THE IRREDUCIBILITY OF THE SPACES OF RATIONAL CURVES ON DEL PEZZO SURFACES

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Abstract

Let X be a del Pezzo surface of degree $d \geq 2$. We prove that the spaces $\mathcal{M}_{0,0}(X,\beta)$ are either empty or irreducible.

Introduction. The general question of the irreducibility of the spaces of maps of curves to varieties with given numerical invariants has a long history: from the irreducibility of the Hurwitz schemes [Fu] and the irreducibility of the moduli space of curves [DM], to the "Severi problem" [Se] and its resolution in [H].

More recently, the study of rationally connected varieties raised more questions on the subject: in particular the articles [dJS1], [dJS2] [HRS] and [HS], examine the structure of the spaces of rational curves on rationally connected varieties. One of the aims is to try to find the correct class of varieties to extend the results of [GHS] in the case where the base is a surface. In a different direction, in [BS] genus zero mapping spaces are used to approach the question of unirationality of some rationally connected varieties.

We establish the irreducibility of the spaces of rational curves on del Pezzo surfaces of degree at least 2.

More specifically, let X be a del Pezzo surface of degree $d \geq 2$. Let $\beta \in \mathrm{H}_2(X,\mathbb{Z})$ be the class of a curve on X. Denote by $R(\beta)$ the subscheme of the linear system $|\beta|$ consisting of the integral nodal curves of geometric genus zero. The Kontsevich mapping space $\overline{\mathcal{M}}_{0,0}(X,\beta)$ is a natural compactification of the space $R(\beta)$: it parametrizes all (stable) maps to the surface X from possibly reducible curves. Some care is required, since the mapping spaces in general have more irreducible components than the corresponding spaces $R(\beta)$, arising from degenerate configurations of curves on the surface. In fact it may happen that $R(\beta) = \emptyset$, while $\overline{\mathcal{M}}_{0,0}(X,\beta) \neq \emptyset$.

Let $\overline{\mathcal{M}}_{bir}(X,\beta)$ be the closure of the subspace of $\overline{\mathcal{M}}_{0,0}(X,\beta)$ consisting of morphisms $f : C \to X$, with $C \simeq \mathbb{P}^1$ and f birational onto its image.

Our main result is that the space $\overline{\mathcal{M}}_{bir}(X,\beta)$ is irreducible or empty.

The idea of the proof is straightforward. First, prove that in the boundary of all the irreducible components of $\overline{\mathcal{M}}_{bir}(X, \beta)$ there are special morphisms of a given type (called in what follows "standard morphisms"). Second, show that the locus of standard morphisms is connected and contained in the smooth locus of $\overline{\mathcal{M}}_{bir}(X, \beta)$. From these two facts we conclude immediately that the smooth locus of $\overline{\mathcal{M}}_{bir}(X, \beta)$ is connected. Since the smooth locus is dense, we deduce that $\overline{\mathcal{M}}_{bir}(X, \beta)$ is irreducible.

The methods used in the proof are of two different kinds. First, there are general techniques, mainly Mori's Bend and Break Theorem, to break curves into components with low anticanonical degree. In the case where $X = \mathbb{P}^2$, this shows that we may specialize a morphism in $\overline{\mathcal{M}}_{bir}(X, \beta)$ so that its image is a union of lines. Second, we need explicit geometric arguments to handle the low degree cases. In the case of $\mathbb{P}^2$, this step is used to bring the domain to a standard form (a chain of rational curves, rather than a general rational tree), while preserving the property that the image of the morphism consists of a union of lines.

Section 1 gives the two main deformation-theoretic tools. The first is a computation of the obstruction space of a stable map to a smooth surface in terms of combinatorial invariants of the map. The second is a lifting result that allows us, given a deformation of a component of a curve, to get a deformation of the whole curve. This is specific to the surface case.

Section 2 is devoted to the analysis of curves of low anticanonical degree on a del Pezzo surface. We use extensively the group of symmetries of the Picard lattice to reduce the number of cases to treat.

Section 3 uses systematically the results of Section 1. We produce curves with many irreducible components in all components of $\overline{\mathcal{M}}_{bir}(X, \beta)$.

Section 4 contains the main techincal result: the irreducibility of all mapping spaces can be reduced to finitely many cases.

Section 5 contains the proof of the main result.

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1. Cohomology Groups and Obstruction Spaces

1.1. The Conormal Sheaf. Let $f : C \to X$ be a morphism from a connected, projective, at worst nodal curve C to a smooth projective variety X.

Definition 1.1. The morphism $f : C \to X$ is a *stable map* if C is a connected, projective, at worst nodal curve and the normalization of every contracted component of geometric genus zero contains at least three points lying over singular points of C and every contracted component of geometric genus one contains at least one singular point of C.

Denote by $L_f^\bullet$ the complex $f^*\Omega_X^1 \to \Omega_C^1$, where $f^*\Omega_X^1$ is in degree -1 and Ω_C^1 is in degree 0. We want to compute the obstruction space to the deformations of the stable map $f : C \to X$. The stability condition is equivalent to the vanishing of the group $\mathrm{Hom}\big(L_f^\bullet, \mathcal{O}_C\big)$. The tangent space to $\overline{\mathcal{M}}_{0,0}\big(X, \beta\big)$ at f is the hypercohomology group $\mathbb{E}\mathrm{xt}^1\big(L_f^\bullet, \mathcal{O}_C\big)$. The obstruction space is a quotient of the hypercohomology group $\mathbb{E}\mathrm{xt}^2\big(L_f^\bullet, \mathcal{O}_C\big)$. Our strategy to compute these groups is to use the short exact sequence of complexes of sheaves

$$\begin{pmatrix} 0 \\ \downarrow \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ \downarrow \\ \Omega_C^1 \end{pmatrix} \longrightarrow \begin{pmatrix} f^*\Omega_X^1 \\ \downarrow \\ \Omega_C^1 \end{pmatrix} \longrightarrow \begin{pmatrix} f^*\Omega_X^1 \\ \downarrow \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ \downarrow \\ 0 \end{pmatrix}$$

apply the functor $\mathrm{Hom}(-, \mathcal{O}_C)$, use the long exact hypercohomology sequence and several standard identifications to obtain

