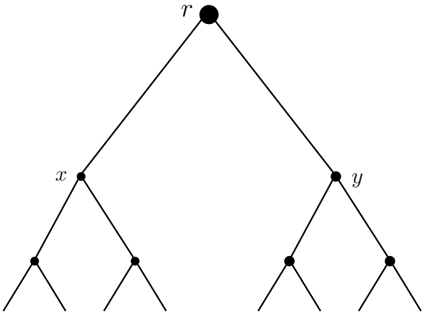




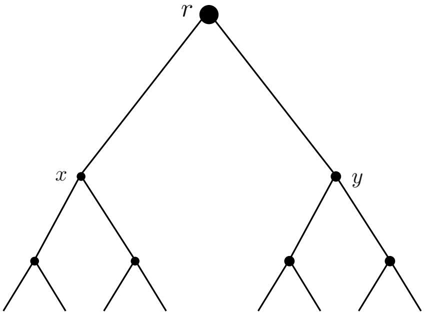
Kaimanovich's construction of the Sierpinski gasket as the hyperbolic boundary of a graph

The Basilica group



The Basilica group

The Basilica group B acts by automorphisms on ———>



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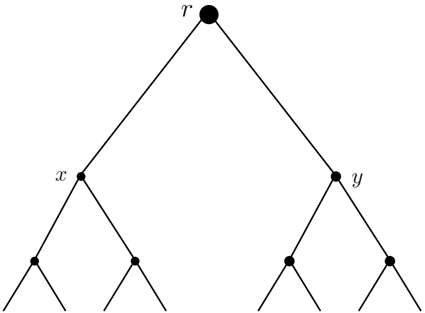

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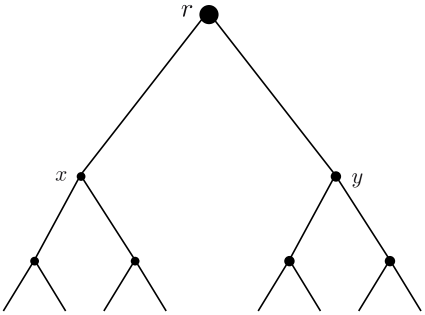

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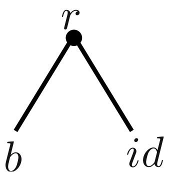

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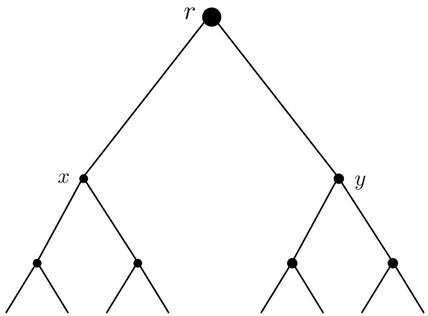

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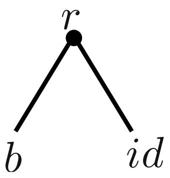
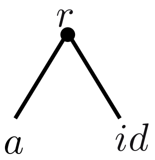
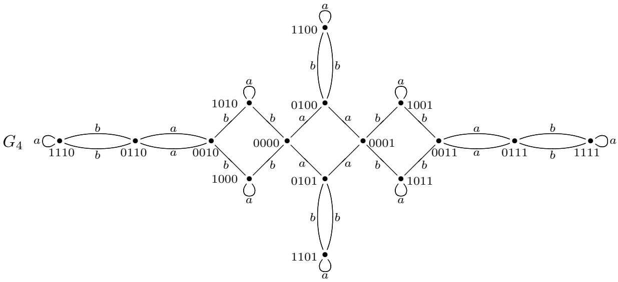


$b := x \leftrightarrows y +$





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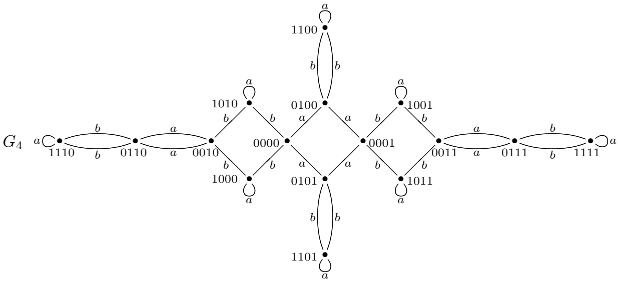

