# Brownian Motion on graph-like spaces

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**Abstract**

We construct Brownian motion on a wide class of metric spaces similar to graphs, and show that its cover time admits an upper bound depending only on the length of the space.

## 1 Introduction

The aim of this paper is to construct the analog of Brownian motion on metric spaces that are similar to graphs in a sense made precise below, and study some of its basic properties. It turns out that, under mild conditions, there is a unique stochastic process qualifying for this.

Figure 1 shows some example spaces on which our process can live; the numbers indicate the lengths of the corresponding arcs.

The first one is the Hawaian earring: an infinite sequence of circles attached to a common point $p$ to which they converge. It might at first sight seem impossible to have a Brownian motion on this space started at $p$, unless we impose some ad-hoc bias as to the probability with which each circle is chosen first. However, there need not be a 'first' circle visited by a continuous path from $p$, and indeed our process will traverse infinitely many of them before moving to any distance $r > 0$ from $p$. Still, each of the finitely many points at distance exactly $r$ from $p$ has the same probability to be reached first. The second example is an $\mathbb{R}$-tree of finite total length. Our Brownian motion will reach the 'boundary' at the bottom after some finite time $\tau$, and will continue its continuous path after this, almost surely visiting infinitely many boundary points in any inteval $[\tau, \tau + \epsilon]$. The third example is obtained from the Sierpinski gasket by replacing articulation points with arcs. This space contains a homeomorphic copy of the second example, and a subspace homotopy equivalent to the first example; our process on it is more complex, combining features of both the above.

In all these examples, and in much greater generality indeed, our process behaves locally like standard Brownian motion on a real interval $I$ on each open

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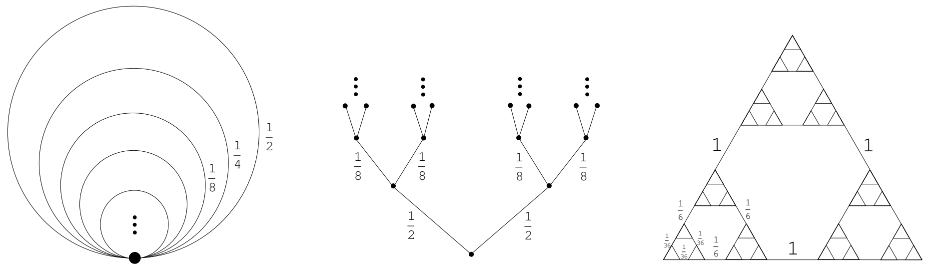

Figure 1: Examples of graph-like spaces.

arc of our space isometric to $I$, its sample paths are continuous, it has the strong Markov property, and it almost surely covers the whole space after finite time.

We call a topological space $X$ *graph-like*, if it contains a set $E$ of pairwise disjoint copies of $\mathbb{R}$, called *edges*, each of which is open in $X$, such that the subspace $X \setminus \bigcup E$ is totally disconnected. This notion was introduced by Thomassen and Vella [30], and was motivated by recent developments in graph theory; see also [9].

Recall that a *continuum* is a compact, connected, non-empty metrizable space (some authors replace 'metric' by Hausdorff). We will use $\mathcal{H}(X)$ to denote the 1-dimensional Hausdorff measure of $X$. Although our processes can be constructed on any graph-like continuum, for its uniqueness it is necessary to have $\mathcal{H}(X) < \infty$.

In order to construct our process, we use a result from [9] stating, roughly speaking, that every graph-like space $X$ can be approximated by a sequence of finite graphs (i.e. 1-complexes) contained in $X$. Such a sequence of graphs is called a graph approximation of $X$; see Section 3 for the precise definition. For example, any sequence $(G_n)_{n \in \mathbb{N}}$ where $G_n$ consists of finitely many of the cicles of the Hawaian earring and each circle appears in almost every $G_n$ is a graph approximation. The main goal of this paper is to show that if $B_n$ denotes Brownian motion on the $n$th member of any graph approximation of $X$, then the $B_n$ converge weakly —in the space of measures on continuous paths on $X$, see Section 2.2— to a stochastic process $B$ on $X$ with all the desired properties, and this $B$ does not depend on the choice of the graph approximation:

**Theorem 1.1.** *Let $G$ be a graph-like continuum with $\mathcal{H}(X) < \infty$, and $o$ a point of $X$. Then there is a stochastic process $B$ on $X$ with continuous sample paths starting at $o$, the strong Markov property, and a stationary distribution proportional to $\mathcal{H}$.*

*Moreover, for every graph approximation $(G_n)_{n \in \mathbb{N}}$ of $X$, and every choice of points $o_n \in G_n$ such that* $\lim o_n = o$, *if $B_n$ is the standard Brownian motion on $G_n$ from $o_n$, then $B_n$ converges weakly to $B$, and $B$ is unique with this property.*

Theorem 1.1 states that our process is unique with the property of being a weak limit of Brownian motions on graph approximations of $X$, but we suspect that it is unique in a stronger sense.

It was shown in [12] that the expected time for Brownian motion on a finite, connected 1-complex $G$ to cover all of $G$ is bounded from above by a value depending only on the total length of $G$ and not on its structure. Applying this to each member of our graph approximations, we prove the corresponding result for our Brownian motion on an arbitrary graph-like continuum:

**Theorem 1.2.** *The expected cover time of the process $B$ of Theorem 1.1 is at most* $20\mathcal{H}(X)^2$.

A related result of Krebs [20] shows that the hitting times for Brownian motion on nested fractals are bounded.

There are many constructions of Brownian motion on spaces similar to the ones considered in this paper: on finite graphs [4], on trees and their boundaries [1, 5, 6, 19] on the Sierpinski gasket [3, 13, 21] and many other fractals [15, 14, 22]. Brownian motion especially on fractals has attracted a lot of interest, with motivation coming both from pure mathematics and mathematical physics (see [21] and references therein), and has many connections to other analytic properties of fractals which also attract a lot of research [18, 28].

The first author had asked for a construction of Brownian motion on a special type of graph-like spaces, namely metric completions of infinite graphs [10, Section 8], and this paper gives a very satisfactory answer to that question.

This paper is structured as follows. After reviewing some definitions and basic facts Section 2, we prove the existence part of Theorem 1.1 in Section 3. The uniqueness part is then proved in Section 5. Then we prove that our process has the strong Markov property (Theorem 6.3), and the bound on the cover time is given in Section 7. Finally, we prove that $\mathcal{H}$ is a stationary distribution and that our process behaves locally like standard Brownian motion inside any edge in Section 8.

# 2 Preliminaries

## 2.1 Graph-like spaces

An *edge* of a topological space $X$ is an open subspace $I \subseteq X$ homeomorhpic to the real interval $(0,1)$ such that the closure of $I$ in $X$ is homeomorphic to $[0,1]$. (We could allow the closure of $I$ to be a circle; it is only for convenience in certain situations that we disallow this.) Note that the frontier of an edge consists of two points, which we call its *endvertices*. An edge-set of a topological space $X$ is a subspace consisting of finitely many, pairwise disjoint, edges of $X$.

A topological space $X$ is *graph-like* if there is an edge-set $E$ of $X$ such that $G \backslash E$ is totally disconnected. In that case, we call $E$ a *disconnecting edge-set*.

The following fact provides an equivalent definition of a graph-like continuum.

**Lemma 2.1** ([9])**.** *A continuum $X$ is graph-like if and only if for every $\epsilon$ there is a finite set of edges $S_\epsilon$ of $X$ such that the diameter of every component of $X \backslash S_\epsilon$ is less than $\epsilon$.*

The following property of graph-like spaces is very useful to us, as it implies that Brownian motion on such a space cannot travel a long distance without traversing a long edge.

**Proposition 2.2.** *If $X$ is a graph-like continuum, then for every $\rho > 0$ there is a finite edge-set $R_\rho$ of $X$ such that for every topological path $p : [0, 1] \to X$ in $X$, if $d(p(0), p(1)) > \rho$ then $p$ traverses an edge in $R_\rho$.*

*Proof.* Applying Lemma 2.1 for $\epsilon = \rho/3$, we obtain a finite set of edges $S$ such that the diameter of every path-component of $X \backslash S$ is less than $\rho/3$. Subdivide each edge $e \in S$ into a finite set of edges each of length at most $\rho/6$, and let $R$ be the set of edges resulting from $S$ after all these subdivisions. Now note that any topological path $p$ as in the assertion has to traverse an element of $R$; to see this, contract each path-component of $X \backslash S$ into a point to obtain a new metric space $X'$, and note that $X'$ is isometric to a finite graph whose edgeset can be identified with $R$. Moreover, after the contractions we have $d(p(0), p(1)) > \rho - 2\rho/3 = \rho/3$, and as each edge of our graph has length at least $\rho/6$, the assertion easily follows by geometric arguments. Thus we can choose $R_\rho = R$. □

Graph-like spaces have nice bases:

**Lemma 2.3** ([9])**.** *Let $X$ be a graph-like metric continuum. Then the topology of $X$ has a basis consisting of connected open sets $O$ such that the frontier of $O$ is a finite set of points each contained in an edge.*

## 2.2 Measures on the space of sample paths and weak convergence

Given a graph-like space $(X, d_X)$, we denote by $C = C_T(X)$ the set of continuous functions from the real interval $[0, T]$ to $X$. We call $C$ the *space of sample paths*; our process will be formally defined as a probability measure on $C$. We endow $C$ with the $L^\infty$ metric $d_C(b, d) := \sup_{t\in[0,T]} d_X(b(t), c(t))$.

Let $\mathcal{M} = \mathcal{M}(C)$ denote the space of all borel probability measures on $C$. The *weak topology* on $\mathcal{M}$ is the topology generated by the open sets of the form

$$O_\mu(f_1, \dots, f_k; \epsilon_1, \dots, \epsilon_k) = \left\{ \nu \in \mathcal{M} : |\int f_i d\nu - \int f_i d\mu| < \epsilon_i, 1 \le i \le k \right\},$$

where $\mu$ ranges over all elements of $\mathcal{M}$, the $f_i$ range over all bounded continuous functions $f_i : C \to \mathbb{R}$, and the $\epsilon_i$ range over $\mathbb{R}_{>0}$. An immediate consequence of this definition is that a sequence of measures $\mu_i \in \mathcal{M}$ converges in this topology to $\mu \in \mathcal{M}$ if and only if $\int f d\mu_i$ converges to $\int f d\mu$ for every bounded continuous function $f : C \to \mathbb{R}$. If such a sequence converges, then the limit is unique [26, Chapter II, Theorem 5.9].

Our main tool in obtaining limits of stochastic processes is the following standard fact, see e.g. [26, Chapter VII, Lemma 2.2].[1]

**Lemma 2.4.** *Let $\Gamma$ be a set of probability measures on $C$. Then $\overline{\Gamma}$ is compact if and only if for every $\epsilon, \rho > 0$ there is $\eta = \eta(\epsilon, \rho) > 0$ such that*

$$\mu(\{p \mid \omega_p(\eta) > \rho\}) < \epsilon \text{ for every } \mu \in \Gamma,$$

*where* $\omega_p(\eta) := \sup_{|t-t'|\le\eta} |p(t) - p(t')|$.