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(\Omega_C^1, \mathcal{O}_C) \longrightarrow \mathrm{H}^0(C, f^*T_X) \longrightarrow \mathbb{E}\mathrm{xt}^1\big(L_f^\bullet, \mathcal{O}_C\big) \longrightarrow \\ \longrightarrow \mathrm{Ext}^1(\Omega_C^1, \mathcal{O}_C) \longrightarrow \mathrm{H}^1(C, f^*T_X) \longrightarrow \mathbb{E}\mathrm{xt}^2\big(L_f^\bullet, \mathcal{O}_C\big) \longrightarrow 0. \end{aligned} \tag{1.1}$$

In particular we see that if $\mathrm{H}^1(C, f^*T_X) = 0$, then the obstruction group $\mathbb{E}\mathrm{xt}^2\big(L_f^\bullet, \mathcal{O}_C\big)$ vanishes as well, i.e. the map is unobstructed, the space of stable maps has the expected dimension at f and the point represented by f is smooth (for the stack). Consider the dual sequence of (1.1) and use Serre duality to obtain

$$\begin{aligned} 0 \longrightarrow \big(\mathbb{E}\mathrm{xt}^2\big)^\vee \longrightarrow \mathrm{H}^0(C, f^*\Omega_X^1 \otimes \omega_C) \xrightarrow{\alpha} \mathrm{H}^0(C, \Omega_C^1 \otimes \omega_C) \longrightarrow \\ \longrightarrow \big(\mathbb{E}\mathrm{xt}^1\big)^\vee \longrightarrow \mathrm{H}^1(C, f^*\Omega_X^1 \otimes \omega_C) \longrightarrow \mathrm{H}^1(C, \Omega_C^1 \otimes \omega_C) \longrightarrow 0 \end{aligned}$$

where the morphism α is induced by $df : f^*\Omega_X^1 \longrightarrow \Omega_C$. Associated to f we define the sheaves $\mathcal{C}_f := \ker(df)$ and $\mathcal{Q}_f := \mathrm{coker}(df)$ on C. Since the dualizing sheaf ω_C is locally free, tensoring by ω_C is exact and taking global sections is left exact. From these remarks we deduce that

$$\mathrm{H}^0\big(C, \mathcal{C}_f \otimes \omega_C\big) \simeq \mathbb{E}\mathrm{xt}^2\big(L_f^\bullet, \mathcal{O}_C\big)^\vee$$

and we conclude that in order to compute the obstruction space of f, it is enough to compute the global sections of the sheaf $\mathcal{C}_f \otimes \omega_C$.

Definition 1.2. The sheaf $\mathcal{C}_f$ defined above is the *conormal sheaf of f*.

We drop the subscript f, when the morphism is clear from the context. Note that the support of every non-zero section of $\mathcal{C}_f$ has pure dimension one.

Definition 1.3. Suppose that f is finite. A point $p \in C$ is a *ramification point*, if it belongs to the support of the sheaf $\mathcal{Q}_f$. The *ramification divisor of* f is the (Weil) divisor whose multiplicity at $p \in C$ is the length of $\mathcal{Q}_f$ at p.

Let $f : C \to X$ be a non-constant morphism from a smooth curve to a smooth surface. Suppose $p \in C$ is a point and let u and v be local coordinates on X near $f(p) = q$ and let x be a local parameter for C near p. Since f is not constant, there exists an integer $k \geq 1$ such that

$$f^* : \begin{cases} u \longmapsto x^k U(x) \\ v \longmapsto x^k V(x) \end{cases}$$

and $\big(U(0), V(0)\big) \neq (0,0)$. We call a *tangent vector to C at p* any non-zero vector in T_qX proportional to $\big(U(0), V(0)\big)$, and *tangent direction to C at p* the point in $\mathbb{P}(T_qX)$ determined by a tangent vector to C at p. We say that the morphism f is *ramified at p* if $k > 1$ and we say it is unramified otherwise. Given two morphisms as above $f_i : C_i \to X$ with $f_i(p_i) = q$, $i \in \{1,2\}$, the curves C_1 and C_2 are *transverse at the point* $q \in X$ if f_i is unramified at p_i and the tangent directions of f_1 at p_1 and of f_2 at p_2 are distinct.

Definition 1.4. Same notation as above, denote by $\tilde{f}_i$ the morphism induced by f_i from C_i to the blow-up of X at q, and assume $\tilde{f}_i(p_i) = \tilde{q}$. We say that the two curves C_1 and C_2 are *simply tangent at q* if C_1 and C_2 are transverse at $\tilde{q}$.

Informally we can say that two curves are simply tangent at q if the morphisms f_1 and f_2 agree to exactly first order and are non-zero. In the next Lemma we will see that being simply tangent is closely related to the local structure of the conormal sheaf.

Lemma 1.5. *Suppose that X is a smooth surface and let $f : C \to X$ be a morphism from a curve C consisting of two irreducible components C_1 and C_2, meeting in a node p. Denote by f_i the restriction of f to C_i and by $p_i \in C_i$ the point $p \in C$, and suppose that f does not contract any component of C. Then there are the following cases:*

(1) *if C_1 and C_2 are transverse at $f(p)$, then $\mathcal{C}_f$ is locally free and the following sequence is exact*

$$0 \longrightarrow \mathcal{C}_f \longrightarrow \mathcal{C}_{f_1}(-p) \oplus \mathcal{C}_{f_2}(-p) \longrightarrow \mathcal{C}_{f,p} \longrightarrow 0;$$