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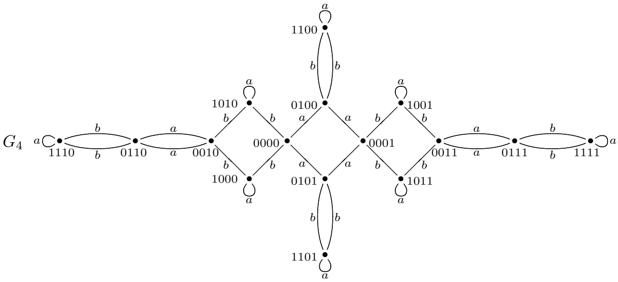

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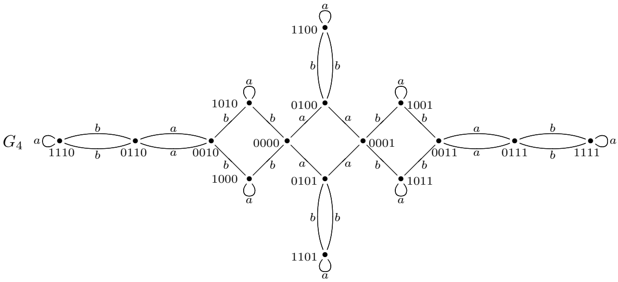

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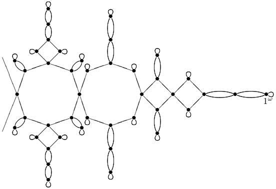

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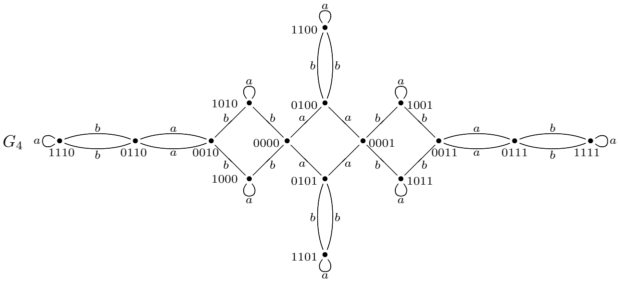

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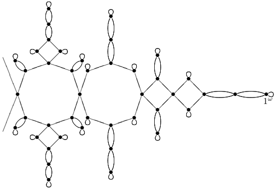

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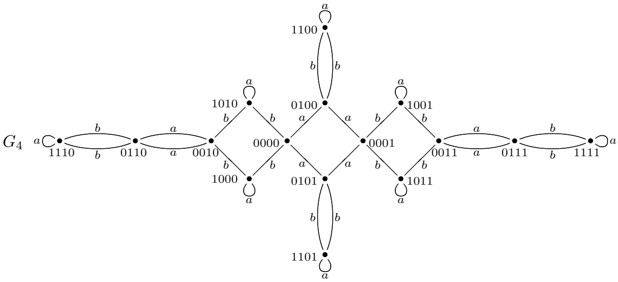

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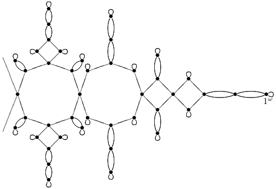

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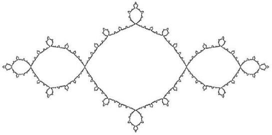

The Julia set of $z^2 - 1$

The 2nd answer

Theorem (Nekrashevych '05)

The hyperbolic boundary of the self-similarity graph of the Basilica group is $J(z^2 - 1)$

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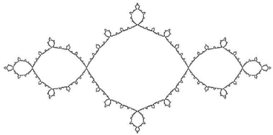

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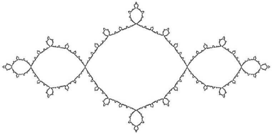

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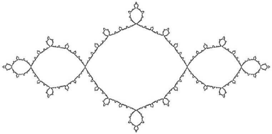

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Nekrashevych proved that certain groups are non-isomorphic by comparing the boundaries of their self-similarity graphs

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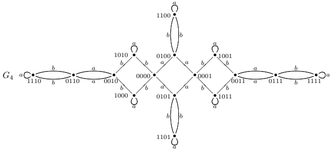

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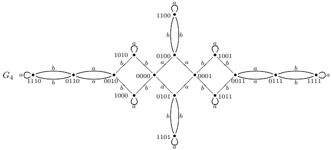

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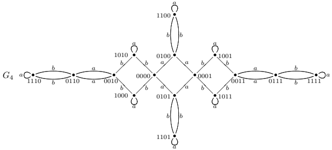

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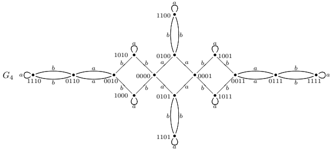

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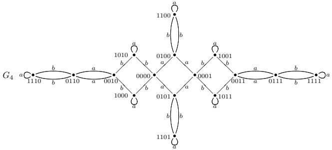