[1] Condition (i) in [26][Chapter VII, Lemma 2.2] is void in our case because our spaces have finite diameter.

## 2.3 Metric graphs and their Brownian motion

In this paper, by a *graph* $G$ we will mean a topological space homeomorhpic to a simplicial 1-complex. We assume that any graph $G$ is endowed with a fixed homeomorphism $h : K \to G$ from a simplicial 1-complex $K$, and call the images under $h$ of the 0-simplices of $K$ the *vertices* of $G$, and the images under $h$ of the 1-simplices of $K$ the *edges* of $G$. Their sets are denoted by $V(G)$ and $E(G)$ respectively. All graphs considered will be *finite*, that is, they will have finitely many vertices and edges.

A metric graph is a graph $G$ endowed with an assignment of lengths $\ell : E(G) \to \mathbb{R}_{>0}$ to its edges. This assignment naturally induces a metric $d_\ell$ on $G$ with the following properties. Edges are locally isometric to real intervals, their lengths (i.e. 1-dimensional Hausdorff measures) with respect to $d_\ell$ coincide with $\ell$, and for every $x, y \in V(G)$ we have $d_\ell(x, y) := \inf_{P \text{ is an } x\text{–}y \text{ arc}} \ell(P)$, where $\ell(P) := \sum_{P \supseteq e \in E(G)} \ell(e)$; see [11] for details on $d_\ell$.

The length $\ell(G)$ of a metric graph $G$ is defined as $\sum_{e \in E(G)} \ell(e)$.

An *interval* of an edge $e$ of $G$ is a connected subspace of $e$.

Brownian motion on $\mathbb{R}$ extends naturally to Brownian motion on a metric graph. The edges incident to a vertex constitute a "Walsh spider" (see, e.g., [31, 24]) with equiprobable legs, and it is easily verified that in such a setting the probability of traversing a particular incident edge (or oriented loop) first is proportional to the reciprocal of the length of that edge, while inside any interval of an edge, it behaves like standard Brownian motion on a real interval of the same length. To make this more precise, it is shown in [4] that there is a probability distribution on the space $C(G)$ of continuous functions from a real interval $[0, T]$ to $G$, which we will call *standard Brownian motion* on $G$, that has the following properties

(i) The strong Markov property;

(ii) for every vertex $v$ of $G$ and any choice of points $p_i, 1 \leq i \leq k$, one inside each edge incident with $v$, the probability to reach $p_j$ before any other $p_i, i \neq j$ when starting at $v$ is $\frac{1/\ell_j}{\sum_{1 \leq i \leq k} 1/\ell_i}$, where $\ell_i$ denotes the length of the interval from $v$ to $p_i$ ([4, §4: Lemma 1 applied with $\tilde{p_i} := 1/k$]);

(iii) for every vertex $v$ of $G$, the expected time to exit the ball of radius $r$ around $v$ when starting at $v$ tends to 0 as $r$ tends to 0 ([4, (3.1)]).

(iv) When starting at a point $p$ inside an edge $e$, the expected time till the first traversal of one of the two intervals of $e$ of length $\ell$ starting at $p$ is $\ell^2$ ([4, (3.4)]).

The expected time for Brownian motion started at a vertex $a$ to visit a vertex $z$ and then return to $a$, i.e., $\mathbb{E}_a[\tau_z] + \mathbb{E}_z[\tau_a]$, is called the *commute time* between $a$ and $z$.

**Lemma 2.5** ([7, 23])**.** *Let $G$ be a finite metric graph, and $a$, $z$ two vertices of $G$. The commute time between $a$ and $z$ equals $2\ell(G)R(a, z)$.*

## 2.4 Electrical network basics

An *electrical network* is a graph $G$ endowed with an assignment of resistances $r : E \to \mathbb{R}_+$ to its edges. The set $\vec{E}$ of *directed edges* of $G$ is the set of ordered pairs $(x, y)$ such that $xy \in E$. Thus any edge $e$ of $G$ with endvertices $x, y$ corresponds to two elements of $\vec{E}$, which we will denote by $\overrightarrow{xy}$ and $\overrightarrow{yx}$. A *$p$–$q$ flow* of strength $I$ in $G$ is a function $i : \vec{E} \to \mathbb{R}$ with the following properties

(i) $i(\overrightarrow{e^0e^1}) = i(\overrightarrow{e^1e^0})$ for every $e \in E$ ($i$ is antisymmetric);

(ii) for every vertex $x \neq p, q$ we have $\sum_{y\in N(x)} i(\overrightarrow{xy}) = 0$, where *N(x)* denotes the set of vertices sharing an edge with $x$ ($i$ satisfies Kirchhoff's node law outside $p, q$);

(iii) $\sum_{y\in N(p)} i(\overrightarrow{py}) = I$ and $\sum_{y\in N(q)} i(\overrightarrow{qy}) = -I$ ($i$ satisfies the boundary conditions at $p, q$).

The *effective resistance* $R_G(p, q)$ from a vertex $p$ to a vertex $q$ of $G$ is defined by

$$R_G(p, q) := \inf_{i \text{ is a } p\text{–}q \text{ flow of strength } 1} E(i),$$

where the *energy* $E(i)$ of $i$ is defined by $E(i) := \sum_{\vec{e}\in\vec{E}} i(\vec{e})^2 r(e)$. In fact, it is well-known that this infimum is attained by a unique $p$–$q$ flow, called the corresponding *electrical current*.

The effective resistance satisfies the following property which justifies its name

**Lemma 2.6.** *Let $G$ be an electrical network contained in an electrical network $H$ in such a way that there are exactly two vertices $p, q$ of $G$ connected to vertices of $H - G$ with edges. Then if $H'$ is obtained from $H$ by replacing $G$ with a $p$–$q$ edge of resistance $R_G(p, q)$, then for every two vertices $v, w$ of $H'$ we have $R_{H'}(v, w) = R_H(v, w)$.*

The proof of this follows easily from the definition of effective resistance. See e.g. [23] for details.

Any metric graph naturally gives rise to an electrical network by setting $r = \ell$, and we will assume this whenever talking about effective resistances in metric graphs.

The importance of effective resistances for this paper is due to the following fact, showing that they determine transition probablities between any two points in a finite set for Brownian motion on a metric graph.

**Lemma 2.7** ([23, Exercise 2.54])**.** *Let $G$ be a metric graph and $U$ a finite set of points of $G$. Start Brownian motion at a point $o \notin U$ of $G$ and stop it upon its first visit to $U$. Then the exit probabilities are determined by the values $\{R(x, y) \mid x, y \in U \cup \{o\}\}$.*

# 3 Existence

In this section we prove the existence part of Theorem 1.1, in other words, the existence of an accumulation point in $\mathcal{M}(C)$ of every sequence $(B_n)_{n\in\mathbb{N}}$ such that $B_n$ is standard Brownian motion on a graph $G_n \subseteq X$ and $(G_n)_{n\in\mathbb{N}}$ is a *graph approximation*: a graph approximation of $X$ is a sequence $(G_n)_{n\in\mathbb{N}}$ of finite graphs that are subspaces of $X$ satisfying the following two properties:

(i) for every edge $e \in E(G_n)$ the length $\ell(e)$ of $e$ in $G_n$ coincides with the length of the corresponding arc of $X$;

(ii) For every finite edge-set $F$ of $X$, and every component $C$ of $X \backslash F$, there is a unique component of $G_i \backslash F$ meeting $C$ for almost all $i$.

The existence of graph approximations was established in [9]. In fact, we can furthermore assume that each $G_n$ is connected, and that $G_n \subseteq G_{n+1}$ for every $n$, although it will not make a formal difference for our proofs. It is also shown in [9] that (ii) implies that $\bigcup G_n$ contains every edge of $X$ and is dense in $X$.

So let us fix such a sequence $(G_n)_{n\in\mathbb{N}}$. For every $n \in \mathbb{N}$ and $o_n \in G_n$, let $\mu_{n,o}$ be the measure on $C$ corresponding to standard Brownian motion on $G_n$ starting at the point $o_n$. Let

$$\Gamma := \{\mu_{n,o} \mid n \in \mathbb{N}, o \in G_n\}.$$

The following result shows that this family of measures has accumulation points in $\mathcal{M}(C)$, which we think of as candidates for our Brownian motion on $X$. We will show in Section 5 that if the $o_n$ converge to a point of $X$, then $\Gamma$ has a unique accumulation point.

**Lemma 3.1.** *The family $\overline{\Gamma}$ is compact (with respect to the weak topology).*

*Proof.* Throughout this proof $B_n$ is a random sample path in $C$ chosen according to some of our measures $\mu \in \Gamma$, and probabilities refer to that measure.

We are going to show that our family $\Gamma$ satisfies the condition of Lemma 2.4, that is, for any $\epsilon > 0$

$$\lim_{\delta\to 0} \mathbb{P}\left(\sup_{t,s<T;|t-s|<\delta} d(B_n(t), B_n(s)) > \epsilon\right) = 0 \text{ uniformly in } n. \tag{1}$$

So fix $\epsilon > 0$. Let $R = R_\epsilon$ be a finite set of edges as in Proposition 2.2, and let $\epsilon_1 = \min\{\ell(e) \mid e \in R\}$.

Thus we have the following bound for the probability appearing in (1):

$$\mathbb{P}[\sup_{t,s<T;|t-s|<\delta} d(B_n(t), B_n(s)) > \epsilon] \leq$$
$$\mathbb{P}[B_n([t, t+\delta]) \text{ traverses an edge } e \in R \text{ for some } t \in [0, T-\delta]].$$

It remains to show that the last probabilities converge to 0 uniformly in $n$ as $\delta \to 0$. For this we will use the fact that each brownian motion $B_n$ in the interior of an edge behaves locally like standard Brownian Motion $W$ on the real line. Let us make this more precise. Let $R'$ be the set of half-edges of $R$, that is, each element of $R'$ is a open subinterval of an edge of $R$ from an endpoint to the midpoint. Let us subdivide the time interval $[0, T]$ into the $\lceil T/\delta \rceil$ subintervals $I_0, I_1, \ldots I_k$ of the form $I_i = [i\delta, (i+1)\delta]$; note that each $I_i$ has duration at most $\delta$. Then, if $B_n$ traverses an edge of $R$ in time $\delta$ at some point, then there is a time intervals $I_i$ during which $B_n$ traverses an element of $R'$. Thus we can write

$$\begin{aligned}&\mathbb{P}[B_n([t,t+\delta]) \text{ traverses an edge } e \in R \text{ for some } t \in [0,T]]\\ &\qquad \leq \sum_i \mathbb{P}[B_n([i\delta,(i+1)\delta]) \text{ traverses an edge } e \in R'].\end{aligned}$$

Now denote by $M$ the set of the midpoints of elements of $R'$, and by $\tau_n^i = \inf\{t \geq i\delta : B_n(t) \in M\}$ the associated hitting times. Then we can bound the last expression by

$$\sum_i \mathbb{P}[B_n([\tau_n^i, \tau_n^i + \delta]) \text{ traverses an edge } e \in R''],$$

where $R''$ is the set of half-edges of $R'$, in other words, the 'quarter-edges' of $R$.

