$\overline{M}_{0,134}$ is not a Mori Dream Space

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$\overline{M}_{0,n}$ seminar, November 22nd, 2021

This talk is an exposition of the main steps in the paper

$\overline{M}_{0,n}$ is not a Mori Dream Space

by Ana-Maria Castravet and Jenia Tevelev.

1. Definition of Mori Dream Spaces.

2. Examples and non-examples.

3. Blow-up presentation of the Losev-Manin spaces $\overline{LM}_n$.
Blow-up presentation of the spaces $\overline{M}_{0,n}$.

4. (Goto, Nishida, Watanabe)
The blow-up of $\mathbb{P}(25, 72, 29)$ at the point $[1, 1, 1]$ is not a Mori Dream Space.

5. Putting it all together.

In many of the previous talks in this seminar, the *modular* interpretation of the spaces of stable curves of genus 0 and n-marked points played a fundamental role.

We also saw how to extend/weaken the conditions to obtain the *Hassett spaces* with their modified modular interpretation.

This allowed us to view the *Losev-Manin spaces* as different, but related, compactifications of $\overline{M}_{0,n}$.

In this talk, the modular interpretation plays a virtually inexistent role.

We work directly with presentations of $\overline{M}_{0,n}$ and of the Losev-Manin spaces $\overline{LM}_n$ as blow-ups of projective spaces.

Mori Dream Spaces (MDS)

A normal projective variety X is called a Mori Dream Space (MDS) if the following conditions hold:

1. X is $\mathbb{Q}$-factorial and $\operatorname{Pic}(X)_{\mathbb{Q}} \simeq \mathrm{N}^1(X)_{\mathbb{Q}}$;
2. $\operatorname{Nef}(X)$ is generated by finitely many semiample line bundles;
3. there is a finite collection of small $\mathbb{Q}$-factorial modifications $\{f_i \colon X \dashrightarrow X_i\}_{i \in \{1,\ldots,r\}}$ such that
    - for each $i \in \{1, \ldots, r\}$, X_i satisfies (1) and (2), and
    - $\operatorname{Mov}(X)$ coincides with the union $\bigcup_i f_i^* \operatorname{Nef}(X_i)$.

The actual definition will not play a big role in this talk.

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Mori Dream Spaces are finitely generated in some sense.

Small $\mathbb{Q}$-factorial modifications crop up sometimes.
If you do not know what they are, think about birational models with the same divisors.

Properties of Mori Dream Spaces

Let X, Y be smooth[1] projective varieties.

- The image of a MDS is a MDS:

  if $X \longrightarrow Y$ is a dominant morphism, then

  $$X \text{ is a MDS} \implies Y \text{ is a MDS.}$$

- Small $\mathbb{Q}$-factorial modifications preserve MDSs:

  if X is a small $\mathbb{Q}$-factorial modification of Y, then

  $$X \text{ is a MDS} \iff Y \text{ is a MDS.}$$

For this talk, the second property is only useful if you know what it means.

[1] normal and $\mathbb{Q}$-factorial is enough.

A toric variety is a Mori Dream Space.

Thus, (weighted) projective spaces $\mathbb{P}^n$ (or $\mathbb{P}(a_0, a_1, \ldots, a_n)$), products of such (weighted) projective spaces, are all MDS.

Losev-Manin spaces $\overline{LM}_n$ are toric varieties and hence are MDS.

Blow ups have a tendency of messing up Mori Dream Spaces.

For $n \geq 5$, the space $\overline{M}_{0,n}$ is not a toric variety.

From now on, we concentrate on $\overline{M}_{0,n}$, $\overline{LM}_n$ and $\mathbb{P}(a, b, c)$.

