Contact in algebraic and tropical geometry

Damiano Testa

University of Warwick

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Outline

1. Overview
   - Background
   - Inflection points and bitangent lines of plane curves
2. An elementary approach
3. Inflection *points* and inflection *lines*
4. Bitangent lines
5. Further directions

Goal

Reconcile

classical constructions in algebraic geometry over the **complex numbers**

and

recent results in **tropical** geometry

via

positive characteristic.

Based on joint work with **Marco Pacini** (UFF, Rio de Janeiro).

Conventions

Curve usually means a projective, plane curve $C \subset \mathbb{P}^2_k$ over a field k.

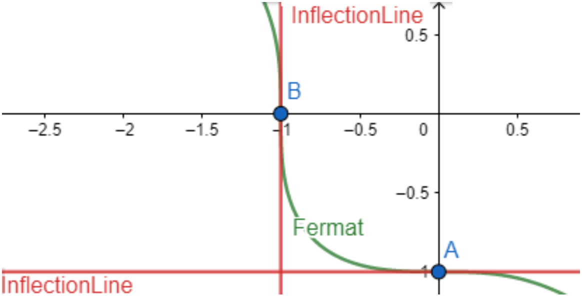
Often, a curve C is **general** among plane curves of the same degree as C. In particular, there is no loss in thinking that curves are smooth.

The field k can be taken to be algebraically closed.

Important **characteristics** of fields in this talk are 0, 3 and 2.

Inflection points of plane curves

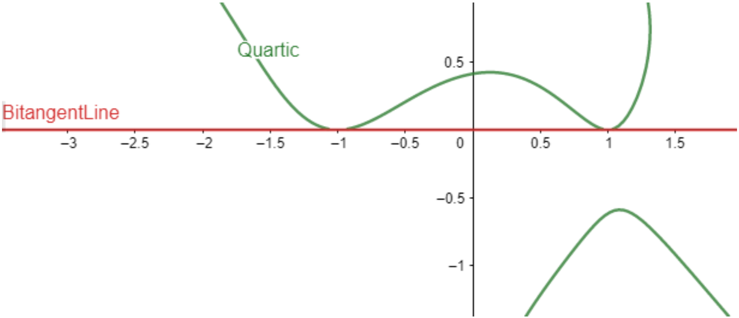
An *inflection point* of a curve C is a point $a \in C$ at which the tangent line to C meets the curve C with multiplicity at least 3.



The Fermat cubic $x^3 + y^3 + z^3 = 0$ and its 9 inflection points:

$$[0, 1, \zeta], \quad [1, 0, \zeta], \quad [1, \zeta, 0], \quad \text{with } \zeta^3 + 1 = 0.$$

A *bitangent line* of a curve C is a line $\ell \subset \mathbb{P}^2_k$ that is tangent to C at two distinct points.



The quartic $(x^2 - z^2)^2 = y(2z^3 - xz^2 + yz^2 - xy^2)$ and its bitangent line $y = 0$.

Complex and tropical geometry

Two starting points.

Theorem.

A $\left\langle \begin{array}{c} \text{plane curve over } \mathbb{C} \\ \text{tropical plane curve} \end{array} \right\rangle$ of degree d has $\left\langle \begin{array}{c} 3d(d-2) \\ d(d-2) \end{array} \right\rangle$ inflection points.

Theorem.

A $\left\langle \begin{array}{c} \text{plane quartic over } \mathbb{C} \\ \text{tropical plane quartic} \end{array} \right\rangle$ has $\left\langle \begin{array}{c} 28 \\ 7 \end{array} \right\rangle$ bitangent lines.

(Recall: **curve** means **general curve**.)

Algebraic and tropical geometry

Let k and l be algebraically closed fields of characteristics 3 and 2. Our observations are underlined.

Theorem.