(2) *if C_1 and C_2 have distinct tangent directions at $f(p)$, f_1 is unramified at p and f_2 is ramified at p, then $\mathcal{C}_f$ is not locally free and*

$$\mathcal{C}_f \cong \mathcal{C}_{f_1}(-p) \oplus \mathcal{C}_{f_2}(-2p);$$

(3) *if* C_1 *and* C_2 *have distinct tangent directions at* $f(p)$ *and both maps* f_1 *and* f_2 *are ramified at* p*, then* $\mathcal{C}_f$ *is not locally free and*

$$\mathcal{C}_f \cong \mathcal{C}_{f_1}(-p) \oplus \mathcal{C}_{f_2}(-p);$$

(4) *if* C_1 *and* C_2 *are simply tangent at* $f(p)$*, then* $\mathcal{C}_f$ *is not locally free and*

$$\mathcal{C}_f \cong \mathcal{C}_{f_1}(-p) \oplus \mathcal{C}_{f_2}(-p);$$

(5) *if* C_1 *and* C_2 *are not transverse nor simply tangent at* $f(p)$*, then* $\mathcal{C}_f$ *is locally free and there is an exact sequence*

$$0 \longrightarrow \mathcal{C}_f \longrightarrow \mathcal{C}_{f_1} \oplus \mathcal{C}_{f_2} \longrightarrow \mathcal{C}_{f,p} \longrightarrow 0.$$

Proof. The proofs of the five cases are similar; we prove only (3). Write

$$f^* : \begin{cases} u \longmapsto x^{k_1} U_1(x) + y^{k_2} U_2(y) \\ v \longmapsto x^{l_1} V_1(x) + y^{l_2} V_2(y) \end{cases}$$

where $l_1 > k_1 \geq 2$, $k_2 > l_2 \geq 2$ and $U_1(0), V_2(0) \neq 0$. Let $\alpha_1(x) := k_1 U_1(x) + xU_1'(x)$, $\alpha_2(y) := l_2 V_2(y) + yV_2'(y)$. We have

$$\begin{aligned} \mathcal{O}_{C,p} \cdot du + \mathcal{O}_{C,p} \cdot dv &\xrightarrow{df} \Big(\mathcal{O}_{C,p} \cdot dx + \mathcal{O}_{C,p} \cdot dy\Big) \Big/ \big(ydx + xdy\big) \\ \tfrac{du}{\alpha_1} &\longmapsto x^{k_1-1} dx + y^{k_2-1} \varphi(y) dy \\ \tfrac{dv}{\alpha_2} &\longmapsto x^{l_1-1} \psi(x) dx + y^{l_2-1} dy \end{aligned}$$

with

$$\begin{aligned} y^{k_2-1}\varphi(y) &= \frac{y^{k_2-1}}{k_1 U_1(0)} \Big(k_2 U_2(y) + yU_2'(y)\Big) \\ x^{l_1-1}\psi(x) &= \frac{x^{l_1-1}}{l_2 V_2(0)} \Big(l_1 V_1(x) + xV_1'(x)\Big). \end{aligned}$$

The elements of the kernel of df are determined by the condition

$$f_1(x,y)\frac{du}{\alpha_1} + f_2(x,y)\frac{dv}{\alpha_2} \longmapsto r(x,y)\big(ydx + xdy\big)$$

which translates to the two equations

$$\begin{aligned} x^{k_1-1}\Big(f_1(x,y) + x^{l_1-k_1} f_2(x,y)\psi(x)\Big) &= yr(x,y) = yr(0,y) \\ y^{l_2-1}\Big(y^{k_2-l_2} f_1(x,y)\varphi(y) + f_2(x,y)\Big) &= xr(x,y) = xr(x,0). \end{aligned} \tag{1.2}$$

The equations imply $r(x,y)=0$, and thus $f_1(x,y) = -x^{l_1-k_1} f_2(x,y)\psi(x) + yh_1(y)$. Substituting back in (1.2), we find

$$y^{l_2-1}\Big(y^{k_2-l_2+1}h_1(y)\varphi(y) + f_2(x,y)\Big) = 0$$

i.e. $f_2(x,y) = xg_2(x) - y^{k_2-l_2+1}h_1(y)\varphi(y)$ and therefore

$$f_1(x,y) = -x^{l_1-k_1+1}g_2(x)\psi(x) + yh_1(y).$$

By inspection we see that choosing $\big(g_2(x), h_1(y)\big) = (1,0)$ or $(0,1)$ yields elements of the kernel of df. Thus near p the kernel of df is generated by

$$x\left(-x^{l_1-k_1}\psi(x)\frac{du}{\alpha_1} + \frac{dv}{\alpha_2}\right) \qquad \text{and} \qquad y\Big(\frac{du}{\alpha_1} + y^{k_2-l_2}\varphi(y)\frac{dv}{\alpha_2}\Big).$$

We conclude that $\mathcal{C}_f$ is not locally free near p. Since the terms in brackets in the previous expression are local generators for $\mathcal{C}_{f_1}$ and $\mathcal{C}_{f_2}$ respectively near p, it follows that $\mathcal{C}_f \cong \mathcal{C}_{f_1}(-p) \oplus \mathcal{C}_{f_1}(-p)$. Thus (3) is established. $\square$

Let $f : C \to X$ be a stable map to a smooth surface X. In view of the previous Lemma, we partition the set of nodes of C in five disjoint sets:

- τ_{uu} is the set of nodes p such that the two components of C meeting at p are transverse at $f(p)$;
- τ_{ur} is the set of nodes p such that the two components of C meeting at p have distinct tangent directions at $f(p)$ and one is unramified and the other one is ramified;
- τ_{rr} is the set of nodes p such that the two components of C meeting at p have distinct tangent directions at $f(p)$ and both are ramified;
- ν_2 is the set of nodes p such that the two components of C meeting at p are simply tangent at $f(p)$;
- ν_l is the set of nodes p such that the two components of C meeting at p are not transverse and not simply tangent at $f(p)$.