Now since inside an edge $B_n$ behaves like standard Brownian Motion $W$, the above sum is at most

$$\lceil T/\delta \rceil \mathbb{P}[\max_{t\in[0,\delta]} |W(t)| > \epsilon_1/4) = \lceil T/\delta \rceil \mathbb{P}[|W(\delta)| > \epsilon_1/4),$$

by reflection principle [25, Theorem 2.21]. This expression converges to 0 with $\delta$, since the second factor decay rapidly with $\delta$. Moreover, it does not depend on $n$, and so it yields (1) as desired.

□

Remark: if $(\mu_n)_{n\in\mathbb{N}}$ is a convergent sequence of elements of $\Gamma$ with limit $\mu$, then for every $x \in X$,

$$\mathbb{E}_\mu[d(b(0),x)] = \lim_n \mathbb{E}_{\mu_n}[d(b(0),x)].$$

In particular, if the starting points of the $\mu_n$ converge to $x$, then the starting point of $\mu$ is $x$ a.s.

# 4 Occupation time of small subgraphs

A *subgraph* $H$ of a graph $G$ is a subspace of $G$ that is a graph itself. If $G$ is a metric graph, then we consider $H$ to be a metric graph as well, with its edge-lengths induced from those of $G$ in the obvious way. Note that the vertices of $H$ need not be vertices of $G$; an interval of an edge of $G$ can be an edge of $H$.

For a (finite) metric graph $G$ and standard Brownian motion $B$ on $G$, the *occupation time* $OT_t(H) = OT_t(H;B)$ of a subgraph $H \subseteq G$ up to time $t$ is defined to be the amount of time $\int_0^t \mathbb{1}_{\{B(s)\in H\}} ds$ spent by $B$ in $H$ in the time interval $[0,t]$. We define the occupation time of $H$ for random walk on $G$ similarly.

The following lemma shows that the occupation time of a subgraph $H$ of $G$ is short with high probability when the length $\ell(H)$ is small compared to $\ell(G)$, and in fact can be bounded above by a function depending only on the proportion of the lenghts but not on the structure of $G$ and $H$.

**Lemma 4.1.** *For every $L, T, \epsilon \in \mathbb{R}_{>0}$ there is a small enough $\ell \in \mathbb{R}_{>0}$ such that for every finite metric graph $G$ with $\ell(G) \geq L$ and every subgraph $H \subseteq G$ with $\ell(H) \leq \ell$, we have $OT_T(H) < \epsilon$ with probability at least $1 - \epsilon$.*

*Proof.* Let $\tau$ be the random time of the first return to the starting point $o$ after time $T$. We claim that

$$\mathbb{E}_B[OT_\tau(H)] = \mathbb{E}_B[\tau]\frac{\ell(H)}{\ell(G)},$$

Where the subscript $B$ stands for the fact that the expectation is taken with respect to standard Brownian motion on $G$. For this, we use the fact that for simple random walk $R$ on $G$ it is well-known [7] that the expected occupation time $\mathbb{E}_R[OT_\tau(K)]$ up to time $\tau$ in any subgraph $K$ equals $\mathbb{E}_R[\tau]$ times the stationary distrubution $\pi$ integrated over $K$ (this follows directly from renewal theory [27, Proposition 7.4.1]). That is, we have

$$\tfrac{\mathbb{E}_R[OT_\tau(K)]}{\mathbb{E}_R[\tau]} = \pi(K), \tag{2}$$

where $\pi(K) = \sum_{v \in V(K)} \pi(v)$.

Now let us assume that all edge lengths of $G$ are rational. Then, we can find a subdivision $G'$ of $G$ such that all edges of $G'$ have the same length. Formally, $G'$ is a metric graph isometric to $G$ as a metric space. Clearly, we can find subgraphs $H_<, H_>$ of $G$ such that $H_< \subseteq H \subseteq H_>$ and each boundary vertex of $H_<$ or $H_>$ is a midpoint of an edge of $G'$, where a boundary vertex of $H_<$ is one incident with the complement of $H_<$, i.e. a point in $H_< \cap \overline{(G \backslash H_<)}$. Thus, since the stationary distribution $\pi$ is proportional to the vertex degree, and since every edge of $G'$ has the same length, we have

$$\pi(H_<) = \tfrac{\ell(H_<)}{\ell(G)} \text{ and } \pi(H_>) = \tfrac{\ell(H_>)}{\ell(G)}. \tag{3}$$

Note that Brownian motion $B$ on $G$ naturally induces a continuous-time random walk $Z(t), t \in \mathbb{R}_+$ on $G'$, and also a discrete time random walk $R(i), i \in \mathbb{N}$. It follows from (ii) in Section 2.3 that the transition probabilities of $Z$ and $R$ coincide with the transition probabilities of the usual random walk on $G'$, where the probability to go from a vertex $v$ to each of its neighbours $w$ is $c(vw)/\sum_{y\tilde{v}} c(vy)$ if we set $c(vy) = 1/\ell(vy)$ for every edge $vy$ incident with $v$.

It is proved in [12, Section 5.1] that, for every subgraph $K$ of $G$, in particular for $K = H_<$ or $K = H_>$, we have

$$\tfrac{\mathbb{E}_R[OT_\tau(K)]}{\mathbb{E}_R[\tau]} = \tfrac{\mathbb{E}_Z[OT_\tau(K)]}{\mathbb{E}_Z[\tau]}. \tag{4}$$

Note that we have $\mathbb{E}_Z[\tau] = \mathbb{E}_B[\tau]$ by the definition of the continuous time random walk $Z$. Moreover, using the fact that each boundary vertex of $H_<$ or $H_>$ is a midpoint of an edge of $G'$, it is possible to prove that

$$\mathbb{E}_Z[OT_\tau(H_<)] = \mathbb{E}_B[OT_\tau(H_<)] \text{ and } \mathbb{E}_Z[OT_\tau(H_>)] = \mathbb{E}_B[OT_\tau(H_>)]$$

because for each edge $e = xy$ of $G'$, the expected number of traversals of $e$ from $x$ to $y$ up to time $\tau$ equal the expected number of traversals of $e$ from $y$ to $x$ (this follows from the same arguments as in the proof of 2), and Brownian motion on an interval from an endpoint is equidistributed with its reflectection around the midpoint. Combining this with (4), (3) and (2), we obtain

$$\frac{\mathbb{E}_B[OT_\tau(H_<)]}{\mathbb{E}_B[\tau]} = \frac{\ell(H_<)}{\ell(G)} \text{ and } \frac{\mathbb{E}_B[OT_\tau(H_>)]}{\mathbb{E}_B[\tau]} = \frac{\ell(H_>)}{\ell(G)}.$$

Since $\mathbb{E}_B[OT_\tau(H_<)] \leq \mathbb{E}_B[OT_\tau(H)] \leq \mathbb{E}_B[OT_\tau(H_>)]$ by the choice of $H_<, H_>$, and $\frac{\ell(H_<)}{\ell(G)}, \frac{\ell(H_>)}{\ell(G)}$ can be made arbitrarily close to $\frac{\ell(H)}{\ell(G)}$ by making the subdivision $G'$ fine enough, our claim $\mathbb{E}_B[OT_\tau(H)] = \mathbb{E}_B[\tau]\frac{\ell(H)}{\ell(G)}$ follows in the case that all edge lengths of $G$ are rational. The general case can now be handled using a standard approximation argument.

Thus if $H, G$ are as in the statement, then, since $T \leq \tau$, we obtain

$$\mathbb{E}_B[OT_T(H)] \leq \mathbb{E}_B[OT_\tau(H)] \leq \mathbb{E}_B[\tau]\frac{\ell}{L}.$$

Now if $\mathbb{P}[OT_T(H) \geq \epsilon] > \epsilon$ then $\mathbb{E}_B[OT_T(H)] > \epsilon^2$. Combined with the above inequality, this yields

$$\mathbb{E}_B[\tau] > \epsilon^2 \frac{L}{\ell}.$$

On the other hand, applying the commute time formula of Lemma 2.5 to the pair of points $o, B(T)$ where $B(T)$ is the random position of the particle at time $T$, we obtain $\mathbb{E}_B[\tau] \leq T + 2L^2$ since, easily, $R(a,z) \leq L$ for every two points $a, z$ of $G$. The latter two inequalities imply $T + 2L^2 \geq \epsilon^2 \frac{L}{\ell}$, and so letting $\ell = \frac{\epsilon^2 L}{T+2L^2}$ proves our assertion. □

The following lemma is of similar flavour

**Lemma 4.2.** *Let $X$ be a graph-like continuum with $\mathcal{H}(X) < \infty$, and $(G_n)_{n\in\mathbb{N}}$ a graph approximation of $X$. For any time $T_0$ and $p \in X$ lying in an edge of $X$, we have*

$$\lim_{r\to 0} \sup_n \mathbb{P}(B_n(T_0) \textit{ is in the ball of radious } r \textit{ centred at } p) = 0.$$

*Proof.* Denote by $p_t^n$ the heat kernel for the Brownian motion $B_n$ i.e. $\mathbb{P}(B_n(t) \in dx) = p_t^n(x)dx$. We will show that for $p \in X$ contained in the interior of some edge, all the functions $p_{T_0}^n$ are uniformly bounded on some neighbourhood of $p$ by the same constant. In fact, we will show that there is an open neighbourhood $U$ of $p$ and $\delta > 0$ such that

$$\sup_n \sup_{(x,t)\in U\times[T_0-\delta, T_0+\delta]} p_t^n(x) < \infty. \tag{5}$$

The proof of the above boundedness is based on the fact that we may identify the interior of edge with an open interval on $\mathbb{R}$ and on these interval $p_t^n$ satisfies the heat equation

$$\partial p_t^n(x) = \partial_x^2 p_t^n(x). \tag{6}$$

This observation allows us to apply a deep result from the theory of partial differential equations. Namely, we can apply Hörmander's theorem (compare with [17] chapter 22.2, see also chapter 15.1 in [29] and Theorem 2.1 in [16]) that says that for any real $s$, positive $l$, $\epsilon$, and compactly supported smooth function $\phi$ there are a natural number $k$ and a smooth function $\psi$ s.t. supp$\psi$ is contained in the $\epsilon$-neighbourhood of supp and satisfies

$$\|\phi f\|_{H(s+l)} \leq \|\psi L^k f\|_{H(s)} + \|\psi f\|_{H(s)}, \tag{7}$$

where $L = -\partial^2_{x_1} + \partial_{x_2}$ and the norm $\|\cdot\|_{H(s)}$ is the Sobolev norm defined by

$$\|f\|^2_{H(s)} = \int_{\mathbb{R}^2} |\hat{f}|^2(\omega)(1+|\omega|^2)^s d\omega. \tag{8}$$

An easy observation gives that $\|\cdot\|_\infty \leq C\|\cdot\|_{H(s)}$, for $s > 1$ and $\|\cdot\|_{H(s)} \leq C\|\cdot\|_1$ for $s < -1$. Taking this into account the Hörmander theorem gives that $\|\phi f\|_\infty \leq \|\psi L^k f\|_1 + \|\psi f\|_1$ for appropriate $\phi$, $\psi$, and $k$.