A $\left\langle \begin{array}{c} \text{plane curve over } \mathbb{C} \\ \text{plane curve } \underline{\text{over } k} \\ \text{tropical plane curve} \end{array} \right\rangle$ of degree d has $\left\langle \begin{array}{c} 3d(d-2) \\ \underline{d(d-2)} \\ d(d-2) \end{array} \right\rangle$ inflection points.

Theorem.

A $\left\langle \begin{array}{c} \text{plane quartic over } \mathbb{C} \\ \text{plane quartic } \underline{\text{over } l} \\ \text{tropical plane quartic} \end{array} \right\rangle$ has $\left\langle \begin{array}{c} 28 \\ \underline{7} \\ 7 \end{array} \right\rangle$ bitangent lines.

Inflection points

Arguing via the real numbers, Klein, Ronga, Schuh, Viro, Brugallé, López de Medrano,... address the factor 3 between the $3d(d-2)$ complex and the $d(d-2)$ tropical inflection points.

Theorem.

A general $\left\langle \begin{array}{c} \text{plane curve over } \mathbb{R} \\ \text{tropical plane curve} \end{array} \right\rangle$ of degree d has at most $d(d-2)$

distinct $\left\langle \begin{array}{c} \text{real} \\ \text{tropical} \end{array} \right\rangle$ inflection points. The upper bound is achieved.

Takeaway

- For each real inflection point, there are two further complex conjugate inflection points.
- Reading off real multiplicities in tropical geometry is hard!

An elementary approach

We propose a *local* approach in *positive characteristic* to explain geometrically the discrepancy between the complex and the tropical counts.

The intuition is that the **contact multiplicities** interact with the **characteristic** of the field.

For instance,

- working with inflection points, we reduce modulo 3;
- working with bitangent lines, we reduce modulo 2.

The method has the potential for broader applications.

Basic computation: multiple roots

Lemma.

Let k be a field and let $f(x) \in k[x]$ a polynomial with a root α of multiplicity m. The $\gcd(f, f')$ is

- divisible by $(x-\alpha)^{m-1}$;
- divisible by $(x-\alpha)^m$ if and only if $\operatorname{char} k \mid m$.

Proof. Write

$$f(x) = (x-\alpha)^m g(x),$$

with $g(x) \in k[x]$, and $g(\alpha) \neq 0$. Compute

$$f'(x) \quad = \quad (x-\alpha)^{m-1}(mg(x) + (x-\alpha)g'(x)).$$

Thus $(x-\alpha)^{m-1}$ divides $f'(x)$ and

$$\begin{aligned}(x-\alpha)^m \text{ divides } f'(x) &\iff (x-\alpha) \text{ divides } mg(x) \\ &\iff m = 0 \text{ in } k.\end{aligned}$$

Basic computation: multiple roots

Conclusion

Let k be a field, let p be a prime number and let $n \geq p$ be an integer. There is a rational function $r_p(f_0, \ldots, f_n)$ in $(n+1)$ variables with the following property.

Assume that the polynomial $f = f_0 x^n + f_1 x^{n-1} + \cdots + f_n$ has a unique root α of multiplicity at least 2 and that the multiplicity of α is p.

- If $\operatorname{char} k \neq p$, then $r_p(f) = \alpha$.
- If $\operatorname{char} k = p$, then $r_p(f) = \alpha^p$.

Proof. Use the previous lemma and induction to show

$$\gcd\left(f, f', \ldots, f^{(p-1)}\right) = \begin{cases} x - \alpha, & \text{if } \operatorname{char} k \neq p; \\ (x-\alpha)^p = x^p - \alpha^p, & \text{if } \operatorname{char} k = p. \end{cases}$$

(Derivatives = Hasse derivatives) Compute gcd via Euclid's Algorithm.

Inflection *points* and inflection *lines*

Back to inflection points of curves in $\mathbb{P}^2_k$.