It follows from the Lemma that the sheaf $\mathcal{C}_f$ is locally free at the nodes τ_{uu} and ν_l, while it is not free at the others. Let $C_1, \ldots, C_\ell$ be the components of C. Then we let τ^i_{uu} denote the divisor on C_i of nodes lying in τ_{uu}, and similarly for the other types of nodes. Note that only one of the definitions above is not symmetric, namely τ_{ur} (and τ^i_{ur}). To take care of this, let us introduce one more divisor on each component of C: let τ^i_{ru} be the divisor on C_i consisting of all nodes p of C on C_i, such that the two components of C through p have distinct tangent directions at $f(p)$, and the restriction of f to these two components is ramified only on C_i. We will denote by the same symbol a divisor on a curve and its degree. Given a coherent sheaf $\mathcal{F}$ on a curve C, let $\tau(\mathcal{F})$ denote the subsheaf generated by the sections whose support has dimension at most 0 and let $\mathcal{F}^{free}$ be the sheaf $\mathcal{F}/\tau(\mathcal{F})$.

Proposition 1.6. *Let $f : C \to X$ be a stable map of genus zero with no contracted components to a smooth surface X and let $C_1, \ldots, C_\ell$ be the irreducible components of C. We have*

$$(1.3)\quad \begin{aligned} \deg\left((\mathcal{C}_f \otimes \omega_C)\big|_{C_i}^{free} \right) &= f_*[C_i] \cdot K_X - \deg \tau_{ru}^i + \deg \nu_l^i + \deg \mathcal{Q}_i \\ \chi\left(\mathcal{C}_f \otimes \omega_C\right) &= f_*[C] \cdot K_X + \deg \tau_{rr} + \deg \nu_2 + \\ &\quad +2 \deg \nu_l + \sum \deg \mathcal{Q}_i + 1. \end{aligned}$$

Moreover, let $\nu : \tilde{C} \to C$ be the normalization of C at the nodes in $\tau_{ur} \cup \tau_{rr} \cup \nu_2$. The sheaf $\mathcal{C}_f$ is the pushforward of a locally free sheaf on $\tilde{C}$.

Proof. Thanks to Lemma 1.5 there is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{C}_f \longrightarrow \oplus_i \mathcal{C}_{f_i}\left(-\tau_{uu}^i - \tau_{ur}^i - 2\tau_{ru}^i - \tau_{rr}^i - \nu_2^i\right) \longrightarrow \mathcal{C}_f|_{\tau_{uu}} \oplus \mathcal{C}_f|_{\nu_l} \longrightarrow 0.$$

We can write the divisor by which we are twisting $\mathcal{C}_{f_i}$ as $-val[C_i] - \tau_{ru}^i + \nu_l^i$ (where $val[C_i]$ is the valence of the vertex $[C_i]$ in the dual graph of C). We have $\deg \mathcal{C}_{f_i} = f_*[C_i] \cdot K_X + 2 + \deg \mathcal{Q}_i$. The Proposition follows by twisting the previous sequence by ω_C and the fact that the dual graph of f is a tree. □

The next Proposition deals with morphisms with contracted components. We introduce two more subsets of the nodes on contracted components, depending on the behaviour of $f : \bar{C} \to X$ near the node:

- ρ_u is the set of nodes p such that f is constant on one of the two components, and it is unramified on the other;
- ρ_r is the set of nodes p such that f is constant on one of the two components, and it is ramified on the other.

Proposition 1.7. *Let $f : \bar{C} \to X$ be a stable map of genus zero to a smooth surface X. Let $\bar{C} = C \cup R$, where $C = C_1 \cup \ldots \cup C_\ell$ is the union of all components of $\bar{C}$ which are not contracted by f, and R is the union of all components of $\bar{C}$ contracted by f. Let r be the number of connected components of the curve R (equivalently, $r = \chi(\mathcal{O}_R)$). We have*

$$(1.4)\quad \begin{aligned} \deg\left((\mathcal{C}_f \otimes \omega_{\bar{C}})\big|_{C_i}^{free} \right) &= f_*[C_i] \cdot K_X + \mathcal{Q}_i - \tau_{ru}^i + \nu_l^i + \rho_u^i + \rho_r^i \\ \chi\left(\mathcal{C}_f \otimes \omega_{\bar{C}}\right) &= f_*[C] \cdot K_X + \sum \mathcal{Q}_i + \tau_{rr} + \nu_2 + 2\nu_l + \\ &\quad +\rho_u + 2\rho_r - 3r + 1. \end{aligned}$$

Proof. The proof is similar to that of Proposition 1.6 and is omitted. □

1.2. Dimension Estimates. We refer to the integer $-C \cdot K_X$ as the anticanonical degree (or simply as the degree) of a curve C in X.

We consistently use the following notational convention: if $f : \bar{C} \to X$ is a morphism and $\bar{C}_1$ denotes a component of $\bar{C}$, we denote the image of $\bar{C}_1$

by C_1, and in general, a symbol with a bar over it denotes an object on the source curve $\bar{C}$, while the same symbol without the bar over it denotes the image of the object in X.

Definition 1.8. ([Ko] II.3.6). Let $f, g \in \mathrm{Hom}(\bar{C}, X)$; we say g is a deformation of f, if there is an irreducible subscheme of $\mathrm{Hom}(\bar{C}, X)$ containing f and g. We say that a general deformation of f has some property if there is an open subset $U \subset \mathrm{Hom}(\bar{C}, X)$ containing f and a dense open subset $V \subset U$ such that all $f' \in V$ have that property.

When we choose a general deformation g of a morphism f, we assume that g is a deformation of f, i.e. that f and g lie in the same irreducible component of $\mathrm{Hom}(\bar{C}, X)$.

Lemma 1.9. *Let $f : \mathbb{P}^1 \to X$ be a free morphism; if f is birational onto its image, then a general deformation of f is free and it is an immersion.*

Proof. The result follows from [Ko] Complement II.3.14.4, and the fact that a general deformation of a free and birational map is free and birational. □

Fix a free rational curve $\beta \subset X$ and let $d := -\beta \cdot K_X$.