Theorem (D' Angeli, Donno, Matter and Nagnibeda '09)

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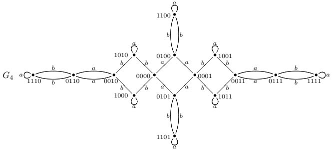

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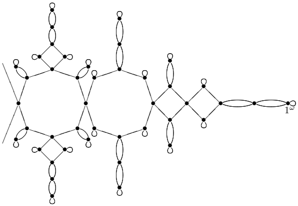

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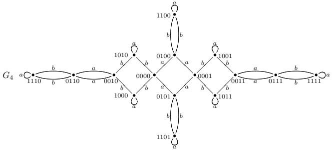

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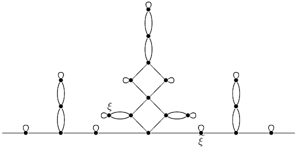

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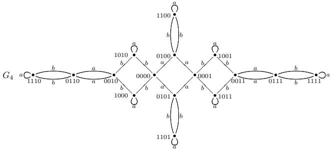

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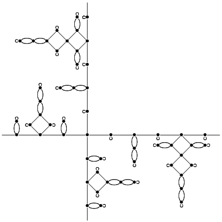

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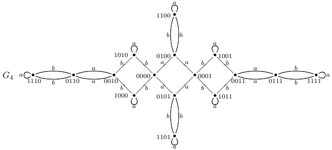
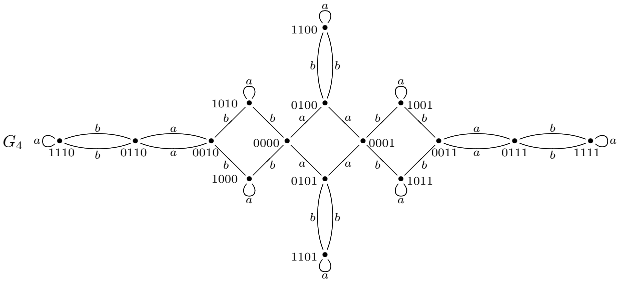


Theorem (D' Angeli, Donno, Matter and Nagnibeda '09)

The limit graph has 1, 2, or 4 ends.

Limit graphs have been used to study the Abelian Sandpile Model.



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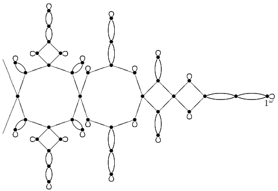

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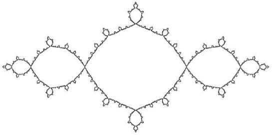
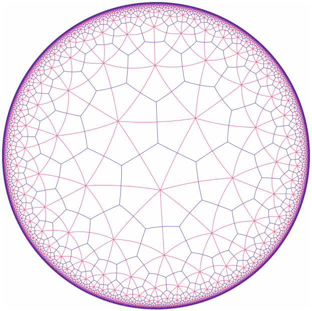


The Julia set of $z^2 - 1$

Definition of hyperbolic boundary revisited



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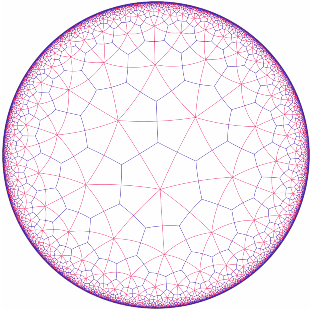

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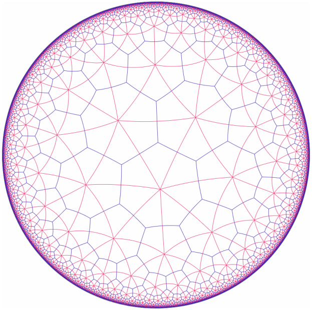

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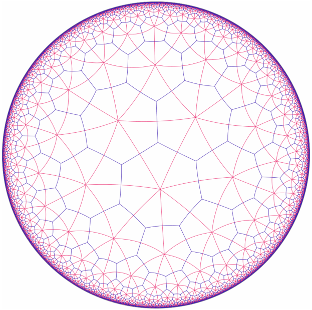

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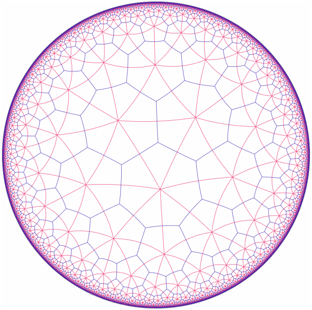