Fix a curve $C \subset \mathbb{P}^2_k$. Define the incidence correspondence:

$$\mathscr{F}_C = \left\{ (x, \ell) \in \mathbb{P}^2_k \times (\mathbb{P}^2_k)^\vee \;\middle|\; \begin{array}{l} x \text{ is an inflection point of } C, \\ \ell \text{ is the tangent line to } C \text{ at } x. \end{array} \right\}$$

where $(\mathbb{P}^2_k)^\vee$ is projective plane dual to $\mathbb{P}^2_k$.

(Points of $(\mathbb{P}^2_k)^\vee$ correspond to lines in $\mathbb{P}^2_k$.)

Question.
Can we reconstruct $\mathscr{F}_C$ from either the set of inflection points or the set of inflection lines of C?

Inflection *points* and inflection *lines*

($\Longrightarrow$) **From inflection points to inflection lines.**

$$\mathscr{F}_C = \left\{ (x,\ell) \in \mathbb{P}^2_k \times (\mathbb{P}^2_k)^\vee \;\middle|\; \begin{array}{l} x \text{ is an inflection point of } C, \\ \ell \text{ is the tangent line to } C \text{ at } x. \end{array} \right\}$$

Fix an inflection *point* x. We can reconstruct the corresponding inflection *line*, by computing the tangent line to C at x.

Thus, $\mathscr{F}_C$ has as many elements as C has inflection *points*:

$$\#\mathscr{F}_C = \#\{\text{inflection points of } C\}.$$

Inflection *points* and inflection *lines*

($\Longleftarrow$) **From inflection lines to inflection points.**

$$\mathscr{F}_C = \left\{ (x,\ell) \in \mathbb{P}^2_k \times (\mathbb{P}^2_k)^\vee \;\middle|\; \begin{array}{l} x \text{ is an inflection point of } C, \\ \ell \text{ is the tangent line to } C \text{ at } x. \end{array} \right\}$$

Fix an inflection line ℓ to the curve C.

As C is *general*, the inflection line ℓ is tangent to C at just one point x and the intersection multiplicity between ℓ and C at x is exactly 3.

Thus, a polynomial F vanishing on C restricts to a polynomial $F|_\ell$ on $\ell \simeq \mathbb{P}^1_k$ with the following properties:

- $F|_\ell$ has a unique repeated root, corresponding to the inflection point of C on ℓ;
- the multiplicity of the repeated root is 3.

Inflection *points* and inflection *lines*

Inflection lines to inflection points.
Fix an inflection line ℓ to C.
As C is *general* $[\dots]$, we find a polynomial $F|_\ell$ satisfying:

- $F|_\ell$ has a unique repeated root α, corresponding to the inflection point of C on ℓ;
- the multiplicity of the repeated root α is 3.

We have seen earlier to what extent we can reconstruct α from F!

If $\operatorname{char} k \neq 3$, then we can reconstruct α from $F|_\ell$ and we deduce that

$$\#\mathscr{F}_C = \#\{\text{inflection lines of } C\}.$$

If $\operatorname{char} k = 3$, then we can reconstruct α^3 from $F|_\ell$ and we deduce that

$$\#\mathscr{F}_C = 3 \cdot \#\{\text{inflection lines of } C\}.$$

Summary for inflection points

$$
\mathscr{F}_C = \left\{ (x,\ell) \in \mathbb{P}^2_k \times (\mathbb{P}^2_k)^\vee \;\middle|\; \begin{array}{c} x \text{ is an inflection point of } C, \\ \ell \text{ is the tangent line to } C \text{ at } x. \end{array} \right\}
$$

$$
\begin{tikzcd}[column sep=small]
& \mathscr{F}_C \arrow[dl, "\text{birational}"'] \arrow[dr, "\substack{\text{char}\,k\neq 3,\ \text{birational} \\ \text{char}\,k=3,\ \text{purely} \\ \text{inseparable} \\ \text{of degree } 3}"] & \\
\left\{ x \in \mathbb{P}^2_k \;\middle|\; \begin{array}{l} x \text{ inflection} \\ \text{point of } C \end{array} \right\} \arrow[rr, "\text{“Gauss map”}"] & & \left\{ \ell \in (\mathbb{P}^2_k)^\vee \;\middle|\; \begin{array}{l} \ell \text{ inflection} \\ \text{line of } C \end{array} \right\}
\end{tikzcd}
$$

