Infinite Cycles in Graphs

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Finite Cycles are not Enough

Theorem (Thomassen)

Every finite $(k+3)$-connected graph contains a cycle C such that $G - C$ is k-connected.

?

What is an infinite cycle?

Tutte's Theorem

Theorem (Tutte '56)

Every finite 4-connected planar graph has a Hamilton cycle

Theorem (Yu '05)

Every locally finite 4-connected planar 3-indivisible graph has a spanning double ray

Fleischner's Theorem

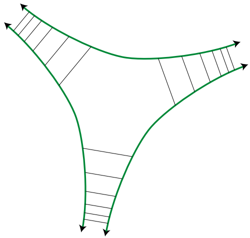
Theorem (Fleischner '74)

The square of a finite 2-connected graph has a Hamilton cycle

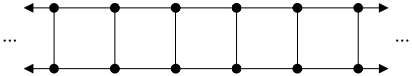
Theorem (Thomassen '78)

The square of a locally finite 2-connected 2-indivisible graph has a spanning double ray

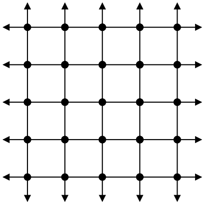
Infinite Cycles



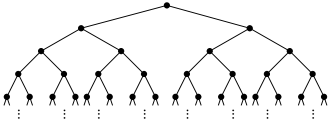
Ends



two ends



one end

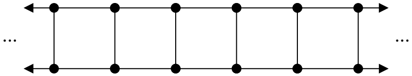


2^{ω} many ends

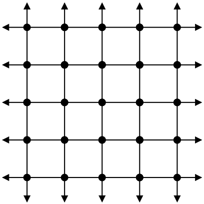
Ends

end: equivalence class of rays

two rays are equivalent if no finite vertex set separates them



two ends



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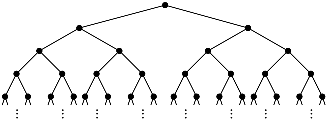
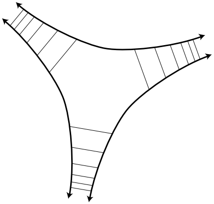


2^ω many ends

The Freudenthal Compactification



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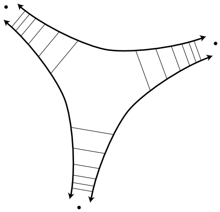

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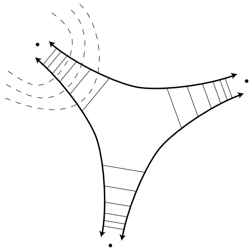

Infinite cycles

Circle:
A homeomorphic image of S^1 in $|G|$.

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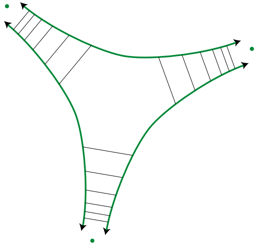
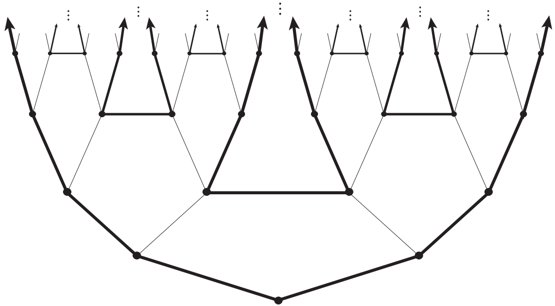


Hamilton circle:
a circle containing all vertices

Infinite cycles

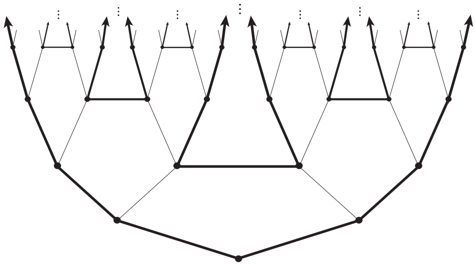
Circle:
A homeomorphic image of S^1 in $|G|$.



Fleischner's Theorem for Locally Finite Graphs

Theorem (G '06)

The square of a locally finite 2-connected graph has a Hamilton circle



Fleischner's Theorem for Locally Finite Graphs

Euler tour: A continuous image from S^1 to $|G|$ traversing each edge exactly once.

Theorem (G '06)

If a locally finite graph has an Euler tour then it also has one visiting each end exactly once.

The Cycle Space of a Finite Graph

$\mathcal{C}(G)$

- A vector space over $\mathbb{Z}_2$
- Consists of sums of circuits

The Cycle Space of an Infinite Graph

Theorem (Tutte)

If G is 3-connected then its peripheral circuits generate $\mathcal{C}(G)$

peripheral: induced and non-separating

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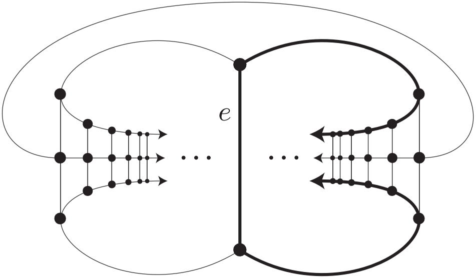

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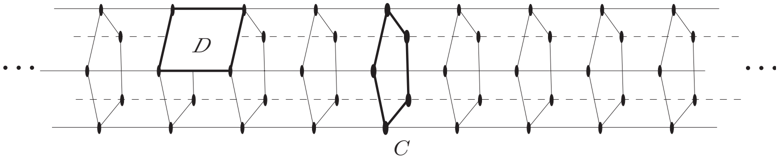

The Cycle Space of an Infinite Graph

$\mathcal{C}(G)$: all thin sums of circuits, i.e. edge sets of **circles**

The Cycle Space of an Infinite Graph

Known facts:

- Every $C \in \mathcal{C}(G)$ is a disjoint union of circuits
- The fundamental circuits of a spanning tree generate $\mathcal{C}(G)$
- $C \in \mathcal{C}(G)$ iff C meets every cut evenly
- A connected graph is eulerian iff every vertex has even degree
- G is planar iff $\mathcal{C}(G)$ has a simple generating set
- If G is 3-connected then its peripheral circuits generate $\mathcal{C}(G)$
- The geodesic cycles generate $\mathcal{C}(G)$

Generalisations:

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The geodesic cycles generate $\mathcal{C}(G)$	G & Sprüssel

Extremal Graph Theory

Theorem (Mader)

If G has average degree at least $4k$ then it has a $(k+1)$-connected subgraph

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Theorem (Stein)

If each vertex of G has degree at least $2k$ and each end has edge-degree at least $2k$ then G has a $(k+1)$-edge-connected region.

Vertex version also exists

Other Problems

Problem

Is every (topologically) connected subspace of $|G|$ path-connected?

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Theorem (G '04)

No.

Open Problems

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If G is $(k+3)$-connected, does $|G|$ contain a circle C such that $G - C$ is k-connected?

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Are there connected (non-locally-finite) graphs G such that $|G|$ has no topological spanning tree?

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Is G^3 hamiltonian for every connected countable graph G?

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Is G^3 hamiltonian for every connected countable graph G?

Problem

If G^k is hamiltonian, is G^{k+1} also hamiltonian?

Your To Do List

- read Diestel's expository paper
  The cycle space of an infinite graph
  or Chapter 8.5. of The Book
- visit the project's webpage:
  http://www.math.uni-hamburg.de
  /home/diestel/papers/TopGrProject.html
- read the proof that if G is connected then G^3 is hamiltonian in
  Infinite hamilton cycles in squares of locally finite graphs
- solve some of the open problems