Random walks on graphs: a survey

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Which problem is harder?

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- mathematicians have studied e.g. cover time of Brownian motion on Riemannian manifolds [Dembo, Peres, Rosen & Zeitouni]

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A lot of questions arise as to more exact bounds for cc

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Proof: Both functions p_r and $v(r)$ are harmonic, i.e.

$$h(r) = \frac{1}{d(r)} \sum_{w \sim r} h(w)$$

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By uniqueness of harmonic functions, p must coincide with v.

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The commute time formula (Chandra et. al. '89):

$$k(x,y) := H_{xy} + H_{yx} = 2mr(x,y)$$

Cover Cost and the Wiener Index

Theorem (G & S. Wagner '12)

For every tree T, and every $r \in V(T)$, we have

$$CC(r) + D(r) = 2W(T)$$

...where $D(r) := \sum_{y \in V(T)} d(r, y)$ is the *centrality* of r and $W(T) := \frac{1}{2} \sum_{x,y \in V(T)} d(x, y)$ is the *Wiener Index* of T.

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in other words:
$\sum_{y \in V(T)} \left(H_{ry} + d(r, y) \right) = 2W(T) := \sum_{x,y \in V(T)} d(x, y).$

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Let $CC_d(r) := \sum_{y \in V(T)} d(y) H_{ry} / 2m$.

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Let $CC_d(r) := \sum_{y \in V(T)} d(y) H_{ry} / 2m$. Then

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and λ ranges over the eigenvalues of the transition matrix.

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[*L. Lovász: "Random Walks on Graphs: A Survey", '93.*]

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[*H. Chen and F. Zhang: "Resistance distance and the normalized Laplacian spectrum", '07*]

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Question: Is there a 'reverse' Kemeny constant?

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Problem (Aldous ’89)

Let G be a graph such that the (random) time of the first return to x by random walk from x has the same distribution for every $x \in V$.
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Problem

Let G be a graph such that $H_{xy} = H_{yx}$ for every $x, y \in V(G)$. Does G have to be regular?

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Theorem

The following are equivalent for every graph G:

1. $H_{xy} = H_{yx}$ *for every* $x, y \in V(G)$;
2. *The hitting time from a random enpoint of a random edge to* x *is independent of* x;
3. *The (weighted) resistance-centrality* $R_d(x) := \frac{\sum_{y \in V(G)} d(y) r(x,y)}{2m}$ *is independent of* x.

Problem

Let G be a graph satisfying one of the above. Does G have to be regular?

Vertex orderings - General graphs

Theorem (G & Wagner '12)

For every graph G, and every vertex $x \in V(G)$, we have

$$\begin{aligned} CC(x) &= mR(x) - \frac{n}{2}R_d(x) + K'_d(G), \\ RC(x) &= mR(x) + \frac{n}{2}R_d(x) - K'_d(G), \\ RC_d(x) &= 2mR_d(x) - K_d(G), \text{ and} \\ CC_d(x) &= K_d(G). \end{aligned}$$

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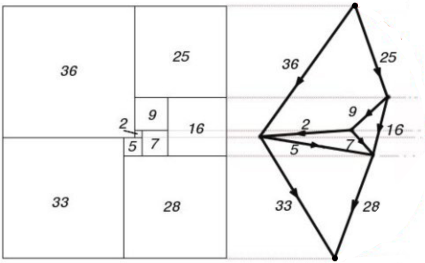
Square Tilings

Theorem (Brooks, Smith, Stone & Tutte '40)

There is a correspondence between finite planar graphs and tilings of rectangles by squares.



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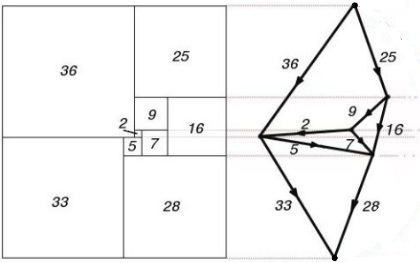
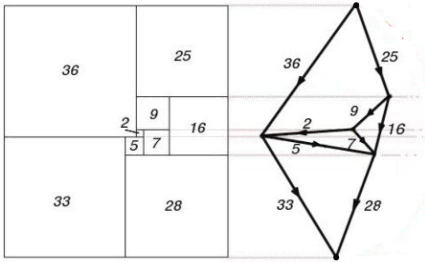


[*Brooks, Smith, Stone & Tutte: "Determinants and current flows in electric networks." Discrete Math. '92*]

Properties of square tilings



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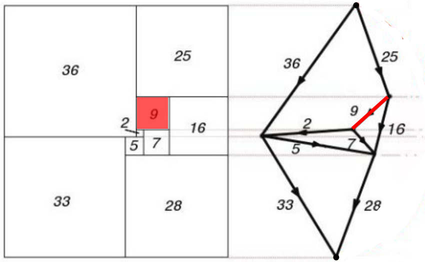

