Infinite Cycles in Graphs

Agelos Georgakopoulos

Mathematisches Seminar
Universität Hamburg

Vancouver, 3-18-2008

?

How do you define an infinite cycle?

Finite cycles not enough!

Things that go wrong if you only allow finite cycles:

Finite cycles not enough!

Things that go wrong if you only allow finite cycles:

- Tutte’s theorem that the peripheral cycles of a 3-connected graph generate all cycles

Finite cycles not enough!

Things that go wrong if you only allow finite cycles:

- Tutte's theorem that the peripheral cycles of a 3-connected graph generate all cycles
- Thomassen's theorem that every $(k+3)$-connected graph contains a cycle C such that $G - C$ is k-connected

Finite cycles not enough!

Things that go wrong if you only allow finite cycles:

- Tutte's theorem that the peripheral cycles of a 3-connected graph generate all cycles
- Thomassen's theorem that every $(k+3)$-connected graph contains a cycle C such that $G - C$ is k-connected
- Hamilton cycles?

Tutte's Theorem

Theorem (Tutte '56)

Every finite 4-connected planar graph has a Hamilton cycle

Tutte's Theorem

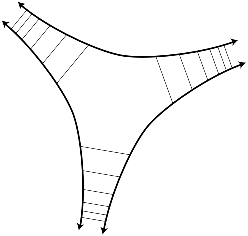
Theorem (Tutte '56)

Every finite 4-connected planar graph has a Hamilton cycle

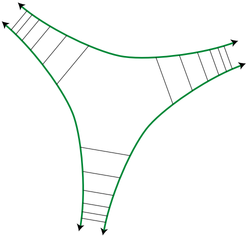
Theorem (Yu '05)

Every locally finite 4-connected planar graph with at most 2 ends has a spanning double ray

Infinite Cycles



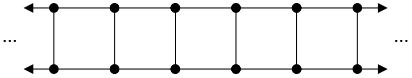
Infinite Cycles



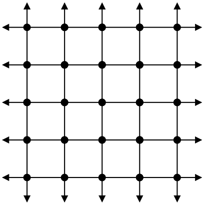
Ends

end: equivalence class of rays

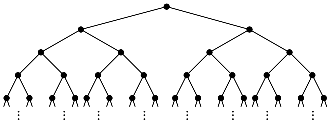
two rays are equivalent if no finite vertex set separates them



two ends



one end



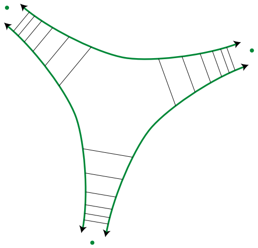
2^ω many ends

The Freudenthal compactification

Circle (Diestel & Kühn):
A homeomorphic image of S^1 in $|G|$.

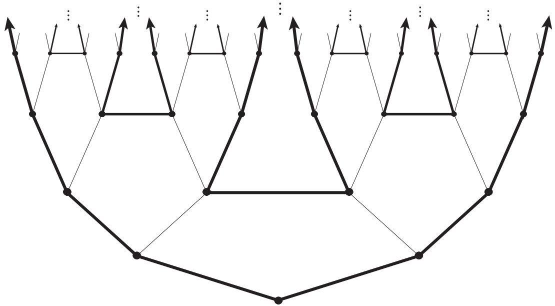
The Freudenthal compactification

Circle (Diestel & Kühn):
A homeomorphic image of S^1 in $|G|$.



Infinite cycles

Circle:
A homeomorphic image of S^1 in $|G|$.