Now we take $V$ an open neighbourhood of $p$ contained in one edge and $U_1$, $U_2$ such that $p \in U_1 \subset \overline{U_1} \subset U_2 \subset \overline{U_2} \subset V$. We set $f^n(x_1, x_2) = p^n_{x_2}(x_1)$ for $(x_1, x_2) \in V \times [T_0 - 3\delta, T_0 + 3\delta]$. Now let $\phi$ be a smooth function that is equal to 1 on $U_1 \times [T_0 - \delta, T_0 + \delta]$ and its support is contained in $U_2 \times [T_0 - 2\delta, T_0 + 2\delta]$. Since $Lf^n$ vanishes on $V \times [T_0 - 3\delta, T_0 + 3\delta]$ the Hörmander theorem gives that there exist smooth $\psi$, supported on $V \times [T_0 - 3\delta, T_0 + 3\delta]$ such that

$$\sup_{(x_1,x_2)\in U_1\times[T_0-\delta,T_0+\delta]} f^n(x_1, x_2) \leq \|\phi f^n\|_\infty \leq \|\psi f\|_1 \leq 6\delta\|\psi\|_\infty. \tag{9}$$

$\square$

# 5 Uniqueness

The following fact implies that if $\mathcal{H}(X) < \infty$ then the Brownian motion we constructed in Section 3 is uniquely determined by $(X, d)$; in particular, it does not depend on the choice of the graph approximation used.

**Theorem 5.1.** *Let $X$ be a graph-like space with $\mathcal{H}(X) < \infty$. Then for every graph approximation $(G_n)_{n\in\mathbb{N}}$, and any convergent sequence $(o_n)_{n\in\mathbb{N}}$ of points of $X$ with $o_n \in G_n$, standard Brownian motion $B^n_{o_n}$ from $o_n$ on $G_n$ converges weakly to an element of $\mathcal{M}$ independent of the choice of $(G_n)_{n\in\mathbb{N}}$.*

This follows immediately from the following lemma. The independence of the limit from $(G_n)$ follows from the fact that if $(H_n)$ is another graph approximation of $X$, then $G_1, H_1, G_2, H_2, \ldots$ is also a graph approximation.

**Lemma 5.2.** *Let $X$ be a graph-like space with $\mathcal{H}(X) < \infty$ and $(G_n)_{n\in\mathbb{N}}$ a graph approximation of $X$. Let $o_i \in G_i$ be a sequence of points that converges to a point $o \in X$. Then for every finite collection of open sets $A_1, \ldots, A_z$ of $|G|$, and every finite collection of time instants $T_1, \ldots T_k \in \mathbb{R}^+$, the probability $\mathbb{P}[B_n(T_i) \in A_i$ for every $1 \leq i \leq k]$ converges, where $B_n$ denotes standard Brownian motion on $G_n$ from $o_i$.*

The rest of this section is devoted to the proof of Lemma 5.2. As it is rather involved, we would like to offer the reader the option of reading a simpler proof of a weaker result that still contains many of the ideas: the case where $X$ contains a disconnecting edge-set $E$ with $\sum_{e\in E} \ell(e) = \mathcal{H}(X)$.

*The reader choosing this option will be guided throughout the proof as to which parts can be skiped.*

## 5.1 Useful facts about graph-like spaces

We will be using the following terminology and facts from [9].

**Theorem 5.3** ([9])**.** *Let $X$ be a graph-like space with $\mathcal{H}(X) < \infty$ and $(G_n)_{n\in\mathbb{N}}$ be a graph approximation of $X$. Then for every two sequences $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ with $p_n, q_n \in G_n$, each converging to a point in $X$, the effective resistance $R_{G_n}(p_n, q_n)$ converges. If $p_n = p, q_n = q$ are constant sequences, then this convergence is from above, i.e. $\lim_n R_{G_n}(p, q) \leq R_{G_i}(p, q)$ for every $i$.*

*The reader that chose to read the simplified version can now skip to Section 5.1.1.*

A *pseudo-edge* of a metric space $X$ is an open connected subspace $f$ such that $|\partial f| = 2$ and no homeomorphic copy of the interval $(0, 1)$ contained in $\overline{f}$ contains a point in $\partial f$. We denote the elements of $\partial f$ by $f^0, f^1$, and call them the *endpoints* of $f$. Note that every edge is a pseudo-edge. See [9] for further examples.

We define the *discrepancy* $\delta(f)$ of a pseudo-edge $f$ by $\delta(f) := \mathcal{H}(f) - d(f^0, f^1)$, which is always non-negative [9].

**Theorem 5.4** ([9])**.** *For every graph-like continuum $X$ with $\mathcal{H}(X) < \infty$, and every $\epsilon > 0$, there is a finite set $\mathcal{F}$ of pairwise disjoint pseudo-edges of $X$ with the following properties*

*(i)* $\sum_{f\in\mathcal{F}} \mathcal{H}(f) > \mathcal{H}(X) - \epsilon$;

*(ii)* $\sum_{f\in\mathcal{F}} \delta(f) < \epsilon$;

*(iii)* *$X\backslash\mathcal{F}$ has finitely many components, each of which is clopen in $X\backslash\mathcal{F}$ and contains a point in $\overline{\mathcal{F}}$;*

*(iv)* *for every $f \in \mathcal{F}$, and every graph approximation $(G_n)_{n\in\mathbb{N}}$, $G_n \cap \overline{f}$ is connected and contains a $f^0$–$f^1$ path for almost every $n$;*

*(v)* *$\overline{\bigcup \mathcal{F}}$ avoids any prescribed point of $X$;*

*(vi)* *$\mathcal{F}$ contains any prescribed finite edge-set.*

### 5.1.1 Beginning of proof

*Proof of Lemma 5.2.* For simplicity we will assume that $k = 1$, letting $A_1 =: A, T_1 =: T$; the same arguments can be used to prove the general case. Given an arbitrarily small positive real number $\epsilon$, we will find an integer large enough that whenever $n, m$ exceed that integer we have

$$|\mathbb{P}[B_n(T) \in A] - \mathbb{P}[B_m(T) \in A]| < \epsilon. \tag{10}$$

This immediately implies the assertion. So let us fix $T, A$ and $\epsilon$.

Note that it suffices to prove the assertion when $A$ is a basic open set of $X$. By Lemma 2.3 we can assume that the frontier $\partial A$ of $A$ consists of finitely many points, which are inner points of edges. Thus we can choose $\delta \in \mathbb{R}^+$ small enough that $(A)_\delta := \bigcup\{Ball_\delta(a) \mid a \in \partial A\}$, where $Ball_\delta(a)$ is the ball of radius $\delta$ around $a$ in $|G|_\ell$, is a disjoint union of edges.

Moreover, by Corollary 4.2 we can make this $\delta$ small enough that

$$\text{for every } n, \text{ we have } \mathbb{P}[B_n(T) \in ((A)_\delta)] < \epsilon/15. \tag{11}$$

Next, we choose a parameter $\beta$, depending on $\delta$, small enough that it is relatively unlikely that standard Brownian motion will traverse one of the intervals in $\partial A_\delta$ in a time interval of length $\beta$. More precisely, denoting by $W(t)$ the standard Brownian motion on $\mathbb{R}$ starting at the origin, we choose $\beta$ so that

$$\mathbb{P}[\max_{t\in[0,\beta]} |B(t)| > \delta] < \epsilon/15, \tag{12}$$

### 5.1.2 Applying the pseudo-edge structure theorem

Fix a graph approximation $(G_n)_{n\in\mathbb{N}}$ of $X$ for the rest of this proof.

*The reader that chose to read the simplified version can now skip to Section 5.1.4, letting $\mathcal{F}$ be a finite subset of $E$ with $\sum_{f\in\mathcal{F}} \mathcal{H}(f) > \mathcal{H}(X) - \epsilon$, assuming $(A)_\delta \subseteq \mathcal{F}$, letting $\mathcal{K}$ be the set of components of $X\backslash\mathcal{F}$ —which is finite [9, Lemma 2.9]— and letting $\Theta := \bigcup_{f\in\mathcal{F}} \partial f = \bigcup_{K\in\mathcal{K}} \partial K$. Moreover, almost every $G_n$ contains $\mathcal{F}$ [9, Proposition 3.4.], hence it also contains $\Theta$ (14). We may assume that $o_n \in \bigcup\mathcal{K}$ for almost every $n$* (13)*, for if $o$ happens to lie in an edge $e$ in $\mathcal{F}$ we can remove from $\mathcal{F}$ a sufficiently small subedge of $e$ cointaining $o$, making sure that $o \in \bigcup\mathcal{K}$ and all the above is still satisfied. By Theorem 5.3 we have large enough that $\lim_n |R(p,q) - R_{G_n}(p,q)| = 0$ for every $p, q \in \Theta$ ((ii)).*

Applying Theorem 5.4 yields a finite set $\mathcal{F}$ of pairwise disjoint pseudo-edges of $X$ with $\sum_{f\in\mathcal{F}} \mathcal{H}(f) \approx \mathcal{H}(X)$ and $\sum_{f\in\mathcal{F}} \delta(f) \approx 0$. Moreover, $X\backslash\mathcal{F}$ has finitely many components, each of which is clopen in $X\backslash\mathcal{F}$ and contains a point in $\overline{\mathcal{F}}$. Let $\mathcal{K}$ be the set of these components. We can also assume by Theorem 5.4 that for every $f \in \mathcal{F}$, the graph $G_n^f := G_n \cap \overline{f}$ is connected and contains a $f^0$–$f^1$ path for almost every $n$. Moreover, we can assume by (vi) that $(A)_\delta \subseteq \mathcal{F}$. Applying (v) to $o$ we can assume that $\mathcal{F}$ avoids an open neighbourhood of $o$, and hence

$$o_n \in \bigcup\mathcal{K} \text{ for almost every } n. \tag{13}$$

Note that for every component $K \in \mathcal{K}$, we have $\partial K \subseteq \bigcup_{f\in\mathcal{F}} \partial f$ because $K$ is clopen in $X\backslash\mathcal{F}$ and $f$ is open in $X$. Thus we can write $\Theta := \bigcup_{f\in\mathcal{F}} \partial f = \bigcup_{K\in\mathcal{K}} \partial K$. Since we know that the subgraph $G_n^f$ of $G_n$ contains a $f^0$–$f^1$ path for almost every $n$, it follows that

$$G_n \text{ contains } \Theta \text{ for almost every } n. \tag{14}$$

It follows easily from the definitions that for every $f \in \mathcal{F}$, the sequence of graphs $(G_n^f)_{n\in\mathbb{N}}$ is a graph approximation of $\overline{f}$. Thus we can apply Theorem 5.3 to this sequence to deduce that their effective resistances $R_{G_n^f}(f^0, f^1)$ converge to a value that we will denote by $R_f$.

By Theorem 5.3 again, the effective resistances $R_{G_n}(p,q)$ between any two points $p, q$ in the boundary $\partial K$ of some component $K \in \mathcal{K}$ converge with $n$ from above to a value that we will denote by $R(p,q)$ (where we used (14)). Thus we have

(i) $\lim_n |R_f - R_{G_n^f}(f^0, f^1)| = 0$ for every $f \in \mathcal{F}$, and

(ii) $\lim_n |R(p,q) - R_{G_n}(p,q)| = 0$ for every $p, q \in \Theta$.