Let $e_1, \ldots, e_{n-2} \in \mathbb{P}^{n-3}$ be the $n-2$ coordinate points

$$\begin{aligned} e_1 &= [1,0,0,\ldots,0,0], \\ e_2 &= [0,1,0,\ldots,0,0], \\ &\vdots \\ e_{n-2} &= [0,0,0,\ldots,0,1], \end{aligned}$$

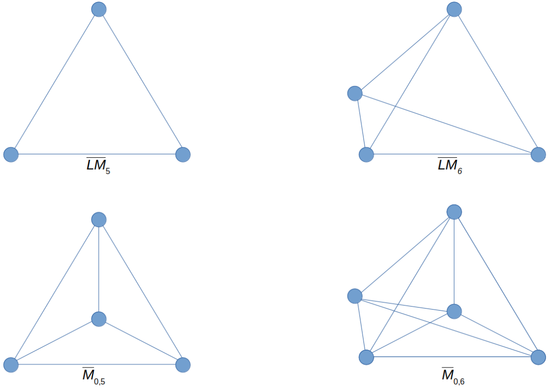
and let $\mathbf{e} \in \mathbb{P}^{n-3}$ be the point

$$\mathbf{e} = [1,1,\ldots,1].$$

Losev-Manin spaces and $\overline{M}_{0,n}$ – Blow ups

Start with $\mathbb{P}^{n-3}$.

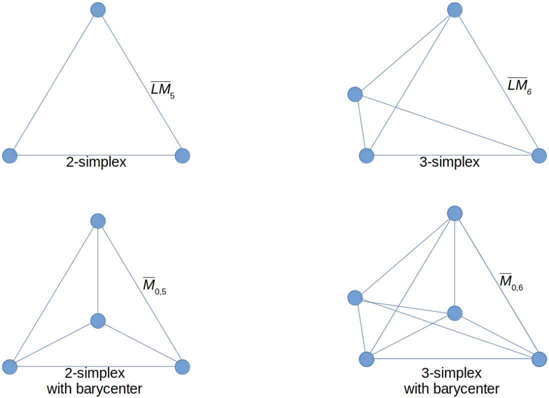
Losev-Manin $\overline{LM}_n$	$\overline{M}_{0,n}$
• Blow up the points $e_1, \dots, e_{n-2}$.	• ... **and e**.
• Blow up the strict transforms of the lines through the points $e_1, \dots, e_{n-2}$.	• ... **and e**.
• Blow up the strict transforms of the planes connecting all triples of points $e_1, \dots, e_{n-2}$.	• ... **and e**.
$\vdots$	$\vdots$
• Blow up the strict transforms of the $(n-4)$-planes connecting all $(n-3)$-tuples of points $e_1, \dots, e_{n-2}$.	• ... **and e**.



$$\overline{LM}_5 \simeq \mathrm{Bl}_{3\text{ pts}}\,\mathbb{P}^2 \qquad \overline{LM}_6 \simeq \mathrm{Bl}_{4\text{ pts} + 6\text{ lines}}\,\mathbb{P}^3$$

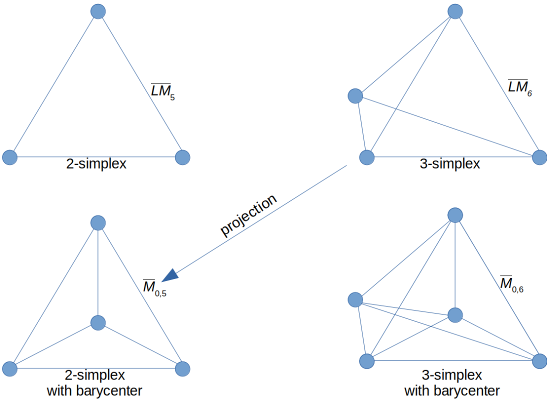
$$\overline{M}_{0,5} \simeq \mathrm{Bl}_{4\text{ pts}}\,\mathbb{P}^2 \qquad \overline{M}_{0,6} \simeq \mathrm{Bl}_{5\text{ pts} + 10\text{ lines}}\,\mathbb{P}^3$$