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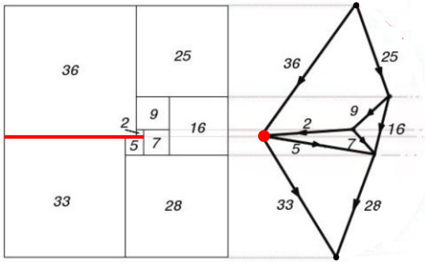

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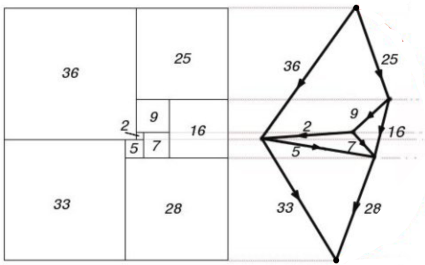

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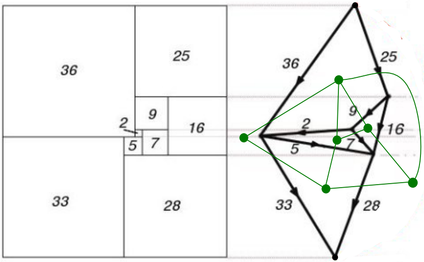

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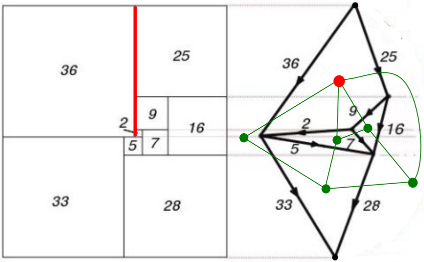

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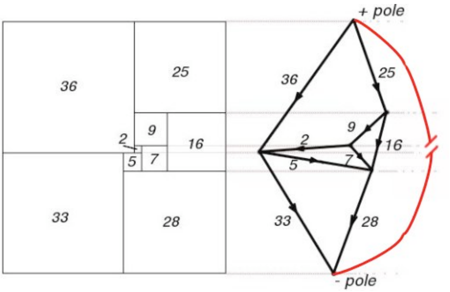
The construction of square tilings



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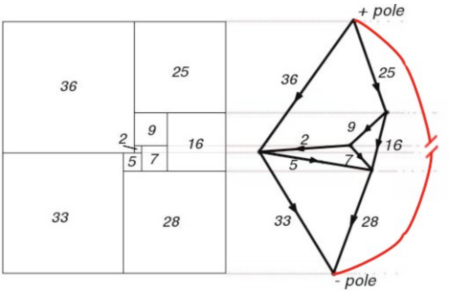

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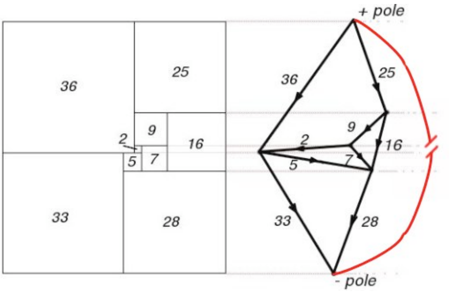

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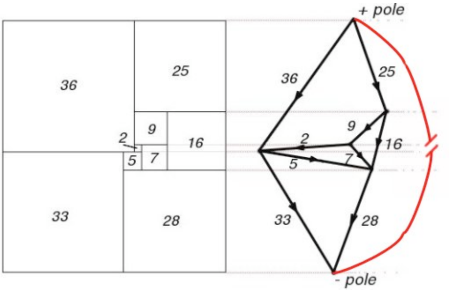

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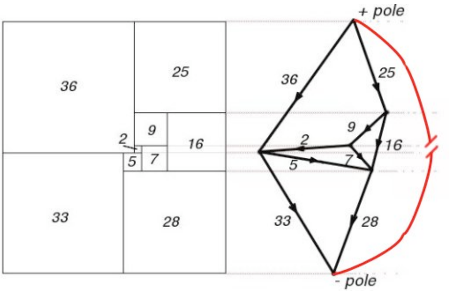

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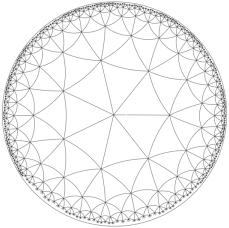
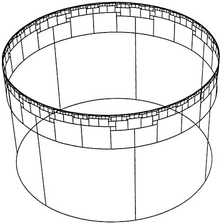
- let the square corresponding to edge e have side length the flow $i(e)$;
- place each vertex x at height equal to the potential $h(x)$;
- use a duality argument to determine the width coordinates.