### 5.1.3 The first coupling

The first step in our proof will be to couple our Brownian motion $B_n$ on $G_n$ with standard Brownian motion $B_n^-$ on a simplified version $G_n^-$ of $G_n$, which can be thought of as being obtained from $G_n$ by turning the pseudo-edges in $\mathcal{F}$ into edges.

*This step can be omitted if $\mathcal{F}$ are edges to begin with, and the reader who chose to read the simplified version of this proof can skip the rest of this subsection.*

Let $G_n^-$ denote the graph obtained from $G_n$ by replacing, for every $f \in \mathcal{F}$, the subgraph $G_n^f = G_n \cap \overline{f}$ with an edge $e_f$ with endvertices $f^0, f^1$ and length $\ell(e_f) = R_{G_n^f}(f^0, f^1)$. Recall that by (13), $o_n \notin \bigcup \mathcal{F}$. If $f$ happens to be an edge to begin with, then it remains an edge of $G_n^-$; in particular, $(A)_\delta$ is still contained in the set of edges of $G_n^-$. Let $B_n^-$ denote standard Brownian motion from $o_n$ on $G_n^-$.

In order to couple $B_n$ with $B_n^-$, we are going to modify $G_n$ into $G_n^-$ in a more elaborate way than described above, using more local changes.

For this, choose some $f \in \mathcal{F}$, and recall that, by the definition of a pseudo-edge, and by (iv), $G_n^f$ is connected, it contains a $f^0$–$f^1$ arc $P$, and both $f^0, f^1$ have degree 1 in $G_n^f$. We can choose $P$ to be the shortest such arc; this is easy to do since $G_n^f$ is a finite graph and so there are only finitely many candidates.

We claim that there is a finite edge-set $\mathcal{P}$ (in the topological sense of Section 2.1) contained in $P$, such that letting $\mathcal{C}$ denote the set of components of $G_n^f - \mathcal{P}$, and letting $\Pi$ denote the finite set $\partial\mathcal{P}\backslash\{f^0, f^1\}$ separating $\mathcal{P}$ from $\mathcal{C}$, we have (see top half of Figure 2)

(i) No $C \in \mathcal{C}$ contains $f^0$ or $f^1$;

(ii) Each $C \in \mathcal{C}$ contains at most 2 elements of $\Pi$, and

(iii) $\sum_{C \in \mathcal{C}} \ell(C) \leq 2\ell(G_n^f\backslash P) \approx 0$.

To show this, for every component $K$ of $G_n^f\backslash P$ we let $P(K)$ denote the minimum subpath of $P$ separating $K$ from $G_n^f\backslash K$; thus $K$ sends at least one edge to each endvertex of $P(K)$ by its minimality. Note that $P(K)$ is trivial, i.e. just a vertex, if that vertex alone separates $K$. Let $B$ denote the union of the $B(K)$ over all such components $K$. Note that $B$ is a disjoint union of subpahts of $P$, some of which might be the union of several intersecting $B(K)$. Let $\mathcal{P}$ be its complement $P\backslash B$, and let $\Pi = \partial\mathcal{P}\backslash\{f^0, f^1\}$ be the set of endvertices of these paths.

It is clear that this choice satisfies (i), since none of the components $K$ above send an edge to $f^0$ or $f^1$ because, since $f$ is a pseudo-edge and $G_n^f$ is contained in it, each of these vertices has only one incident edge, and that edge must be in $P$.

To see that (ii) is satisfied, suppose $C$ contains 3 vertices $x, y, z \in \Pi$ lying in that order on $P$, let $e$ be an edge in $\mathcal{P}$ incident with $y$, and let $R$ be an $x$–$z$ arc in $C$. Let $x'$ be the last point on $R$ in the component of $P\backslash e$ containing $x$, and $z'$ the first point on $R$ in the component of $P\backslash e$ containing $z$. Then the subarc of $R$ from $x'$ to $z'$ avoids $P$ and hence shows that $e$ is contained in $B$. This contradicts our choice of $\mathcal{P}$, and proves (ii).

Finally, (iii) is tantamount to saying that the subgraph $P \backslash \mathcal{P}$ of $P$ contained in $\bigcup \mathcal{C}$ has length at most $\ell(G_n^f \backslash P)$. This follows from our choice of $P$ as a shortest $f^0$–$f^1$ arc: for if we contract each component $K$ of $G_n^f \backslash P$ together with $P(K)$ (as defined above), then we are left with a path of length $\ell(\mathcal{P})$ at the end, and for each contracted subarc $R$ of $P$ we have contracted a subgraph of $G_n^f \backslash P$ of length at least $\ell(R)$.

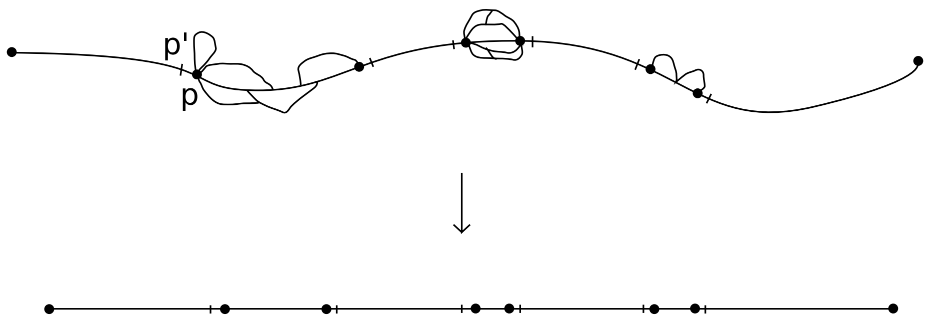


Figure 2: Replacing the components $\mathcal{C}$ of $G_n^f - \mathcal{P}$ with equivalent edges.

Now replace each component $C \in \mathcal{C}$ containing two elements $v, w$ of $\Pi$ with a $v$-$w$ edge of length $R_C(v, w)$ (Figure 2). Then contract any $C \in \mathcal{C}$ that contains only one element of $\Pi$ into that point. Note that this modifies $G_n^f$ into a $f^0$–$f^1$ arc $P'$.

Note that $R_{P'}(f^0, f^1) = R_{G_n^f}(f^0, f^1)$ by Lemma 2.6. Since we chose $\ell(e_f) = R_{G_n^f}(f^0, f^1)$ in the above definition of $G_n^-$, it follows that if we perform these modifications on each $f \in \mathcal{F}$ then the resulting graph will be isometric to $G_n^-$.

In order to couple $B_n$ with Brownian motion $B_n^-$ on $G_n^-$, we pick a set of points $\Pi'$ on $P$ as follows. By definition, every $p \in \Pi$ is incident with exactly one element $R_p$ of $\mathcal{P}$, which is a subpath of $P$. We choose a point $p'$ on $R_p$ that is very close to $p$; more precisely, we choose these points $p'$ in such a way that, letting $r_p$ denote the subarc of $R_p$ between $p$ and $p'$, we have

$$\sum_{p \in \Pi} \ell(r_p) < \ell(G_n^f \backslash P). \tag{15}$$

Since we can choose the $p'$ as close as we wich to $p$, there is no difficulty in satisfying this.

In order to perform the desired coupling, we separate the sample path of $B_n$ into excursions by stopping at first visit to $\Pi$, then at the first visit to $\Pi \cup \Pi'$ thereafter (there are always 2 candidate points at which we can stop, one in $\Pi$ and one in $\Pi'$), then at the next visit to $\Pi$, and so on. To couple with Brownian motion on $G_n^-$, replace each such excursion $R$ starting at a point $p$ in $\Pi$ by an excursion on $G_n^-$ with same starting point $p$ and stopping upon its first visit to $\Pi \cup \Pi'$ (again, there are 2 candidate points at which we can stop), conditioned on stopping at the same point where $R$ stopped.

Since transition probabilities are the same by Lemma 2.7, the resulting process is equidistributed with Brownian motion $B_n^-$ on $G_n^-$. The two graphs differ in that $\bigcup \mathcal{C}$ is replaced by edges. The coupling is such that the two processes only differ as to the time they spend in $\bigcup \mathcal{C} \cup \bigcup_{p \in \Pi} r_p$ or the part of $G_n^-$ that replaces it respectively. We will use Lemma 4.1 to bound this time.

We claim that $B_n$ behaves similarly to $B_n^-$ with respect to our open set $A$; more precisely, we claim that

$$\mathbb{P}[\{B_n^-(T) \in A \text{ and } B_n(T) \notin A\} \text{ or } \{B_n^-(T) \notin A \text{ and } B_n(T) \in A\}] < \epsilon/5. \tag{16}$$

To prove this, suppose that the event appearing in (16) occured. Recall that the two graphs $G_n, G_n^-$ differ in that $\bigcup \mathcal{C}$ is replaced by a set of edges $E_\mathcal{C}$. Let $\Delta_1 := OT_{2T}(\bigcup \mathcal{C})$ and $\Delta_2 := OT_{2T}^-(E_\mathcal{C})$ denote the occupation time of this difference $\bigcup \mathcal{C}$ or $E_\mathcal{C}$ by $B_n$ and $B_n^-$ respectively up to time $2T$ (the reason for the factor 2 will become apparent below). We claim that in this case, at least one of the following (unlikely) events occured as well:

(i) $\Delta_1 > \beta$ or $\Delta_2 > \beta$ (large occupation time of a small set);

(ii) $B_n(T)$ or $B_n^-(T)$ is in $(A)_\delta$ (particle in a small set at time $T$);

(iii) $B_n([T, T+\beta])$ or $B_n^-([T, T+\beta])$ crosses an edge in $(A)_\delta$ (fast crossing of an edge).

To see this, let $\tau_i$ denote the time $t$ that $B_n(t)$ has just crossed $(A)_\delta$ for the $i$th time; thus $B_n(\tau_i)$ is an endpoint of $(A)_\delta$, and $B_n[t, \tau_i]$ is contained in some edge in $(A)_\delta$ for sufficiently large $t$. Define $\tau_i^-$ similarly for $B_n^-(t)$. Note that if $o_n \in A$, then $B_n(\tau_{2i+1}) \in A^c$ and $B_n(\tau_{2i}) \in A$ for every $i \in \mathbb{N}^*$ since $(A)_\delta$ separates $A$ from its complement $A^c$, and so in order to 'change sides' from $A$ to $A^c$ the particle has to cross $(A)_\delta$.

Let $k$ denote the largest integer such that $\tau_k < T$, and $m$ the largest integer such that $\tau_k^- < T$; since $(A)_\delta$ is a finite edge-set, these numbers are well-defined since $B_n[0, T] \in C$ is continuous and can therefore only cross $(A)_\delta$ finitely often. Now if the event appearing in (16) occured, but (ii) did not, then $k \neq m$. Suppose that $k > m$; the other case is similar. This means that $\tau_k < T$ and $\tau_k^- \geq T$.

Let us assume without loss of generality that $T < \beta$, which we can because we can choose $\beta$ as small as we wish. It is not hard to see that, unless (i) occured, $\tau_k^- < 2T$ holds, since the two processes only differ in their excursions inside $\bigcup \mathcal{C}$ or $\bigcup \mathcal{C}$, and their duration yields a bund on how much $\tau_k^-$ can differ from $\tau_k$.