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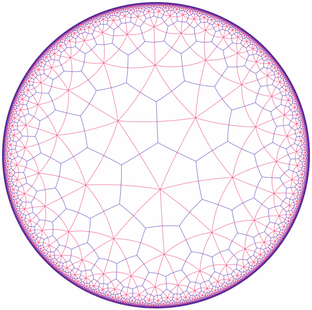

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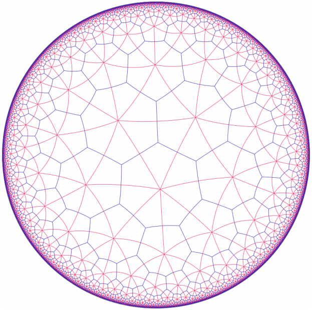

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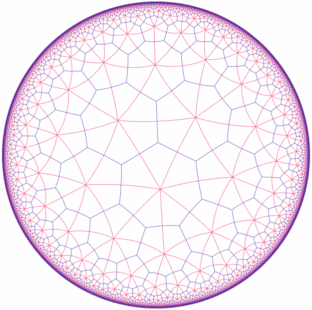

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Holds for non-hyperbolic graphs too, and no "spherical symmetry" needed.

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All above authors “discovered” $|G|_\ell$ independently!

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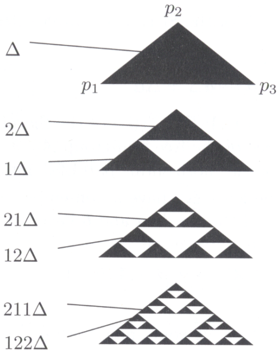
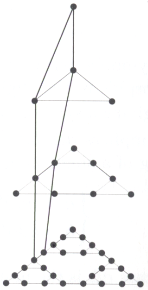
Can you use this to get topological results?

Hyperbolic boundary and topology

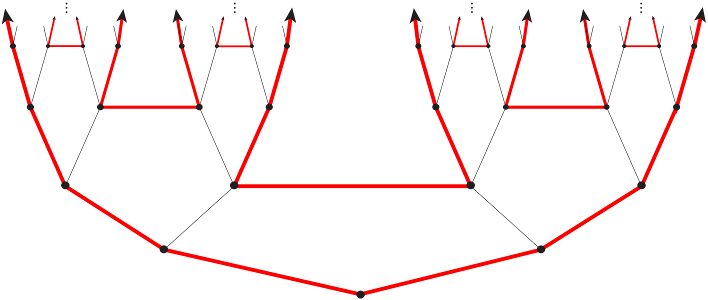




Kaimanovich's construction of the Sierpinski gasket as the hyperbolic boundary of a graph

Topological paths/circles in $|G|$

Circle:
A homeomorphic image of S^1 in $|G|$.



the wild circle of Diestel & Kühn

The Hahn-Mazurkiewicz Theorem

Theorem (The Hahn-Mazurkiewicz Theorem)

A Hausdorff space is a continuous image of the real unit interval iff it is a compact, connected, locally connected metrizable space.

Outlook

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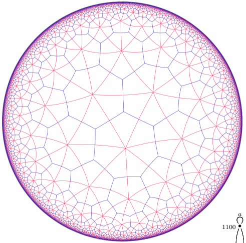
- Construct limits of finite random graphs and study phase transition

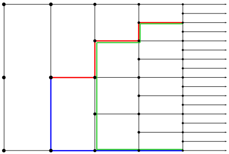
References

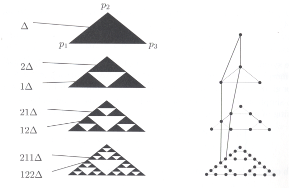
Further reading:

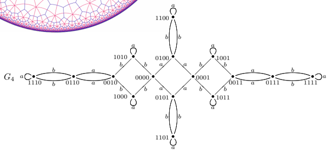
- Ends, $|G|$, etc.: Diestel, http://www.math.uni-hamburg.de/home/diestel/papers/TopSurvey.pdf
- Definitions and basic facts on hyperbolic graphs: H. Short, http://www.cmi.univ-mrs.fr/~hamish/
- Survey on hyperbolic boundaries of groups: Kapovich & Benakli (with 200 further references)
- Basilica group: Nagnibeda et. al., http://arxiv.org/abs/0911.2915
- Self-similarity graphs, Julia sets: Nekrashevych' book, http://www.ams.org/bookpages/surv-117/
- Survey on ℓ-TOP: Georgakopoulos, http://arxiv.org/abs/0903.1744

Summary









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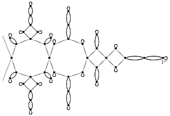

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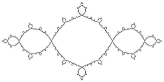

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