Definition 1.10. Denote by $\overline{\mathcal{M}}_{bir}\big(X, \beta\big)$ the closure in $\overline{\mathcal{M}}_{0,0}\big(X, \beta\big)$ of the set of free morphisms $f : \mathbb{P}^1 \to X$ such that f is birational onto its image.

We want to prove that given $r \leq d-1$ general points $p_1, \ldots, p_r \in X$, in all irreducible components of $\overline{\mathcal{M}}_{bir}\big(X, \beta\big)$ there is a stable map whose image contains all the p_i's.

Proposition 1.11. *Let $f : \mathbb{P}^1 \to X$ be an immersion, and let d be the degree of the image of f. Let $c_1, \ldots, c_r$ be distinct points where f is an embedding. The natural morphism*

$$\begin{aligned} F^{(r)} : (\mathbb{P}^1)^r \times \mathrm{Hom}(\mathbb{P}^1, X) & \longrightarrow X^r \\ \big(d_1, \ldots, d_r; [g]\big) & \longmapsto \big(g(d_1), \ldots, g(d_r)\big) \end{aligned}$$

is smooth at the point $\big(c_1, \ldots, c_r; [f]\big)$ if and only if $r \leq d-1$.

Proof. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \oplus T_{c_i}\mathbb{P}^1 & \longrightarrow & T_{(\underline{c};[f])}(\mathbb{P}^1)^r \times \mathrm{Hom}(\mathbb{P}^1, X) & \longrightarrow & T_{[f]}\mathrm{Hom}(\mathbb{P}^1, X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle dF^{(r)}} & & \downarrow{\scriptstyle \delta} & & \\ 0 & \longrightarrow & \oplus T_{f(c_i)}f(\mathbb{P}^1) & \longrightarrow & \oplus T_{f(c_i)}X & \longrightarrow & \oplus \mathcal{N}_{f,c_i} & \longrightarrow & 0 \end{array}$$

where the second row is obtained by restricting the normal sequence of f to $\{c_1, \ldots, c_r\}$. The first vertical arrow is induced by f, while δ is the quotient map, followed by the evaluation map ([Ko] Proposition II.3.5):

$$\begin{tikzcd} T_{[f]}\mathrm{Hom}(\mathbb{P}^1,X) \simeq \mathrm{H}^0(\mathbb{P}^1,f^*T_X) \arrow[r, "q"] \arrow[dr, "\delta"'] & \mathrm{H}^0(\mathbb{P}^1,\mathcal{N}_f) \arrow[d, "ev"] \\ & \oplus\mathcal{N}_{f,c_i} \end{tikzcd}$$

The morphism q is induced by the normal sequence and is surjective. Observe that $dF^{(r)}$ is surjective if and only if δ is surjective, and finally, δ is surjective if and only if the evaluation map ev is surjective. Consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{N}_f(-c_1-\ldots-c_r) \longrightarrow \mathcal{N}_f \longrightarrow \oplus\mathcal{N}_{f,c_i} \longrightarrow 0. \tag{1.5}$$

Since $\deg\mathcal{N}_f = d-2$ and f is an immersion, $\mathcal{N}_f \simeq \mathcal{O}_{\mathbb{P}^1}(d-2)$. The sequence on global sections induced by (1.5) is exact if and only if $\deg\mathcal{N}_f(-c_1-\ldots-c_r) = d-2-r \geq -1$. Therefore $dF^{(r)}$ is surjective if and only if $r \leq d-1$. □

Let $f : \mathbb{P}^1 \to X$ be an immersion representing an element of $\overline{\mathcal{M}}_{bir}(X,\beta)$, and denote by ${}_f\overline{\mathcal{M}}_{bir}(X,\beta)$ the irreducible component of $\overline{\mathcal{M}}_{bir}(X,\beta)$ containing f. Denote by $\mathcal{H}^f \subset \mathrm{Hom}(\mathbb{P}^1,X)$ the irreducible component of $\mathrm{Hom}(\mathbb{P}^1,X)$ containing $[f]$ (remember that $\mathrm{Hom}(\mathbb{P}^1,X)$ is smooth at $[f]$). The action

$$\begin{aligned} \mathrm{Aut}(\mathbb{P}^1) \times (\mathbb{P}^1)^r \times \mathrm{Hom}(\mathbb{P}^1,X) &\longrightarrow (\mathbb{P}^1)^r \times \mathrm{Hom}(\mathbb{P}^1,X) \\ \big(\varphi,(c_1,\ldots,c_r\,;\,[g])\big) &\longmapsto \big(\varphi(c_1),\ldots,\varphi(c_r)\,;\,[g\circ\varphi^{-1}]\big) \end{aligned}$$

clearly preserves the irreducible components of $\mathrm{Hom}(\mathbb{P}^1,X)$. Since f is not constant, the action of $\mathrm{Aut}(\mathbb{P}^1)$ has finite stabilizers. Consider the diagram

$$\begin{tikzcd} & (\mathbb{P}^1)^r \times \mathcal{H}^f \arrow[dl, "F^{(r)}"'] \arrow[dr, "M"] & \\ X^r & & {}_f\overline{\mathcal{M}}_{bir}(X,\beta) \end{tikzcd}$$

where M is the projection onto the factor $\mathcal{H}^f$ followed by the natural map that quotients out the action of $\mathrm{Aut}(\mathbb{P}^1)$. The morphism M is obviously dominant, while Proposition 1.11 (together with Lemma 1.9) implies that $F^{(r)}$ is dominant if $r \leq d-1$. Thus we may compute

$$\dim\big({}_f\overline{\mathcal{M}}_{bir}(X,\beta)\big) = \dim\Big((\mathbb{P}^1)^r \times \mathcal{H}\Big) - r - 3 = -f(\mathbb{P}^1)\cdot K_X - 1 = d-1.$$

