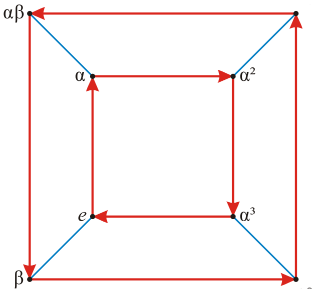
The planar cubic Cayley graphs

Agelos Georgakopoulos

Technische Universität Graz

13.9.10

Cayley graphs



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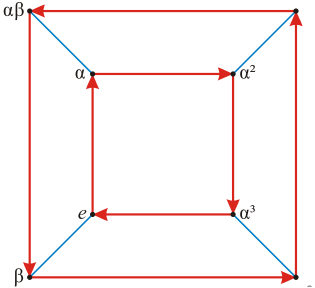

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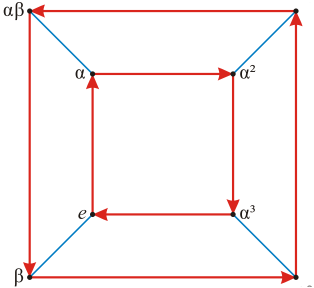

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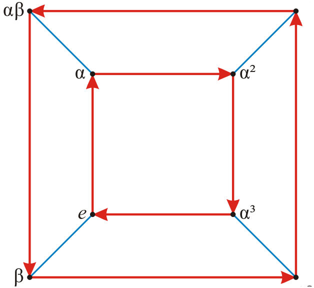

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- for every $g \in \Gamma$ and $s \in \{a, b, c, \ldots\}$, put in an edge:

$$g \xrightarrow{s} gs$$

Sabidussi's Theorem

Theorem (Sabidussi's Theorem)

An edge-coloured digraph is a Cayley graph iff for every $x, y \in V(G)$ there is a colour-preserving automorphism mapping x to y.

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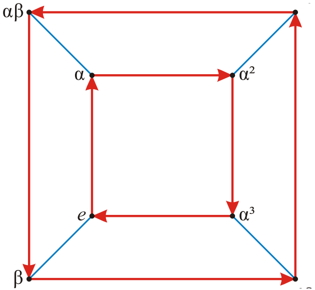
Charactisation of the finite planar groups

Theorem (Maschke 1886)

Every finite planar group is a group of isometries of S^2.



The Cayley complex

Let $\Gamma = \langle a, b, c, \ldots \mid R_1, R_2 \ldots \rangle$ be a group presentation. Define the corresponding Cayley complex $CC \langle a, b, c, \ldots \mid R_1, R_2 \ldots \rangle$ by:

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Given a planar Cayley graph, can you find a presentation in which the relators induce precisely the face boundaries?

Yes! :

Theorem (Whitney)

Let G be a 3-connected plane graph. Then every automorphism of G extends to a homeomorphism of the sphere.

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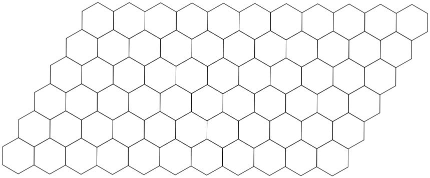
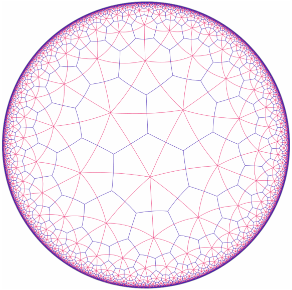
The 1-ended planar groups

Theorem

Every 1-ended planar group is a group of isometries of $\mathbb{R}^2$ or $\mathbb{H}^2$.





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What about the non VAP-free ones?

Theorem (Thomassen '80)

Let G be an infinite 2-connected graph. Then $\mathcal{C}_{fin}(G)$ has a 2-basis if and only if G is VAP-free planar.

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... and what about all the classical theory?

