From mafia expansion to analytic functions in percolation theory

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Joint work with John Haslegrave,
and with Christoforos Panagiotis

These slides are on-line

A "social" network evolves in
(continuous or discrete)
time according to the following rules

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Is $M(\lambda)$ finite or infinite?

It is finite almost surely

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Is its expected size finite or infinite?
finite in the synchronous case,
we don't know in the asynchronous case

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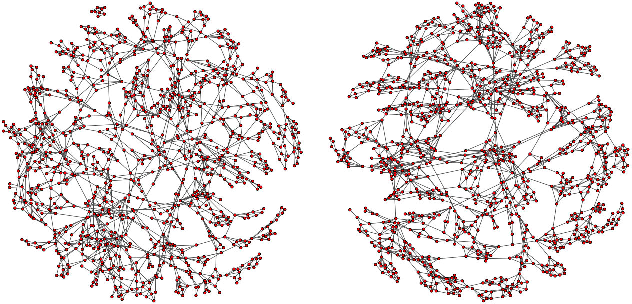
How does the expected size depend on λ?

Random Graphs from trees

Simulations by C. Moniz (Warwick).



The expected size of $M(\lambda)$

Let $\chi(\lambda) := \mathbb{E}(|M(\lambda)|)$

Theorem (G & Haslegrave '18+)

$$e^{c\lambda} \leq \chi(\lambda) \leq e^{e^{C\lambda}}$$

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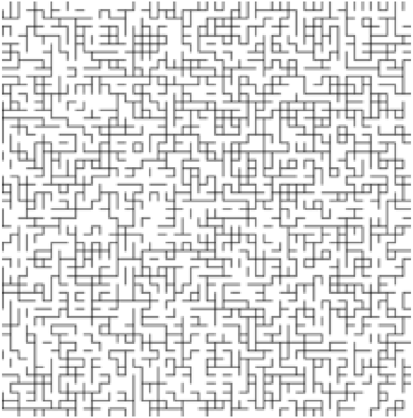
Is $\chi(\lambda)$ continuous in λ?

Percolation model



Bernoulli bond percolation on an infinite graph, i.e.

Each edge
-present with probability p,
and
-absent with probability $1 - p$
independently of other edges.

Percolation threshold:

$$p_c := \sup\{p \mid \mathbb{P}_p(\text{ component of } o \text{ is infinite }) = 0\}$$

Historical remarks on percolation theory

Classical era:
Introduced by physicists Broadbent & Hammersley '57

$p_c(\text{square grid}) = 1/2$ (Harris '59 + Kesten '80)

Many results and questions on phase transitions, continuity, smoothness etc. in the '80s:
Aizenman, Barsky, Chayes, Grimmett, Hara, Kesten, Marstrand, Newman, Schulman, Slade, Zhang ... (apologies to many!)

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Thought of as part of statistical mechanics

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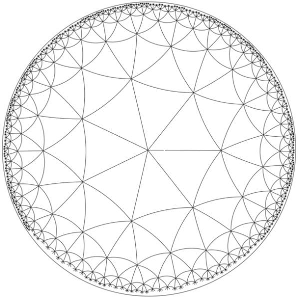
... for example, percolation can characterise amenability:

Theorem ($\Leftarrow$ Aizenman, Kesten & Newman '87,
$\Rightarrow$ Pak & Smirnova-Nagnibeda '00)

A finitely generated group is non-amenable iff it has a Cayley graph with $p_c < p_u$.



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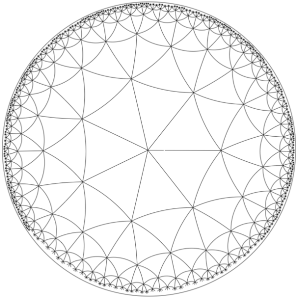
Theorem (Kesten '59)

A finitely generated Cayley graph is non-amenable iff spectral radius of Laplacian < 1 *iff n-step return probability of random walk decays exponentially in n.*



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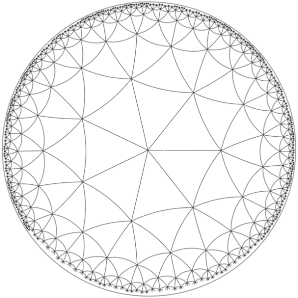
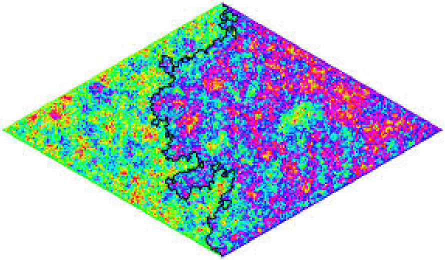


See the textbooks [*Lyons & Peres '15*], [*Pete '18+*] for more.

Historical remarks on percolation theory

Post-modern era:
Scaling limits of critical percolation in the plane
Conformal invariance thereof
SLE



Lawler-Schramm-Werner, Smirnov ... (apologies to many!)

Not covered in this talk.

Back to classics: analyticity below p_c

$\chi(p) := \mathbb{E}_p(|C(o)|)$,

i.e. the expected size of the component of the origin o.