Let $c_1,\ldots,c_r \in \mathbb{P}^1$ be $r \leq d-1$ distinct points at which f is an embedding and let $p_i = f(c_i)$. Let $p := (c_1,\ldots,c_r;[f]) \in (\mathbb{P}^1)^r \times \mathrm{Hom}(\mathbb{P}^1,X)$; it follows from Proposition 1.11 that

$$\dim\big(F^{(r)}\big)^{-1}(p_1,\dots,p_r) = r + \dim\mathcal{H}^f - 2r = -f(\mathbb{P}^1)\cdot K_X + 2 - r = d - r + 2.$$

Let $\overline{\mathcal{M}}_{bir}\big(X,\beta\big)(p_1,\dots,p_r)$ be the closure of $M\Big(\big(F^{(r)}\big)^{-1}(p_1,\dots,p_r)\Big)$. Since $\mathrm{Aut}(\mathbb{P}^1)$ has finite stabilizers on $\big(F^{(r)}\big)^{-1}(p_1,\dots,p_r)$, we find

$$\dim\overline{\mathcal{M}}_{bir}\big(X,\beta\big)(p_1,\dots,p_r) = d - r - 1. \tag{1.6}$$

1.3. Independent Points. The next Lemma analyzes curves containing $d-1$ general points.

Lemma 1.12. *For a general $(p_1,\dots,p_{d-1})\in X^{d-1}$, all the morphisms in $\overline{\mathcal{M}}_{bir}\big(X,\beta\big)(p_1,\dots,p_{d-1})$ are immersions.*

Proof. Let $\mathcal{I}\subset\big(\mathbb{P}^1\big)^{d-1}\times\mathcal{H}^f$ be the set of all $d-$tuples $(c_1,\dots,c_{d-1};[g])$ such that g is not an immersion; Lemma 1.9 implies that $\mathcal{I}$ is a proper closed subset of $\big(\mathbb{P}^1\big)^{d-1}\times\mathcal{H}^f$. Note that $\mathcal{I}$ is $\mathrm{Aut}(\mathbb{P}^1)-$invariant. By Proposition 1.11 and Lemma 1.9, $F^{(d-1)}$ is dominant, hence the general fiber of this morphism has dimension $d-1-f(\mathbb{P}^1)\cdot K_X+2-2(d-1) = d+2-d+1 = 3$, thus the fibers of this morphism are $\mathrm{Aut}(\mathbb{P}^1)-$orbits, since they are stable under the action of $\mathrm{Aut}(\mathbb{P}^1)$. If the general fiber of $F^{(d-1)}$ met $\mathcal{I}$, then we would have

$$\dim\mathcal{I}\geq 2(d-1)+3 = 2d+1 = (d-1)+(d+2) = \dim\Big(\big(\mathbb{P}^1\big)^{d-1}\times\mathcal{H}^f\Big)$$

and $\mathcal{I}$ would equal $\big(\mathbb{P}^1\big)^{d-1}\times\mathcal{H}^f$, which contradicts Lemma 1.9. Thus there is an open dense subset $\mathcal{U}$ in X^{d-1} not meeting the image of $\mathcal{I}$. For any $(d-1)-$tuple $(p_1,\dots,p_{d-1})\in\mathcal{U}$ we have that

$$\overline{\mathcal{M}}_{bir}\big(X,\beta\big)(p_1,\dots,p_{d-1}) = M\Big(\big(F^{(d-1)}\big)^{-1}(p_1,\dots,p_{d-1})\Big)\subset\overline{\mathcal{M}}_{bir}\big(X,\beta\big)$$

consists only of (finitely many) immersions. □

We now want to prove that for a general choice of $d-2$ points on X, all the resulting morphisms in $\overline{\mathcal{M}}_{bir}\big(X,\beta\big)$ through them have reduced image.

Definition 1.13. We say that r points $p_1,\dots,p_r$ in X are *independent* if the following conditions hold for all k:

(1) no k of them are contained in a rational curve of degree k;
(2) the normalization of a rational curve of degree $k+1$ in X through k of them is an immersion.

Proposition 1.11, Lemma 1.12 and the dimension estimates (1.6) imply that for any $r\geq 1$ there are $r-$tuples of independent points if there are free rational curves of anticanonical degree $d\geq r+1$, and that there are rational curves of anticanonical degree d through r independent points if $d\geq r+1$.

Lemma 1.14. *Let $C \subset X$ be a divisor of anticanonical degree $d \geq 3$ all of whose reduced irreducible component are rational. Let $p_1, \dots, p_{d-2} \in X$ be $d-2$ independent points contained in C. Then the divisor C has at most two irreducible components and it is reduced.*

Proof. Denote by $C_1, \dots, C_\ell$ the reduced irreducible components of C. For each curve C_i let d_i be the degree of C_i, m_i be the multiplicity of C_i in C and δ_i be the number of points $p_1, \dots, p_{d-2}$ lying on C_i. Then we have $\sum m_i d_i = d$ and $\delta_i \leq d_i - 1$. Therefore

$$d - 2 = \sum \delta_i \leq \sum d_i - \ell \leq \sum m_i d_i - \ell = d - \ell$$

Thus $\ell \leq 2$, and the Lemma follows at once. $\square$

Lemma 1.15. *Let $p_1, \dots, p_r \in X$ be $r \geq 2$ independent points, and let $\alpha \subset X$ be an integral curve of degree $r+2$ and geometric genus zero containing $p_1, \dots, p_r$. Let B be a smooth connected projective curve and let $F : B \to \overline{\mathcal{M}}^{\alpha}_{bir}(p_1, \dots, p_r)$ be a non-constant morphism. The reducible curves in the family parametrized by B cannot always contain a component mapped isomorphically to a curve of anticanonical degree strictly smaller than two.*

Proof. Let $S \to B$ be the pull-back of the universal family $\overline{\mathcal{M}}_{0,1}(X, \alpha) \to \overline{\mathcal{M}}_{0,0}(X, \alpha)$: $S \to B$ is a ruled surface with singularities of type A_k. The reducible fibers consist of exactly two smooth rational curves (Lemma 1.14) meeting at a point, possibly singular on S. By hypothesis $S \to B$ admits r contractible sections. If all the reducible fibers of S contained a component mapped to a curve of anticanonical degree strictly smaller than two, then the r contractible sections would meet only smooth points of S and would always be contained in the same component of each fiber (by definition of independent points). This is impossible since a $\mathbb{P}^1-$bundle has at most one negative section. $\square$

1.4. Sliding moves. The next Lemma and its Corollary allow us to construct irreducible subschemes in the boundary of the spaces $\overline{\mathcal{M}}_{0,0}\big(X, \beta\big)$.