Note that $\tau_k^- - \tau_k \leq OT_{\tau_k^-}(E_\mathcal{C}) - OT_{\tau_k}(\bigcup \mathcal{C}) \leq OT_{2T}(E_\mathcal{C}) - OT_{\tau_k}(\bigcup \mathcal{C})$ by the above argument. Thus if the event (i) did not occur, then $\tau_k^- - T \leq \beta$ holds since $\tau_k < T$. Since $B_n^-(\tau_k^-)$ has just crossed $(A)_\delta$, this means that either $B_n^-(T)$ is in $(A)_\delta$, or $B_n^-([T, T+\beta])$ traversed an edge in $(A)_\delta$; but this is event (ii) or (iii) respectively.

This proves our claim that the event appearing in (16) implies one of the above events. The probability of each of these 3 events can be shown to be less than $\epsilon/15$: firstly, by Lemma 4.1, and by (iii) and (15), given $L, T, \epsilon$ and $\beta$ we can make the expectation of $\Delta_1$ and $\Delta_2$ arbitrarily small if we can make $\ell(G_n^f \backslash P)$ small enough. We can make the latter arbitrarily small indeed because it is bounded from above by the discrepancy $\delta(f)$ of $f$, which we can make arbitrarily small by (ii) in Theorem 5.4; here, we use the fact that $\ell(G_n^f) \leq \mathcal{H}(f)$ and $\ell(P) \geq d(f^0, f^1)$. Thus the probability of (i) can be made less than $\epsilon/15$.

Secondly, (11) shows that the probability of (ii) is less than $\epsilon/15$ as well. Finally, the choice of $\beta$ (recall (12)) makes (iii) equally unlikely. This completes the proof of (16), which implies in particular

$$|\mathbb{P}[B_n(T) \in A] - \mathbb{P}[B_n^-(T) \in A]| < \epsilon/5. \tag{17}$$

#### 5.1.4 The second coupling

*The reader who chose to read the simplified version can assume that* $G_n^- = G_n$ *and* $B_n^- = B_n$*. This reader will also need the following definitions. Let* $\Theta' := \Theta$ *and* $e'_f = e_f = f$*. For each point* $p \in \Theta$*, choose a further point* $p''$ *inside* $f$ *that is close to* $p$ *(Figure 3); more precisely, we choose these points* $p''$ *in such a way that, letting* $e_p$ *be the interval of* $f$ *between* $p$ *and* $p''$*, we have* $\sum_{p \in \Theta} \ell(e_p) \approx 0$ (18)*. Let also* $e_{p'} = e_p$ *and skip to Definition 5.5.*

In this section we will couple the processes $B_n^-$ with jump process $B_n^*$, which we will later show that can be coupled between them for various values of $n$.

Recall that the effective resistance $R_{G_n^f}(f^0, f^1)$, which we assigned to each edge $e_f$ as its length $\ell(e_f)$, converges to a value $R_f$ from above. Thus for every such edge $e_f, f \in \mathcal{F}$, we can choose an interval $e'_f$ with length $\ell(e'_f) = R_f$ independent of $n$.

Let $\Theta'$ denote the set of endpoints $\partial \bigcup_{f \in \mathcal{F}} e'_f$ of these edges, and note that each point $p' \in \Theta'$ is close to a point $p \in \Theta$ by (i); more precisely, letting $e_p$ be the interval of $e_f$ between $p$ and $p'$, we have

$$\textstyle\sum_{p \in \Theta} \ell(e_p) < h, \tag{18}$$

where $h = h(\epsilon, T)$ is a parameter that we can choose to be as small as wish by choosing $n$ large enough.

For each such point $p' \in \partial e'_f$ we choose a further point $p''$ inside $e_f$ that is close to $p'$ (Figure 3); more precisely, we choose these points $p''$ in such a way that, letting $e_{p'}$ be the interval of $e_f$ between $p'$ and $p''$, we have

$$\textstyle\sum_{p' \in \Theta} \ell(e_{p'}) < h. \tag{19}$$

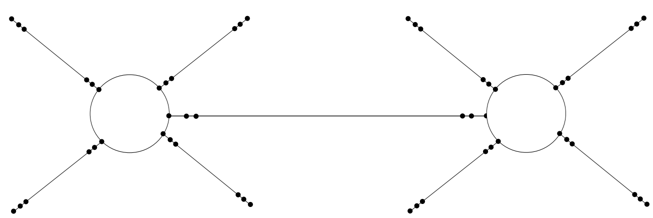

Figure 3: The sets $\Theta, \Theta'$ and $\Theta''$ around two components in $\mathcal{K}$.

Let $\Theta'' := \{p'' \mid p' \in \Theta'\}$. We will use the points in $\Theta'$ and $\Theta''$ similarly to the sets $\Pi, \Pi'$ in the previous section to produce a new process $B_n^*$ coupled with $B_n^-$.

**Definition 5.5.** *Let* $G_n^*$ *be the metric graph obtained from* $G_n^-$ *by contracting each component of* $G_n^- \backslash \Theta'$ *containing an element* $K$ *of* $\mathcal{K}$ *—recall that this was the (finite) set of components of* $X \backslash \mathcal{F}$*— into a vertex* $v_K$*.*

Thus each contracted set comprises a $K \in \mathcal{K}$ and a short subedge of each edge of $G_n^-$ incident with $K$.

Note that $G_n^*$ is isometric to $G_m^*$ for $n, m$ large enough, because $\mathcal{K}$ and $\mathcal{F}$ are fixed and so are the lengths of the edges $e'_f$ of $G_n^*$. We can thus denote by $G^*$ a metric graph isometric to all $G_n^*$, and let $\iota_n : G_n^* \to G^*$ be the corresponding isometry. Moreover, if $f \in \mathcal{F}$ happens to be an edge, e.g. one of the edges in $(A)_\delta$, then we have $e'_f = f$ in the above definition; this means that $\iota_n(\partial A) = \iota_m(\partial A)$.

We now modify $B_n^-(t)$ into a jump process $B_n^*(t)$ on $G_n^-$, that can also be thought of as a jump process on $G_n^*$. The jumps are always performed from $\Theta' \cup \{o_n\}$ to $\Theta''$ and are quite local, so that $B_n^*(t)$ is similar to $B_n^-(t)$. The advantage of $B_n^*(t)$ is that we can couple these processes for various values of $n$ more easily, since they can be projected to the fixed graph $G^*$ via $\iota_n$. Moreover, it will turn out that the event we are interted in, namely whether $B_n^-(T)$ lies in $A$ or not, is tantamount to the projected particle being in the right side of $G^* \backslash \iota(\partial A)$.

To obtain $B_n^*(t)$ from $B_n^-(t)$, we first sample the path of the latter, then we go through this path and each time we visit a point $x$ in $\Theta'$, we jump from $x$ directly to the first point $y$ in $\Theta''$ visited afterwards, removing the corresponding time interval from the domain of $B_n^-(t)$ to obtain $B_n^*(t)$ (at the time instant $t$ where this jump occurs we set $B_n^*(t) = y$, say, so that $B_n^*(t) = y$ is right-continuous).

Recall that $o_n \in \bigcup \mathcal{K}$ for every $n$ (13). When constructing $B_n^*(t)$ from $B_n^-(t)$, we thus also jump over the initial subpath of $B_n^-(t)$ from $o_n$ to the first point $y$ in $\Theta''$ visited, so that $B_n^*(0) = y \in \Theta''$.

Note that $\Theta'$ and $\Theta''$ are finite sets, whence closed in $X$, and so for any topological path (like $B_n^-(t)$) the first visit to any of them is well-defined by elementary topology. Note moreover that we have only finitely many such jumps in the time interval $[0, T]$ because $B_n^-$ is continuous.

As mentioned above, $B_n^*(t)$ can be thought of as a jump process on $G_n^*$ or $G^*$; the jumps occur whenever a vertex of $G^*$ is visited, and lead to a nearby point of an edge incident with that vertex. From then on, the process behaves like standard Brownian motion untill the next visit to a vertex. We will use Lemma 4.1 to show that the time intervals jumped by $B_n^*(t)$ are relatively short, and so the two processes $B_n^-(t)$ and $B_n^*(t)$ are very similar.

### 5.1.5 The jump process $B_n^*$ is similar to $B_n^-$

Our next aim is to show that $B_n^-$ behaves similarly to $B_n^*$ with respect to our open set $A$; more precisely, we claim that

$$\mathbb{P}[\{B_n^*(T) \in A \text{ and } B_n^-(T) \notin A\} \text{ or } \{B_n^*(T) \notin A \text{ and } B_n^-(T) \in A\}] < \epsilon/5. \tag{20}$$

The proof of this is almost identical to the proof of (16), but we will reproduce it for the convenience of the reader.

Let $\Delta$ denote the total duration of the intervals 'jumped' by $B_n^*$ in the time interval $[0, 2T]$. In order for the event appearing in (20) to occur, at least one of the following events must occur:

(i) $\Delta > \beta$;

(ii) $B_n^-(T)$ is in $(A)_\delta$;

(iii) $B_n^-([T, T+\beta])$ traverses an edge in $(A)_\delta$.

To see this, let $\tau_i$ denote the time $t$ that $B_n^-(t)$ has just crossed $(A)_\delta$ for the $i$th time; thus $B_n^-(\tau_i)$ is an endpoint of $(A)_\delta$. Define $\tau_i^*$ similarly for $B_n^*(t)$. Again, if $o_n \in A$, then $B_n^-(\tau_{2i+1}) \in A^c$ and $B_n^-(\tau_{2i}) \in A$ for every $i \in \mathbb{N}^*$ since $(A)_\delta$ separates $A$ from its complement $A^c$, and so in order to 'change sides' from $A$ to $A^c$ the particle has to cross $(A)_\delta$.

Let $k$ denote the largest integer such that $\tau_k < T$, and $m$ the largest integer such that $\tau_m^* < T$; since $(A)_\delta$ is a finite edge-set, these numbers are well-defined since $B_n^-[0,T] \in C$ is continuous and can therefore only cross $(A)_\delta$ finitely often. Now if the event appearing in (20) occured, but (ii) did not, then $k \neq m$, hence $k < m$ since $B_n^*(t)$ is by definition faster than $B_n^-(t)$. This means that $\tau_m \geq T$ although $\tau_m^* < T$.

Let $Y := \bigcup \mathcal{K} \cup \bigcup_{p \in \Theta}(e_p \cup e_{p'})$, and recall that this is the subgraph of $G^-$ inside which $B_n^*(t)$ performs its jumps. Let us assume without loss of generality that $T < \beta$, which we can because we can choose $\beta$ as small as we wish. It is not hard to see that, unless (i) occured, $\tau_m < 2T$ holds, since $\tau_m^* < T$ and the duration of the excursions inside $Y$ yields a bound on how much $\tau_m$ can differ from $\tau_m^*$.