Classification of the cubic planar Cayley graphs

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Let G be a planar cubic Cayley graph. Then G is colour-isomorphic to precisely one element of **the list***.*

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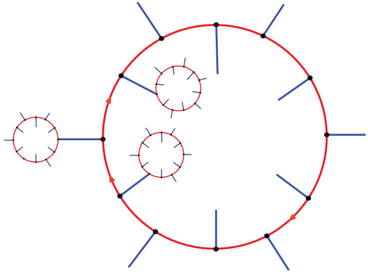
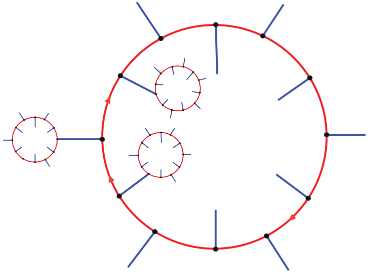
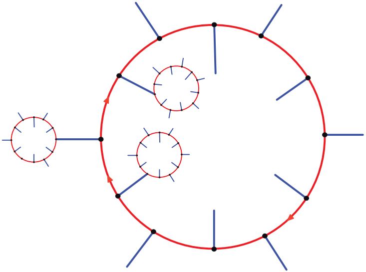
Conversely, for every element of the list and any choice of parameters, the corresponding Cayley graph is planar.





$\langle a, b \mid b^2, a^{2n}, (a^2 b)^m \rangle$



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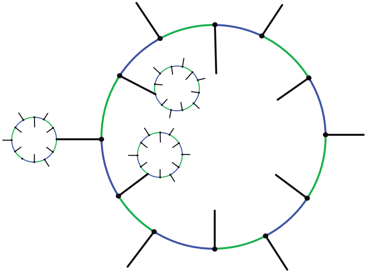
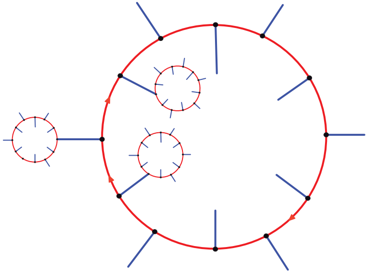


$\langle b, c, d \mid b^2, c^2, d^2, (bcd)^m; (bc)^n \rangle$



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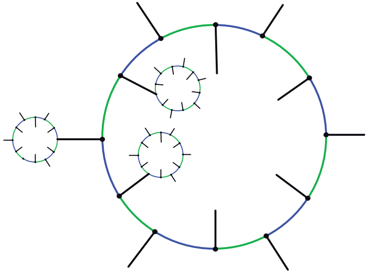

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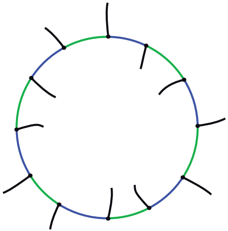

Planar presentations of groups

Corollary

Every planar cubic Cayley graph has a **planar presentation**.

... a presentation of a planar group is called *planar with respect to an assignment f* of spin flags if no two relations cross in *f*. It is called just planar if there is an assignment of spin flags with respect to which it is planar.

Stallings' Theorem

Theorem (Stallings)

Every group with >1 ends can be written as an amalgamation product or an HNN-extension over a finite subgroup.

Group splittings by contractible presentations

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*Let $G = Cay\langle s_1, \ldots, s_k \mid \mathcal{R}\rangle$ be a Cayley graph with >1 end. Then G has a k-**contractible** presentation.*

... a presentation $G = Cay\langle s_1, \ldots, s_k \mid \mathcal{R}'\rangle$ is k-contractible, if there are words $S_1, \ldots, S_k$ with letters $s_1, \ldots, s_k$, such that every relator in $\mathcal{R}'$ is a concatenation of the words S_i.

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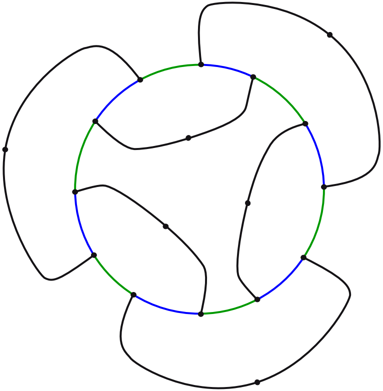
Corollary

True for planar cubic Cayley graphs.

Cayley graphs without finite face boundaries



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