Let $f : \bar{C} \to X$ be a stable map of genus zero to the smooth surface X. Let $\bar{C}_0$ be a connected subcurve, let $\bar{C}_1, \dots, \bar{C}_\ell$ be the connected components of the closure of $\bar{C} \setminus \bar{C}_0$. Let $\bar{C}_{0i}$ be the irreducible component of $\bar{C}_0$ meeting $\bar{C}_i$, and let $\bar{C}_{i,1}$ be the irreducible component of $\bar{C}_i$ meeting $\bar{C}_0$ and let the intersection point of $\bar{C}_{0i}$ and $\bar{C}_{i,1}$ be $\bar{p}_i$. Denote by f_i the restriction of f to $\bar{C}_i$, for $i \in \{0, \dots, \ell\}$.

Let $V \subset \overline{\mathcal{M}}_{0,\ell}\big(X, f_*[\mathbb{P}^1]\big) \times \big(\bar{C}_1 \times \cdots \times \bar{C}_\ell\big)$ be the subscheme consisting of all points $\big([g\,; \bar{c}_1, \dots, \bar{c}_\ell]\,; \bar{c}'_1, \dots, \bar{c}'_\ell\big)$, such that $g(\bar{c}_i) = f(c'_i)$ and $[g\,; \bar{c}_1, \dots, \bar{c}_\ell]$ is in the same irreducible component of $\overline{\mathcal{M}}_{0,\ell}\big(X, f_*[\mathbb{P}^1]\big)$ as $[f\,; \bar{p}_1, \dots, \bar{p}_\ell]$.

Lemma 1.16. *With notation as above, assume also that a general deformation of f_0 is generated by global sections, $\bar{C}_{0i}$ is not contracted by f and*

$f(\bar{C}_{0i}) \not\supset f(\bar{C}_{i,1})$, for all i's. It follows that every irreducible component of V containing $\big([f_0\,;\bar{p}_1,\dots,\bar{p}_\ell]\,;\bar{p}_1,\dots,\bar{p}_\ell\big)$ surjects onto the irreducible component of $\overline{\mathcal{M}}_{0,0}\big(X,f_*[\mathbb{P}^1]\big)$ containing $[f]$.

Proof. Let Φ be an irreducible component of $\overline{\mathcal{M}}_{0,0}\big(X,f_*[\mathbb{P}^1]\big)$ containing (the stable reduction of) $[f]$. Define $\mathcal{C}$ by the Cartesian square on the left and $\underline{ev}$ as the composite of the maps in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[rr, bend left, "\underline{ev}:=(ev_1,\dots,ev_\ell)"] & \overline{\mathcal{M}}_{0,\ell}\big(X,f_*[\mathbb{P}^1]\big) \arrow[r] \arrow[d] & X^\ell \\ \Phi \arrow[r, hook] & \overline{\mathcal{M}}_{0,0}\big(X,f_*[\mathbb{P}^1]\big). & \end{tikzcd}$$

Thus V fits in the diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\iota"] \arrow[dd, bend right=60, "\pi"'] & \big(\bar{C}_1\times\cdots\times\bar{C}_\ell\big) \arrow[d, "(f_1,\dots,f_\ell)"] \\ \mathcal{C} \arrow[r, "\underline{ev}"] \arrow[d] & X^\ell \\ \Phi & \end{tikzcd}$$

and we have $V\subset W:=\mathcal{C}\times\big(\bar{C}_1\times\cdots\times\bar{C}_\ell\big)\xrightarrow{P}\mathcal{C}$. Obviously P is flat and since $\mathcal{C}\longrightarrow\Phi$ is flat, it follows that $W\longrightarrow\Phi$ is flat. The fiber of π at the point $[g]$ is given by

$$\pi^{-1}\big([g]\big)=\Big\{\big([\tilde{g}\,;\bar{c}_1,\dots,\bar{c}_\ell]\,;\bar{c}'_1,\dots,\bar{c}'_\ell\big)\;\Big|\;\tilde{g}(\bar{c}_i)=f_i(\bar{c}'_i)\Big\}$$

where the stable reduction of $\tilde{g}$ is g. If g has irreducible domain, and if the image of g does not contain any singular point of (the reduced scheme) $f\big(\bar{C}_1\cup\ldots\cup\bar{C}_\ell\big)$, nor does it contain any component of $f\big(\bar{C}_i\big)$, then the scheme $\pi^{-1}\big([g]\big)$ is finite. Thanks to [Ko] Theorem II.7.6 and Proposition II.3.7, a general deformation g of f_0 satisfies the previous conditions; thus the general fiber of π in a neighbourhood of $[f]$ is finite and hence, letting $v_0:=\big([f_0\,;\bar{p}_1,\dots,\bar{p}_\ell]\,;\bar{p}_1,\dots,\bar{p}_\ell\big)$, we conclude that $\dim_{v_0}V=\dim\Phi=\dim\mathcal{C}-\ell$. Let $\kappa_i\in\mathcal{O}_{X,f(\bar{p}_i)}$ be a local equation of $f_i\big(\mathbb{P}^1\big)$; clearly the ℓ equations $P^*ev_1^*(\kappa_1),\dots,P^*ev_\ell^*(\kappa_\ell)$ define V near v_0. Since $\dim V=\dim\mathcal{C}-\ell$, it follows that $\mathcal{O}_{V,v_0}$ is a Cohen-Macaulay $\mathcal{O}_{W,v_0}-$module. Using [EGA4] Proposition 6.1.5, we deduce that $\mathcal{O}_{V,v_0}$ is a flat $\mathcal{O}_{\Phi,[f_0]}-$module. The result follows. □