Now note that $\tau_m - \tau_m^* \leq OT_{\tau_m}(Y; B_n^-)$. Thus if the event (i) did not occur, then $\tau_m - T \leq \beta$ holds since $\tau_m^* < T$. Since $B_n^-(\tau_m)$ has just crossed $(A)_\delta$, this means that either $B_n^-(T)$ is in $(A)_\delta$, or $B_n^-([T, T+\beta])$ traversed an edge in $(A)_\delta$; but this is event (ii) or (iii) respectively.

This proves our claim that the event appearing in (20) implies one of the above events. The probability of each of these 3 events can be shown to be less than $\epsilon/15$: firstly, by Lemma 4.1, given $L, T, \epsilon$ and $\beta$ we can make the expectation of $\Delta$ arbitrarily small if we can make $\ell(\bigcup \mathcal{K} \cup \bigcup_{p \in \Theta}(e_p \cup e_{p'}))$ small enough. We can make the latter arbitrarily small indeed by (18), (19) and by (i) in Theorem 5.4 since $\bigcup \mathcal{K}$ is the complement of $\mathcal{F}$. Thus the probability of (i) can be made less than $\epsilon/15$. Secondly, (11) shows that the probability of (ii) is bounded by $\epsilon/15$ as well. Finally, the choice of $\beta$ (recall (12)) makes (iii) equally unlikely. This completes the proof of (20), which implies in particular

$$|\mathbb{P}[B_n^-(T) \in A] - \mathbb{P}[B_n^*(T) \in A]| < \epsilon/5. \tag{21}$$

#### 5.1.6 $B_n^*$ is similar to $B_m^*$ for $n, m$ large; the last coupling

We have thus shown that $\mathbb{P}[\{B_n^-(T) \in A]$ is very close to $\mathbb{P}[B_n^*(T) \in A]$. It remains to show that the dependence of the latter on $n$ can be ignored: we claim that

$$|\mathbb{P}(B_n^*(D) \in A) - \mathbb{P}(B_m^*(D) \in A)| < \epsilon/5. \tag{22}$$

Combined with (17) (*which the reader of the simpler version can take for trivially true* ) and (21), this would imply (10).

For this, we would first like to bound the number of times that $B_n^*(T)$ commutes between $\Theta''$ and $\Theta'$. But this is easy to achieve: Let $r := min_{p' \in \Theta'} \ell(e_{p'})$. We claim that there is a constant $M = M(r)$ large enough that the probability that $B_n^-$ commutes between $\Theta''$ and $\Theta'$ more than $M$ times in the time interval $[0,T]$ is $< \epsilon/15$. Indeed, as $r > 0$, there is a positive probability $q$, depending only on $r$, that the time it takes $B_n^-$ to traverse any of the edges $e_{p'}, p' \in \Theta'$ is at least $T$. Since any commute between $\Theta''$ and $\Theta'$ involves such a traversal, commuting between $\Theta''$ and $\Theta'$ more than $M$ times in the time interval $[0, T]$ thus happens with probability at most $(1-q)^M$. Choosing $M$ large enough we can make this probability as small as we wish. As $\Delta$ is probably small (see previous section), we may assume that the probability that $B_n^*([0,T])$ commutes between $\Theta''$ and $\Theta'$ more than $2M$ times is also less than $\epsilon/15$.

For the proof of (22) it is useful to considered $B_n^*$, or rather $\iota_n(B_n^*)$, as a jump process on $G^*$, for then $B_n^*$ and $B_m^*$ take place on the 'same' metric graph and are easier to couple. To achieve this coupling, we first construct a more convenient realisation of $B_n^-$ as follows. Pick for every $p \in \Theta''$ a sequence $C_{n,1}^p(t), C_{n,2}^p(t) \ldots$ of i.i.d. sample paths of Brownian motion on $G_n^-$, each distributed like $B_n^-(t)$ starting from $p$ and stopping upon their first visit to $\Theta'$. Similarly, pick for every $q \in \Theta'$ a sequence $D_{n,1}^q(t), D_{n,2}^p(t) \ldots$ of i.i.d. sample paths of Brownian motion on $G_n^-$, each distributed like $B_n^-(t)$ starting from $q$ and stopping upon their first visit to $\Theta''$. These sample paths can be glued together to produce a path distributed identically to $B_n^-(t)$: start a Brownian motion at $o_n$, and stop it upon its first visit to a point $q$ in $\Theta''$. Append to this random path the path $C_{n,1}^p$. If the last point visited by the latter is $q$, then append $D_{n,1}^q$. Continue like this, appending paths of the form $C_{n,i}^p$ and $D_{n,j}^q$ alternatingly, each time choosing the right $p$ or $q$ and the smallest $i$ or $j$ for which the path $C_{n,i}^p$ or $D_{n,j}^q$ has not been used yet. As Brownian motion has the Markov property, the random path thus obtained has indeed the same distribution as $B_n^-$.

The advantage of this realisation of $B_n^-$ is that the paths $C_{n,i}^p$ can be coupled with the $C_{m,i}^p$ for every $n, m$. Now note that by construction, the process $B_n^*$ is obtained from $B_n^-$ by discarding all the $D_{n,i}^q$ in the above construction, as well as the initial path from $o_n$ to the first visit to $\Theta''$.

This means that another realisation of $B_n^*$ can be constructed directly by concatenating random paths of the form $C_{n,i}^p$ rather than first constructing $B_n^-$ as above, and then discarding some of its subpaths. For this, we choose a random starting point $p \in \Theta''$ according to the distribution $P_n^o$ of the first point in $\Theta''$ visited by Brownian motion from $o_n$ in $G_n^-$, and use the path $C_{n,1}^p$. Then we recursively concatenate this path with further paths of this form. In order to decide which path $C_{n,i}^p$ to use next, let $q \in \Theta'$ be the last point visited by the last such path used, choose a random $p \in \Theta''$ according to the distribution $P_n^q$ of the first point in $\Theta''$ visited by Brownian motion from $q$ on $G_n^-$, and use $C_{n,i}^p$ for the least $i$ for which this path has not been used yet to extend the path obtained so far.

The probability distributions $P_n^q, q \in \Theta' \cup \{o_n\}$ used above depend little on $n$: note that $P_n^q$ and $P_m^q$ have the same finite domain $\Theta''$. By Lemmas 2.7 and 5.3, these distributions converge. This means that we can couple the experiments of choosing one point in $\Theta''$ according to $P_n^q$ and one according to $P_m^q$ in such a way that the probability that the two experiments yield a different point is smaller than $\epsilon/15(2M)$, say, if $n, m$ are sufficiently large (this remains true if $q = o_n$ in the first case and $q = o_m$ in the second).

Combining this coupling with that of the $C_{n,i}^p$, we deduce that $B_n^*$ can be coupled with $B_m^*$ in such a way that they coincide up to the first time that they jump to a distinct element of $\Theta''$, an event occuring with probability smaller than $\epsilon/15(2M)$ each time that a jump is made. The choice of $M$ now implies that $B_n^*$ coincides with $B_m^*$ up to time $T$ with probability at least $1-\epsilon/10$ when the processes are so coupled. This proves (22).

Combining this with (17) and (21), each applied once for $l=n$ and once for $l=m$, yields $|\mathbb{P}(B_n^-(T)\in A)-\mathbb{P}(B_m^-(T)\in A)| << \epsilon$, and so $\mathbb{P}(B_n^-(T)\in A)$ converges indeed. $\square$

# 6 Strong Markov Property

By the previous section we know that for any open $A$ in $G$ and $x_n \to x$ the probabilities $\mathbb{P}_{x_n}[B_n(t)\in A]$ converge to $\mathbb{P}_x[B(t)\in A]$. For any continuous function $f$ on $G$, by portmanteau theorem, $P_t^n f(x_n)$ also converge to $P_t f(x) := \mathbb{E}_x[f(B(t))]$ where $P_t^n f(y) = \mathbb{E}_y[f(B_n(t))]$ for $y \in G_n$ and it is extended by 0 to $G$.

The strong Markov property follows by similar methods as in [2]. We start with elementary lemma

**Lemma 6.1.** *Suppose $f$ and $f_n$ are functions on $G$ with the property that $f_n(x_n)\to f(x)$ whenever $x_n \in G_n$, $x_n \to x$. Then $f$ is continuous and*

$$\sup_{y\in G_n} |f_n(y) - f(y)| \to 0.$$

*Proof.* In order to prove continuity observe that, by density of $\bigcup G_n$ in $G$, it is enough that show that for $x_n \in G_n$, $x_n \to x$ we have $f(x_n)\to f(x)$. Since $f_m(x_n)\to f(x_n)$, we can take an increasing sequence $m_n$ such that $f_{m_n}(x_n) - f(x_n)$ goes to zero. Since $f_{m_n}(x_n)$ is a subsequence of $f_k(x'_k)$, where $x'_k = x_n$ when $k\in[m_n, m_{n+1})$, $f_{m_n}(x_n)\to f(x)$. This gives that $f$ is continuous.

Suppose that the second part of the theorem fails. Then we have a subsequence $n_k$ and $x_{n_k}\to x$ with $|f_{n_k}(x_{n_k}) - f(x_{n_k})| > \epsilon$ for some $\epsilon > 0$. But

$$|f_{n_k}(x_{n_k}) - f(x_{n_k})| \le |f_{n_k}(x_{n_k}) - f(x)| + |f(x) - f(x_{n_k})|$$

goes to zero by assumption and the continuity of $f$. This contractions proves the theorem. $\square$

**Corollary 6.2.** *For $t > 0$ and a continuous function $f$ on $G$, $P_t f$ is also continuous and*

$$sup_{y\in G_n} |P_t f(y) - P_t^n f(y)| \to 0.$$

**Theorem 6.3.** *$P_t$ is a Feller semigroup. In particular the process $B(t)$ satisfies the strong Markov property.*

*Proof.* By the previous corollary we know that $P_t$ maps $C(G)$ into $C(G)$. First we show that it the family $\{P_t\}$ is a semigroup.

From the Markov property of $B_n$ we have that $P_{t+s}^n = P_t^n P_s^n$. Therefore it is enough to show that $P_t^n P_s^n f(x_n)$ converge to $P_t P_s f(x)$ whenever $x_n \to x$.

$$\begin{aligned} &|P_t^n P_s^n f(x_n) - P_t P_s f(x)| \\ \le |P_t^n P_s^n f(x_n) - P_t^n P_s f(x_n)| + &|P_t^n P_s f(x_n) - P_t P_s f(x_n)| \\ &+ |P_t P_s f(x_n) - P_t P_s f(x)| \end{aligned}$$

Since the first term is bounded by $\sup_{y \in G_n} |P_s^n f(y) - P_s f(y)|$ it goes to 0 by the previous corollary. Similarly, the second term converge to zero since $P_s f$ is continuous. The last term vanishes since $P_t P_s f$ is continuous.