Theorem (Kesten '82)

$\chi(p)$ is an analytic function of p for $p \in [0, p_c)$ when G is a lattice in $\mathbb{R}^d$.

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$\chi(p)$ is an analytic function of p for $p \in [0, p_c)$ when G is a lattice in $\mathbb{R}^d$.

'Trying to think of negative probabilities gave me cultural shock at first...'

—Richard Feynman,
from the paper *Negative Probability (1987).*

Let's just extend p to the complex numbers...

—Harry Kesten '81; blatantly paraphrased

Some complex analysis basics

Theorem (Weierstrass): Let $f = \sum f_n$ be a series of analytic functions which converges uniformly on each compact subset of a domain $\Omega \subset \mathbb{C}$. Then f is analytic on Ω.

Weierstrass M-test: Let (f_n) be a sequence of functions such that there is a sequence of 'upper bounds' M_n satisfying

$$|f_n(z)| \leq M_n, \forall x \in \Omega \quad \text{and} \quad \sum M_n < \infty.$$

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Theorem (Aizenman & Barsky '87)

In every vertex-transitive percolation model,

$$\mathbb{P}_p(|C| \geq n) \leq c_p^{-n},$$

for every $p < p_c$ and some $c_p > 1$.

Conjectures on the percolation probability

$\theta(p) := \mathbb{P}_p(|C| = \infty)$,

i.e. the percolation probability.

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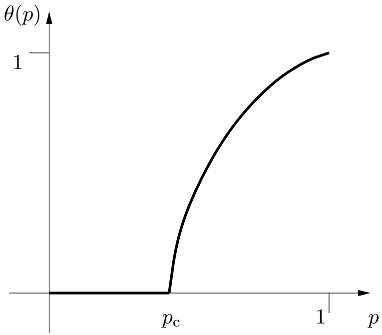

Fig. 1.1. It is generally believed that the percolation probability $\theta(p)$ behaves roughly as indicated here. It is known, for example, that θ is infinitely differentiable except at the critical point p_c. The possibility of a jump discontinuity at p_c has not been ruled out when $d \geq 3$ but d is not too large.

$\theta(p)$ analytic?

Open problem:

$$\text{Is } \theta(p) \text{ analytic for } p > p_c?$$

Appearing (for $G = \mathbb{Z}^d$) in the textbooks
Kesten '82, Grimmett '96, Grimmett '99.

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'...this in not just an academic matter. For instance, there are examples of disordered systems in statistical mechanics that develop a Griffiths singularity, i.e., systems that have a phase transition point even though their free energy is a C^∞ function.'
–Braga, Proccaci & Sanchis '02

Trees

$\theta(p) := \mathbb{P}_p(|C| = \infty)$,
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For percolation on the *d*-**regular tree**, we have

$$\theta(p) = 1 - (1 - p\theta_0(p))^d$$

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Trivial for binary tree, but what about higher degrees?

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We deduce this from

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Which builds upon

Theorem (Benjamini & Schramm '96)

$p_c \leq \frac{1}{1+h}$ on any such graph.

Analyticity for planar lattices

Theorem (G & Panagiotis '18+)

$\theta(p)$ is analytic for $p > p_c$ on any planar lattice.

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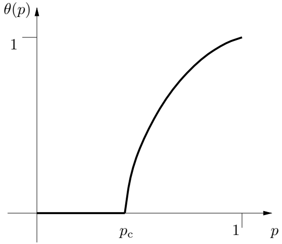

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Partitions of n

Theorem (Hardy & Ramanujan 1918)

The number of partitions of the integer n is of order

$$exp(\sqrt{n}).$$

Elementary proof: [*P. Erdös, Annals of Mathematics '42*]

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–Similar arguments, but we had to generalise *interfaces* to all graphs.

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Theorem (Duminil-Copin, Goswami, Raoufi, Severo & Yadin '18+)

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Proof involves the Gaussian Free Field.

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Theorem (Häggström '00)

Every bounded degree graph exhibits a phase transition in all or none of the following models:
bond/site percolation, Ising, Widom-Rowlinson, beach model.

Analyticity in $\mathbb{Z}^d$

Theorem (G & Panagiotis '18+)

$\theta(p)$ is analytic for $p > 1 - p_c$ for site percolation on any 'triangulated' lattice in $\mathbb{Z}^d$, $d \geq 2$.

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Is $\theta(p)$ analytic at $1 - p_c$?
Continuous at p_c?

Further reading:

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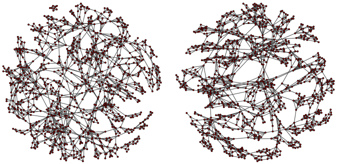
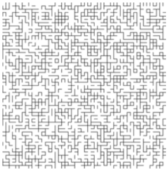
[*G. & Panagiotis, Analyticity results in Bernoulli Percolation*]





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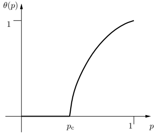

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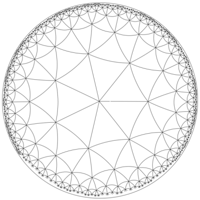

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