I-TOP

An equivalent definition of $|G|$:

- Assign a length $\ell(e) > 0$ to each edge e;

I-TOP

An equivalent definition of $|G|$:

- Assign a length $\ell(e) > 0$ to each edge e;
- This induces a metric $d_\ell(v, x)$;

l-TOP

An equivalent definition of $|G|$:

- Assign a length $\ell(e) > 0$ to each edge e;
- This induces a metric $d_\ell(v, x)$;
- Let ℓ-*TOP*(G) be the completion of (G, d_ℓ).

l-TOP

An equivalent definition of $|G|$:

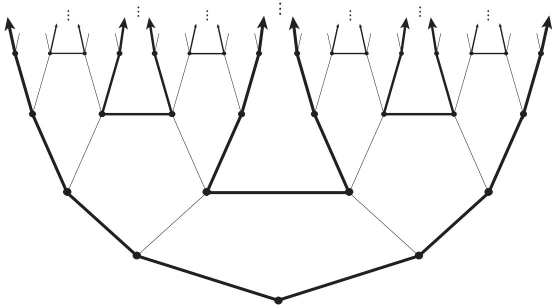
- Assign a length $\ell(e) > 0$ to each edge e;
- This induces a metric $d_\ell(v, x)$;
- Let ℓ-TOP(G) be the completion of (G, d_ℓ).

Proposition (G)

If $\sum_{e \in E(G)} \ell(e) < \infty$ *then* ℓ-$TOP(G) \approx |G|$.

Infinite cycles

Circle:
A homeomorphic image of S^1 in $|G|$.

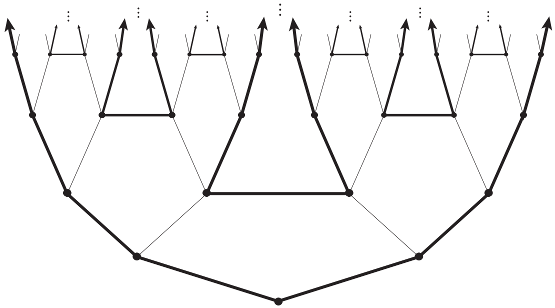


Things that go wrong if you only allow finite cycles:

- Tutte's theorem that the peripheral cycles of a 3-connected graph generate all cycles
- Thomassen's theorem that every $(k+3)$-connected graph contains a cycle C such that $G - C$ is k-connected
- Hamilton cycles?

Infinite cycles

Circle:
A homeomorphic image of S^1 in $|G|$.



Hamilton circle:
a circle containing all vertices.

Fleischner's Theorem

Theorem (Fleischner '74)

The square of a finite 2-connected graph has a Hamilton cycle

Fleischner's Theorem

Theorem (Fleischner '74)

The square of a finite 2-connected graph has a Hamilton cycle

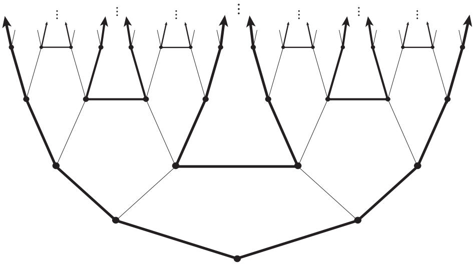
Theorem (Thomassen '78)

The square of a locally finite 2-connected 1*-ended graph has a spanning double ray.*

Fleischner's Theorem for Locally Finite Graphs

Theorem (G '06)

The square of a locally finite 2-connected graph has a Hamilton circle



Cycle Space

The cycle space $\mathcal{C}(G)$ of a finite graph:

- A vector space over $\mathbb{Z}_2$
- Consists of all sums of circuits

Cycle Space

The cycle space $\mathcal{C}(G)$ of a finite graph:

- A vector space over $\mathbb{Z}_2$
- Consists of all sums of circuits

The topological cycle space $\mathcal{C}(G)$ of a locally finite graph G is defined similarly but:

Cycle Space

The cycle space $\mathcal{C}(G)$ of a finite graph:

- A vector space over $\mathbb{Z}_2$
- Consists of all sums of circuits

The topological cycle space $\mathcal{C}(G)$ of a locally finite graph G is defined similarly but:

- Allows edge sets of infinite circles;

Cycle Space

The cycle space $\mathcal{C}(G)$ of a finite graph:

- A vector space over $\mathbb{Z}_2$
- Consists of all sums of circuits

The topological cycle space $\mathcal{C}(G)$ of a locally finite graph G is defined similarly but:

- Allows edge sets of infinite circles;
- Allows infinite sums (whenever well-defined).