Since, $B(t)$ is continuous and $B(0) = x$, we have $P_t f(x) \to f(x)$ for any continuous function $f$. □

# 7 Cover Time

The (expected) *cover time* $CT_o(G)$ of a finite metric graph $G$ from a point $o \in G$ is the expected time untill standard Brownian motion from $o$ on $G$ has visited every point of $G$. The cover time of $G$ is $CT(G) := \sup_{o \in G} CT_o(G)$. It is proved in [12] that there is an upper bound on $CT(G)$ depending only on the total length $\ell(G)$ of $G$ and not on its structure

**Theorem 7.1** ([12]). *For every finite graph $G$ and $\ell : E(G) \to \mathbb{R}_{>0}$, we have $CT(G) \leq 2\ell(G)^2$.*

In this section we use this fact to deduce the corresponding statement for our Brownian motion $B$ on a graph-like continuum $X$: defining $CT(X)$ as above, with standard Brownian motion replaced by our process $B$, we prove

**Theorem 7.2.** *For every graph-like continuum $X$ with $\mathcal{H}(X) = L < \infty$, we have $CT(X) \leq 20L^2$.*

In order to prove it we will need the following bound on the second moment of the cover time in terms of its expectation.

**Lemma 7.3.** *Let $G$ be a finite metric graph, and denote by $\tau_x$ the (random) cover time from $x \in G$. Suppose that for a constant $Q \in \mathbb{R}$ we have $\mathbb{E}[\tau_x] \leq Q$ for every $x \in G$. Then $\mathbb{E}[\tau_x^2] \leq 24Q^2$ for every $x \in G$.*

*Proof.* By the Chebyshev inequality we have

$$\mathbb{P}\left[\tau_x \geq s\right] \leq \mathbb{E}[\tau_x]/s \leq Q/s,$$

for every $s$; setting $s = 2Q$, we obtain

$$\mathbb{P}\left[\tau_x \geq 2Q\right] \leq 1/2. \tag{23}$$

We claim that for every $k \in \mathbb{N}$ we have

$$\mathbb{P}\left[\tau_x \geq 2Qk\right] \leq (1/2)^k. \tag{24}$$

To see this, we subdivide time into intervals of length $2Q$. Since (23) holds for every starting point $x$, the probability that in the $i$th time interval $[(i-1)2Q, i2Q]$ the process fails to cover the whole space $G$ is at most $1/2$. Thus, if we run the process up to time $2Qk$, in which case we have $k$ such 'trials', the probability of not covering $G$ in any of them is at most $(1/2)^k$, proving our claim. Note that we have been generous here, as we are ignoring the part of $G$ that was covered before the $i$th interval begins.

Using this, we can bound the second moment of $\tau$ as follows

$$\mathbb{E}[\tau_x^2] = \int_0^\infty 2t\mathbb{P}\left[\tau_x \geq t\right] dt,$$

by Fubini's theorem. Splitting time $t$ into intervals of length $2Q$, the last integral can be rewritten as

$$\sum_{k=0}^{\infty}\int_{k2Q}^{(k+1)2Q} 2t\mathbb{P}\left[\tau_x \geq t\right] dt \leq 2\sum_{k=0}^{\infty}\int_{k2Q}^{(k+1)2Q} t\mathbb{P}\left[\tau_x \geq k2Q\right] dt$$

$$= 2\sum_{k=0}^{\infty}(2Q)^2(k+1/2)\mathbb{P}\left[\tau_x \geq 2kQ\right] \leq 8Q^2\sum_{k=0}^{\infty}(k+1/2)(1/2)^k = 24Q^2.$$

□

Using our bound for the second moment of $\tau$ from Lemma 7.3 we can now bound the first moment:

**Lemma 7.4.** *Let $(G_n)_{n\in\mathbb{N}}$ be a graph approximation of a graph-like continuum $X$. Suppose that for a constant $Q \in \mathbb{R}$ we have $\mathbb{E}_{B_n}\left[\tau_x\right] \leq Q$ for every $x \in G_n$. Then for every $x \in X$*

$$\mathbb{E}_B\left[\tau_x\right] \leq 10Q.$$

*Proof.* We would like to use the weak convergence of the law $\mu_n$ of Brownian motion $B_n$ on $G_n$ to the law $\mu$ of our limit process $B$ (Theorem 1.1) to deduce that $\mathbb{E}_x\left[\tau\right]$ is finite from Theorem 7.1. However, we cannot do so directly as the cover time $\tau$ is not a continuous function from $C$ to $\mathbb{R}$. To overcome this difficulty, we introduce a function $h(t,\omega) : C \to \mathbb{R}$ (parametrised by time $t$) that is continuous and is closely related to $\tau$.

Let $r > 0$ be some (small) real number. For a path $\omega \in C$, denote by $h'_r(t)[\omega]$ the total length of the set $\{x \in G \mid d(x, \omega(s)) > r \text{ for every } s \leq t\}$; in other words, if we thing of $\omega$ as the trajectory of a particle of 'width' $r$, then $h'_r(t)[\omega]$ is the length of the part of $G$ that this particle has not covered by time $t$. We also define the normalised version $h_r(t)[\omega] := h'_r(t)[\omega]/L$, where $L$ is again the total length of $G$. It is no loss of generality to assume that $L = 1$.

For every fixed $T, M \in \mathbb{R}$, the function

$$\omega \mapsto \left(\int_0^T (h_r(t)[\omega])^{1/M} dt\right)^2$$

as a mapping from $C$ to $\mathbb{R}$ is continuous. We can now use the weak convergence of $\mu_{n,o}$ to $\mu_o$ to obtain

$$\mathbb{E}_x\left[\left(\int_0^T (h_r(t))^{1/M} dt\right)^2\right] = \lim_{n\to\infty} \mathbb{E}_x^n\left[\left(\int_0^T (h_r(t))^{1/M} dt\right)^2\right]$$

$$\leq \lim_{n\to\infty} \mathbb{E}_x^n\left[\left(\int_0^T \mathbb{1}_{[h_r(t)>0]} dt\right)^2\right],$$

where we used the fact that $h_r(t) \leq 1$. Since $\ell(G) - \ell(G_n)$ converges to 0, we deduce that if a path $\omega$ covers $G_n$ at time $t$, for sufficiently large $n$ compared to $r$, then $h_r(t)[\omega] = 0$. It follows that the expression in parenthesis can be bounded from above by $\tau$, and so by Lemma 7.3 we conclude that

$$\mathbb{E}_x\left[\left(\int_0^T (h_r(t))^{1/M} dt\right)^2\right] \leq \mathbb{E}_x\left[\tau^2\right] \leq 24Q^2. \tag{25}$$

Now let $\epsilon > 0$. Note that if $h_r(T) > \epsilon$, then $h_r(t) > \epsilon$ holds for every $t < T$ since $h_r(t)$ is decreasing in $t$. This easily implies

$$\mathbb{E}_x\left[T^2\epsilon^{2/M}\mathbb{1}_{[h_r(T)>\epsilon]}\right] \leq \mathbb{E}_x\left[\left(\int_0^T (h_r(t))^{1/M} dt\right)^2\right],$$

which combined with (25) yields

$$T^2\epsilon^{2/M}\mathbb{P}_x\left[h_r(T) > \epsilon\right] \leq 24Q^2.$$

As $M$ can be chosen arbitrarily large independently of $\epsilon$, we have

$$\mathbb{P}_x\left[h_r(T) > \epsilon\right] \leq 24Q^2/T^2.$$

Letting $\epsilon$ tend to 0 we deduce

$$\mathbb{P}_x\left[h_r(T) > 0\right] \leq 24Q^2/T^2.$$

Observe that the events $\{h_r(T) > 0\}$ decrease to $\{h_0(T) > 0\} = \{\omega : \tau(\omega) > T\}$ as $r$ goes to 0. Hence

$$\mathbb{P}_x\left[\tau > T\right] \leq 24Q^2/T^2.$$

Finally, we have

$$\mathbb{E}_x\left[\tau\right] = \textstyle\int_0^\infty \mathbb{P}_x\left[\tau > t\right] dt \leq Q\sqrt{24} + \int_{Q\sqrt{24}}^\infty 24Q^2/t^2 dt = 2\sqrt{24}Q < 10Q. \quad (26)$$

□

To prove Theorem 7.2, let $(G_n)_{n\in\mathbb{N}}$ be any graph approximation of $X$. Note that $\ell(G_n) \leq \mathcal{H}(X) =: L$ for every $n$ by the definition of $\mathcal{H}$. Thus we can plug the constant $Q = 2L^2$ from Lemma 7.1 into Lemma 7.4 to obtain the bound $10Q = 20L^2$ on the cover time of $X$.

**Corollary 7.5.** *$B_t$ is positive recurrent.*

# 8 Further properties

In this section we show that Hausdorff measure on $X$ is stationary for our process, and that our process behaves locally like standard Brownian motion on $\mathbb{R}$ inside any edge of $X$.

Recall, that any edge $e \subset X$ can be viewed as an interval contained in the real line, that is, there is $F : e \to \mathbb{R}$ which is an isometry onto its image. The next lemma shows that our process $B$ locally coincides with the standard Brownian motion $W$.

**Proposition 8.1.** *Let $e$ be an edge in $X$. For any continuous function $\phi$ with $k-1$ arguments each taking values in $e$, any increasing sequence $t_1, t_2, \ldots, t_k$, and any $x \in e$, we have*

$$\mathbb{E}_x[\phi(F(B(t_1)), \ldots, F(B(t_{k-1}))1_{t_k<\tau_{\partial e}}] = \mathbb{E}_{F(x)}[\phi(W(t_1), \ldots, W(t_{k-1}))1_{t_k<\tau_{\partial F(e)}}]$$

*Proof.* Since the equation is true for $B_n$, we would like to pass to limit with $n$ to prove that $B$ also satisfies this, but first we have to deal with the discontinuity of the indicator under the expectation sign. For any $\delta > 0$ and $n$ we have

$$\mathbb{E}_x[\phi(F(B^n(t_1)), \dots, F(B^n(t_{k-1}))dist(B^n[0,t_k], \partial e)^\delta] = \\ \mathbb{E}_{F(x)}[\phi(W(t_1), \dots, W(t_{k-1}))dist(W[0,t_k], \partial F(e))^\delta]$$

Since the function under the expectation sign is continuous, now we can pass to a limit with $n$ and next, by Lebesgue theorem, with $\delta$ to 0 proving the desired equality. □

**Proposition 8.2.** *The Hausdorff measure $\mathcal{H}$ on $X$ is the stationary measure for process $B$.*

*Proof.* Let $(G_n)_{n\in\mathbb{N}}$ be a graph approximation of $X$. Then $\mathcal{H}_n := \mathcal{H}(G_n)$ is a sum of lengths of edges of $G_n$, and it is proved in [9] that $\mathcal{H}(X) = \lim_n \mathcal{H}(G_n)$. Moreover, it is not hard to check that the measure $\mathcal{H}_n$ is invariant for $P_t^n$. Hence, by Lebesgue theorem, for any bounded continuous $f$, we have

$$\langle P_t f, \mathcal{H}\rangle = \lim_n \langle P_t^n f, \mathcal{H}\rangle = \lim_n \langle 1_{G_n} P_t^n f, \mathcal{H}\rangle \\ = \lim_n \langle P_t^n f, \mathcal{H}_n\rangle = \lim_n \langle f, \mathcal{H}_n\rangle = \lim_n \langle 1_{G_n} f, \mathcal{H}\rangle = \langle f, \mathcal{H}\rangle.$$

□

# 9 Outlook

It would be interesting to relate our Brownian motion with the theory of Dirichlet forms of [8].

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