Losev-Manin spaces and $\overline{M}_{0,n}$ – Simplices



$$\begin{array}{c} \overline{LM}_5 \simeq \mathrm{Bl}_{3\text{ pts}}\,\mathbb{P}^2 \\ \uparrow \\ \overline{M}_{0,5} \simeq \mathrm{Bl}_{4\text{ pts}}\,\mathbb{P}^2 \end{array} \qquad \begin{array}{c} \overline{LM}_6 \simeq \mathrm{Bl}_{4\text{ pts} + 6\text{ lines}}\,\mathbb{P}^3 \\ \uparrow \\ \overline{M}_{0,6} \simeq \mathrm{Bl}_{5\text{ pts} + 10\text{ lines}}\,\mathbb{P}^3 \end{array}$$

Losev-Manin spaces and $\overline{M}_{0,n}$ – Projections



$$\overline{LM}_5 \simeq \mathrm{Bl}_{3\text{ pts}}\,\mathbb{P}^2 \qquad\qquad \overline{LM}_6 \simeq \mathrm{Bl}_{4\text{ pts} + 6\text{ lines}}\,\mathbb{P}^3$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

$$\overline{M}_{0,5} \simeq \mathrm{Bl}_{4\text{ pts}}\,\mathbb{P}^2 \dashleftarrow \qquad \overline{M}_{0,6} \simeq \mathrm{Bl}_{5\text{ pts} + 10\text{ lines}}\,\mathbb{P}^3$$

Sequences

Denote by $\mathbf{e} = [1, 1, \ldots, 1]$ the barycenter of the standard simplex.

$$
\begin{array}{ccccc}
\overline{LM}_5 & & \overline{LM}_6 & & \overline{LM}_n & & \overline{LM}_{n+1} \\
\uparrow & \swarrow & \uparrow & \Longrightarrow & \uparrow & \swarrow & \uparrow \\
\overline{M}_{0,5} & & \overline{M}_{0,6} & & \overline{M}_{0,n} & & \overline{M}_{0,n+1}
\end{array}
$$

$$\overline{LM}_{n+1} \dashrightarrow \boxed{\overline{M}_{0,n} \longrightarrow \mathrm{Bl}_{\mathbf{e}} \overline{LM}_n}$$

$$\widetilde{LM}_{n+1} \longrightarrow \boxed{\overline{M}_{0,n} \longrightarrow \mathrm{Bl}_{\mathbf{e}} \overline{LM}_n}$$

where $\widetilde{LM}_{n+1}$ is[2] $\mathrm{Bl}_{\mathbf{e}} \overline{LM}_{n+1}$.

[2] a small $\mathbb{Q}$-factorial projective modification of

Sequences

The sequence

$$\widetilde{LM}_{n+1} \longrightarrow \overline{M}_{0,n} \longrightarrow \mathrm{Bl}_{\mathbf{e}}\, \overline{LM}_n$$

gives the implications:

- if $\overline{M}_{0,n}$ is a MDS, then $\mathrm{Bl}_{\mathbf{e}}\, \overline{LM}_n$ is a MDS;
- if $\mathrm{Bl}_{\mathbf{e}}\, \overline{LM}_{n+1}$ is a MDS, then $\overline{M}_{0,n}$ is a MDS.

Recall that $\widetilde{LM}_{n+1}$ is[3] $\mathrm{Bl}_{\mathbf{e}}\, \overline{LM}_{n+1}$.

[3] a small $\mathbb{Q}$-factorial projective modification of

Recurring question

Let X be a toric variety and let $\mathbf{e} \in X$ be a point contained in the open torus orbit. Denote by $\operatorname{Bl}_{\mathbf{e}} X$ the blow-up of X at the point $\mathbf{e}$.

Question

When is $\operatorname{Bl}_{\mathbf{e}} X$ a Mori Dream Space?

Imprecisely, "When is the blow-up of a toric variety a MDS?"

Remark

It does not matter which point $\mathbf{e}$ in the open orbit we choose.

(Why?)

Spoiler

Let X be a toric variety and let $\mathbf{e} \in X$ be a point contained in the open torus orbit. Denote by $\mathrm{Bl}_{\mathbf{e}} X$ the blow-up of X at the point $\mathbf{e}$.