The construction of square tilings

Square tilings can be generalised to all finite planar graphs, and even beyond



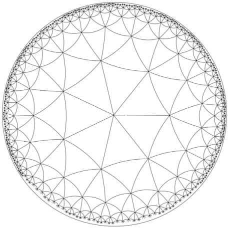


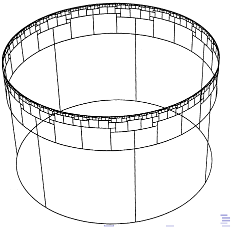
[*Benjamini & Schramm: "Random Walks and Harmonic Functions on Infinite Planar Graphs Using Square Tilings" Ann. Probab., '96*]

Probabilistic interpretation of the tiling's geography

Lemma (G '12)

Let C be a **'parallel circle'** *in the tiling T of G, and let B the set of points of G at which C 'dissects' T. Then the* **widths** *of the points of B in T* **coincide with the probability distribution** *of the first visit to B by brownian motion on G starting at p.*





Probabilistic interpretation of the tiling's geography

Lemma (G '12)

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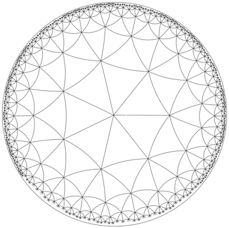
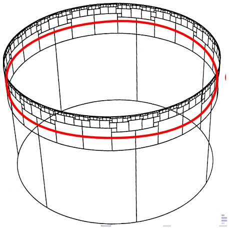
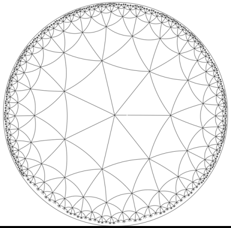


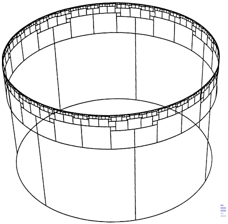


Probabilistic interpretation of the tiling's geography

Lemma

For every 'meridian' M in T, the expected net number of crossings of M by brownian motion on G starting from p is 0.





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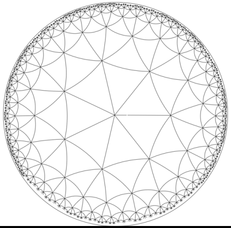
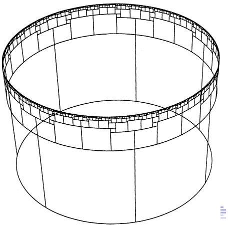
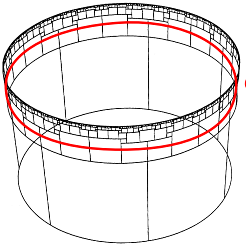




Sketch of proof of the Riemann Mapping Theorem

- Think of Ω as a metal plate;



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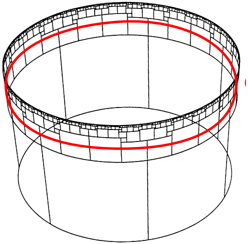

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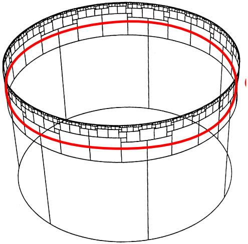

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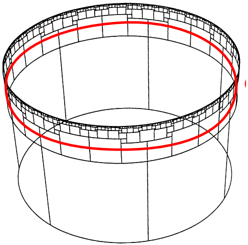

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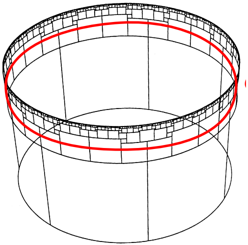

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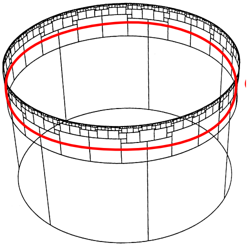

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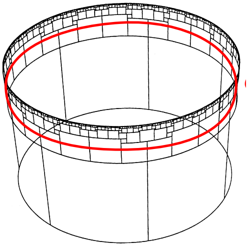

-Map equipotential curves into corresponding concentric circles;
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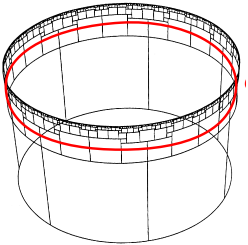

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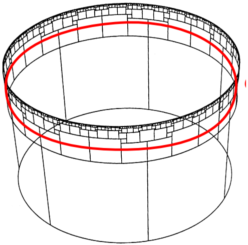

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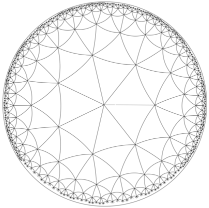
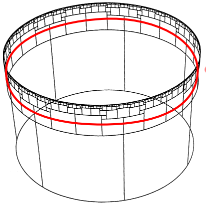
Harmonic functions on an infinite graph via a Poisson-like integral

Question (Benjamini & Schramm '96)

Does the Poisson boundary of every graph as above coincide with the boundary of its square tiling?





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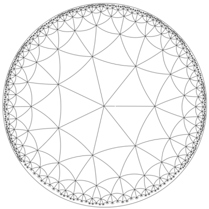

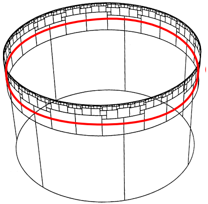

Theorem (G ’12)

Yes!

Summary

$$CC(r)+D(r) = 2W(T)$$

$$CC_d(r) = K_d(G)$$