Amusing consequence. (If you happen to like imperfect fields)
Let k be a separably closed field of characteristic 3.
Let C be a general plane curve defined over k.
The coordinates of the inflection lines are contained in k.
The coordinates of the inflection points are contained in $k^{\frac{1}{3}}$.

Bitangent lines

Similarly for bitangent lines. Fix a plane quartic $C \subset \mathbb{P}^2_k$.

$$\begin{tikzcd}[column sep=small]
& \left\{ (x,\ell) \in \mathbb{P}^2_k \times (\mathbb{P}^2_k)^\vee \;\middle|\; \begin{array}{c} x \text{ is point of bitangency of } C, \\ \ell \text{ is the (bi)tangent line to } C \text{ at } x. \end{array} \right\} \arrow[dl, "\text{birational}"'] \arrow[dr, "\substack{\text{char}\, k\neq 2,\text{ double cover} \\ \text{char}\, k=2,\text{ purely} \\ \text{inseparable} \\ \text{of degree } 4}"] & \\
\left\{ x \in \mathbb{P}^2_k \;\middle|\; \begin{array}{l} x \text{ bitangent} \\ \text{point of } C \end{array} \right\} \arrow[rr, "\text{``Gauss map''}"] & & \left\{ \ell \in (\mathbb{P}^2_k)^\vee \;\middle|\; \begin{array}{l} \ell \text{ bitangent} \\ \text{line of } C \end{array} \right\}
\end{tikzcd}$$

In char 2, contact points contribute 2 each to the inseparable degree. Thus, *bi*tangents give a $4:1$ degree ratio.
The 28 bitangents of plane quartics over $\mathbb{C}$, correspond to the $7 = \frac{28}{4}$ bitangents in characteristic 2.

From bitangents to theta-characteristics

Bitangent lines to plane quartics generalize to *odd theta-characteristics* of curves C of genus g. The inseparable degree works out to be 2^{g-1}.

The curve C has

- $2^{g-1}(2^g - 1)$ odd theta-characteristics over $\mathbb{C}$.
- $2^g - 1$ ~~odd~~ effective theta-characteristics in char 2.

Even theta-characteristics also work, but are slightly more involved.

Open tropical questions already in genus 5.

Further directions

Steiner's conic problem
Number of conics simultaneously tangent to 5 general plane conics.

3264 over $\mathbb{C}$ (Steiner, Chasles, de Jonquières, Fulton, MacPherson);

$51 = \frac{3264}{2^6}$ over $\operatorname{char} 2$ (Vainsencher).

de Jonquières formula
A formula for counting the number of hyperplanes with prescribed contact multiplicities with a given curve.

Gromov-Witten invariants

...

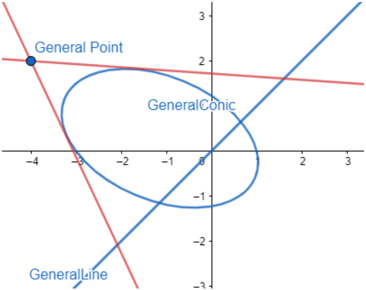
Further directions

Gromov-Witten invariants

The number of plane, nodal, rational curves of degree d containing a general point and meeting a fixed line and conic in a single point each is

$$\binom{2d}{d}$$

(Bousseau, Brini, van Garrel).



Connection: choose d to be a prime number p. The congruence

$$\binom{2p}{p} \equiv 2 \pmod{p^2}$$

holds. It even holds modulo p^3, for $p \geq 5$.

Merci!