Question

When is $\mathrm{Bl}_{\mathbf{e}} X$ a Mori Dream Space?

Example (Goto, Nishida, Watanabe)

Over a field of characteristic zero, the surface $\mathrm{Bl}_{\mathbf{e}} \mathbb{P}(25, 72, 29)$ is not a MDS.

If there is time, ask me about this result.

The summary so far

- For every $n \geq 3$ there are morphisms

$$\widetilde{LM}_{n+1} \longrightarrow \overline{M}_{0,n} \longrightarrow \mathrm{Bl}_{\mathbf{e}} \overline{LM}_n.$$

- The surface $\mathrm{Bl}_{\mathbf{e}} \mathbb{P}(25, 72, 29)$ is not a MDS.

Wanted

A morphism: $\mathrm{Bl}_{\mathbf{e}} \overline{LM}_{134} \longrightarrow \mathrm{Bl}_{\mathbf{e}} \mathbb{P}(25, 72, 29).$

With this, we conclude that $\overline{M}_{0,134}$ is not a MDS.

Using the small $\mathbb{Q}$-factorial stuff, this also proves that, for $n \geq 134$, the space $\overline{M}_{0,n}$ is not a MDS.

A morphism $\mathrm{Bl}_{\mathbf{e}} \overline{LM}_{134} \to \mathrm{Bl}_{\mathbf{e}} \mathbb{P}(25, 72, 29)$

To construct the missing morphism

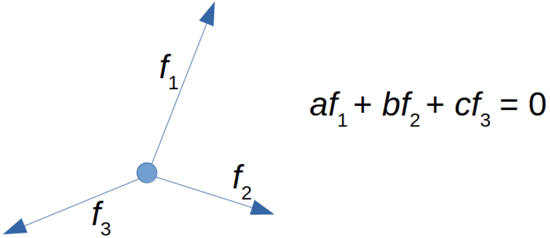
$$\mathrm{Bl}_{\mathbf{e}} \overline{LM}_{134} \to \mathrm{Bl}_{\mathbf{e}} \mathbb{P}(25, 72, 29)$$

we first compare the toric data of $\mathbb{P}(a, b, c)$ and of $\overline{LM}_n$, for general choices of a, b, c and n.

Toric data for $\mathbb{P}(a, b, c)$

Let $a, b, c \in \mathbb{N}$ be pairwise coprime.
Let $f_1, f_2, f_3 \in \mathbb{R}^2$ be three vectors spanning $\mathbb{R}^2$ and satisfying

$$af_1 + bf_2 + cf_3 = 0.$$



The vectors f_1, f_2, f_3 span the extremal rays of the fan associated to the toric variety $\mathbb{P}(a, b, c)$.

Toric data for $\overline{LM}_n$

Let $e_1, \ldots, e_{n-3}$ be a basis of $\mathbb{R}^{n-3}$ and set $e_{n-2} = -(e_1 + \cdots + e_{n-3})$.

Let $N \subset \mathbb{R}^{n-3}$ be the lattice spanned by the vectors $e_1, \ldots, e_{n-2}$.

The extremal rays of the fan associated to $\overline{LM}_n$ are the rays spanned by the primitive vectors

$$\sum_{i \in I} e_i, \qquad \text{for all subsets } I \subset \{1, \ldots, n-2\} \text{ with } 1 \leq |I| \leq n-3.$$

The higher dimensional cones of the fan correspond to higher codimensional torus-stable subvarieties of $\overline{LM}_n$: we need not worry about them, due to the small $\mathbb{Q}$-factorial stuff.