The topological Cycle Space

Known facts:

- A connected graph has an Euler tour iff every edge-cut is even (Euler)
- G is planar iff $\mathcal{C}(G)$ has a simple generating set (MacLane)
- If G is 3-connected then its peripheral circuits generate $\mathcal{C}(G)$ (Tutte)

Generalisations:

Bruhn & Stein

Bruhn

Bruhn & Stein

Failure in “continuous” problems

Theorem

The geodetic cycles of a finite graph G generate its cycle space.

Failure in "continuous" problems

Theorem

The geodetic cycles of a finite graph G generate its cycle space.

Theorem (G & Sprüssel)

The geodetic circles of a locally finite graph generate $\mathcal{C}(G)$

Failure in “continuous” problems

Theorem

The geodetic cycles of a finite graph G generate its cycle space.

Theorem (G & Sprüssel)

The geodetic circles of a locally finite graph generate $\mathcal{C}(G)$

... provided the edges are assigned lengths ℓ that respect $|G|$, i.e. ℓ-TOP(G) $\approx |G|$.

l-TOP

- Assign a length $\ell(e) > 0$ to each edge e;
- This induces a metric $d_\ell(v, x)$;
- Let ℓ-*TOP*(G) be the completion of (G, d_ℓ).

Proposition (G)

If $\sum_{e \in E(G)} \ell(e) < \infty$ *then* ℓ-$TOP(G) \approx |G|$.

Infinite electrical networks

Similarly:

Theorem

In a locally finite electrical network the infinite circles satisfy Kirchhoff's 2nd law if ℓ-TOP$(G) \approx |G|$, where $\ell(e)$ is the resistance of e.

Infinite electrical networks

Similarly:

Theorem

In a locally finite electrical network the infinite circles satisfy Kirchhoff's 2nd law if ℓ-TOP$(G) \approx |G|$, where $\ell(e)$ is the resistance of e.

Even stronger:

Theorem (G)

In a locally finite electrical network with resistances $\ell(e)$, all "proper" circles in ℓ-TOP(G) satisfy Kirchhoff's 2nd law.

Geodetic circles

Theorem (G & Sprüssel)

The geodetic circles of a locally finite graph G generate $\mathcal{C}(G)$

... provided the edges are assigned lengths ℓ that respect $|G|$,
i.e. ℓ-TOP(G) $\approx |G|$.

Open problems

Conjecture

Every locally finite 4-connected line graph has a Hamilton circle.

Hamiltonicity in Cayley graphs

Problem

Does every finite Cayley graph have a Hamilton cycle?

Hamiltonicity in Cayley graphs

Problem

Does every finite Cayley graph have a Hamilton cycle?

Problem

Does every 1*-ended Cayley graph have a Hamilton circle (i.e. a spanning double ray)?*

Hamiltonicity in Cayley graphs

Problem

Does every finite Cayley graph have a Hamilton cycle?

Problem

Does every 1-ended Cayley graph have a Hamilton circle (i.e. a spanning double ray)?

Problem

Prove that a Cayley graph of a finitely generated group Γ has a Hamilton circle unless Γ is the amalgamated product of more than k groups over a subgroup of order k.

Open Problems

Theorem (Thomassen)

Every finite $(k+3)$-connected graph contains a cycle C such that $G - C$ is k-connected.

Open Problems

Theorem (Thomassen)

Every finite $(k+3)$-connected graph contains a cycle C such that $G - C$ is k-connected.

Problem (Diestel)

If G is a locally finite $(k+3)$-connected graph, does $|G|$ contain a circle C such that $G - C$ is k-connected or $|G| - C$ is topologically k-connected?