Map

Given $a, b, c \in \mathbb{N}$, set

$$n = (a+2) + (b+2) + (c+2) + 2 = a + b + c + 8.$$

Toric variety	$\mathbb{P}(a, b, c)$
Lattice	$\mathbb{Z}\text{-}\operatorname{span}\{f_1, f_2, f_3\} \big/ (af_1 + bf_2 + cf_3) \simeq \mathbb{Z}^2$
Rays	$f_1,\ f_2,\ f_3$

Toric variety	$\overline{LM}_n$
Lattice	$\mathbb{Z}\text{-}\operatorname{span}\{e_1, \ldots, e_{n-2}\} \big/ (e_1 + \cdots + e_{n-2}) \simeq \mathbb{Z}^{n-3}$
Rays	all non-zero sums of the vectors $e_1, \ldots, e_{n-2}$

Given $a, b, c \in \mathbb{N}$, set $n - 2 = (a+2) + (b+2) + (c+2)$.

Partition $S = \{e_1, \ldots, e_{n-2}\} = S_1 \sqcup S_2 \sqcup S_3$ in three parts, with

$$|S_1| = a + 2 \qquad |S_2| = b + 2 \qquad |S_3| = c + 2.$$

Fix $\qquad e_{n_1} \in S_1, \qquad e_{n_2} \in S_2, \qquad e_{n_3} \in S_3.$

Define a linear map $\qquad \mathbb{Z}\text{- span}\, S \longrightarrow \mathbb{Z}\text{- span}\{f_1, f_2, f_3\}$
by assigning to each vector $e \in S$

$$e \longmapsto \begin{cases} f_i, & \text{if } e \in S_i,\ e \neq e_{n_i}, \\ -f_i, & \text{if } e = e_{n_i}, \end{cases}$$

and extending linearly. The kernel of such map is generated by

$$\Big\{ e + e_{n_i} \mid i \in \{1,2,3\} \text{ and } e \in S_i \setminus \{e_{n_i}\} \Big\}.$$

The relation $\qquad \displaystyle\sum_{e \in S} e = \sum_{i=1}^{3} \sum_{e \in S_{n_i}} e \longmapsto af_1 + bf_2 + cf_3 \qquad$ holds.

We obtain a homomorphism of lattices

$$\text{lattice of } \overline{LM}_n \longrightarrow \text{lattice of } \mathbb{P}(a,b,c)$$

where

$$\begin{aligned} \text{lattice of } \overline{LM}_n &= \mathbb{Z}\text{- span}\{e_1,\ldots,e_{n-2}\}/(e_1+\cdots+e_{n-2}) \\ \text{lattice of } \mathbb{P}(a,b,c) &= \mathbb{Z}\text{- span}\{f_1,f_2,f_3\}/(af_1+bf_2+cf_3). \end{aligned}$$

This induces a rational maps

$$\overline{LM}_n \dashrightarrow \mathbb{P}(a,b,c) \qquad \text{and} \qquad \mathrm{Bl}_{\mathbf{e}}\,\overline{LM}_n \dashrightarrow \mathrm{Bl}_{\mathbf{e}}\,\mathbb{P}(a,b,c).$$

Using some slightly involved lattice-theoretic reasoning, this is enough to prove the implication

$$\mathrm{Bl}_{\mathbf{e}}\,\mathbb{P}(a,b,c) \text{ is not a MDS} \Longrightarrow \mathrm{Bl}_{\mathbf{e}}\,\overline{LM}_n \text{ is not a MDS.}$$

Conclusion

Summarizing, there are maps

$$\overline{M}_{0,n} \longrightarrow \mathrm{Bl}_{\mathbf{e}}\,\overline{LM}_n \qquad \text{and} \qquad \mathrm{Bl}_{\mathbf{e}}\,\overline{LM}_n \dashrightarrow \mathrm{Bl}_{\mathbf{e}}\,\mathbb{P}(a,b,c),$$

where $n = a + b + c + 8$.

The blow-up $\mathrm{Bl}_{\mathbf{e}}\,\mathbb{P}(25, 72, 29)$ is not a Mori Dream Space.

Since $25 + 72 + 29 + 8 = 134$, neither $\mathrm{Bl}_{\mathbf{e}}\,\overline{LM}_{134}$ is a Mori Dream Space.

Finally, $\overline{M}_{0,134}$ is not a Mori Dream Space.

Thank you!!

Questions?