Construction. If $f:\bar{C}\to X$ is a stable map satisfying the hypotheses of Lemma 1.16, then there is a proper irreducible subscheme $\mathrm{Sl}_f(\bar{C}_0)$ of $\overline{\mathcal{M}}_{0,0}\big(X,f_*[\bar{C}]\big)$, consisting of morphisms $g:\bar{C}'\to X$ such that:

- there is a decomposition $\bar{C}' = \bar{C}'_0 \cup \ldots \cup \bar{C}'_\ell$, where $\bar{C}'_i$ are connected curves;
- there are isomorphisms $g|_{\bar{C}'_i} \simeq f|_{\bar{C}_i}$;
- there is a morphism $res : \mathrm{Sl}_f(\bar{C}_0) \to \overline{\mathcal{M}}_{0,0}\big(X, f_*[\bar{C}_0]\big)$, which is surjective on the irreducible component containing $f|_{\bar{C}_0}$;
- there are dominant morphisms $a_i : \mathrm{Sl}_f(\bar{C}_0) \to \bar{C}_i$, for $i \in \{1, \ldots, \ell\}$.

We say that a stable map $[g] \in \mathrm{Sl}_f(\bar{C}_0)$ such that $a_i([g]) = p$ is obtained from $[f]$ by *sliding $\bar{C}_0$ along $\bar{C}_i$ until it reaches p.* We say that a stable map $[g] \in \mathrm{Sl}_f(\bar{C}_0)$ such that $\pi([g]) = [h]$ is obtained from $[f]$ by *sliding $\bar{C}_0$ fixing the remaining components until it reaches h.*

A typical application of this construction can be found in the proof of Lemma 3.1 as well as in many of the later proofs.

2. Divisors of Small Degree: the Picard Lattice

2.1. The Nef Cone. We collect here some results on the nef cone of a del Pezzo surface. We prove a "numerical" decomposition of any nef divisor on a del Pezzo surface in Corollary 2.3. In the later sections we will show how to realize geometrically this decomposition.

Definition 2.1. Let X_δ be a del Pezzo surface of degree $9-\delta$. An integral basis $\{\ell, e_1, \ldots, e_\delta\}$ of $\mathrm{Pic}(X_\delta)$ is a *standard basis* if there is a presentation $b : X_\delta \to \mathbb{P}^2$ of X_δ as the blow up of $\mathbb{P}^2$ at δ points such that ℓ is the pull-back of the class of a line and the e_i's are the exceptional divisors of b.

The following well-known Proposition gives a criterion to detect nef divisor classes on del Pezzo surfaces.

Proposition 2.2. *Let X be a del Pezzo surface of degree $d \leq 7$. A divisor class $C \in \mathrm{Pic}(X)$ is nef if and only if $C \cdot L \geq 0$ for all $(-1)-$curves $L \subset X$.*

From this Proposition we deduce immediately the following Corollary.

Corollary 2.3. *Let X_δ be a del Pezzo surface of degree $9 - \delta \leq 8$. Let $D \in \mathrm{Pic}(X_\delta)$ be a nef divisor. There are non-negative integers $n_2, \ldots, n_\delta$, a sequence of contraction of $(-1)-$curves $X_\delta \to X_{\delta-1} \to \cdots \to X_1$, and a nef divisor $D' \in \mathrm{Pic}(X_1)$ such that*

$$D = n_\delta(-K_{X_\delta}) + n_{\delta-1}(-K_{X_{\delta-1}}) + \ldots + n_2(-K_{X_2}) + D'.$$

Proof. Proceed by induction on δ. If $\delta \leq 1$, there is nothing to prove. Suppose that $\delta \geq 2$ and let $n_\delta := \min\big\{L \cdot D \mid L \subset X \text{ a } (-1)-\text{curve}\big\}$. By assumption $n_\delta \geq 0$. Let $\bar{D} := D + n_\delta K_{X_\delta}$; then $\bar{D} \cdot L \geq 0$ for every $(-1)-$curve $L \subset X_\delta$. By Proposition 2.2 $\bar{D}$ is nef and by construction there is a $(-1)-$curve $L_0 \subset X$ such that $\bar{D} \cdot L_0 = 0$. Thus $\bar{D}$ is the pull-back of a nef divisor on the del Pezzo surface $X_{\delta-1}$ obtained by contracting L_0. The Corollary follows thanks to the inductive hypothesis. $\square$

2.2. First Cases of the Main Theorem.

Proposition 2.4. *Let X_δ be a del Pezzo surface of degree $9-\delta \geq 3$. The scheme $\overline{\mathcal{M}}_{bir}\big(X_\delta, -K_{X_\delta}\big)$ is birational to a $\mathbb{P}^{6-\delta}-$bundle over X_δ; in particular, it is rational and irreducible.*

Proof. The rational map $\overline{\mathcal{M}}_{bir}\big(X_\delta, -K_{X_\delta}\big) \dashrightarrow X_\delta$ is obtained by assigning to a stable map with irreducible domain the unique singular point of its image. The rest of the proof is straightforward. $\square$

Proposition 2.5. *Let X be a del Pezzo surface of degree two. The scheme $\overline{\mathcal{M}}_{bir}\big(X, -K_X\big)$ is isomorphic to a smooth plane quartic.*

Proof. Let $\kappa : X \to \mathbb{P}^2$ be the morphism associated to the anticanonical sheaf. The branch curve $R \subset \mathbb{P}^2$ is a smooth quartic. The points of every irreducible component of $\overline{\mathcal{M}}_{bir}\big(X, -K_X\big)$ correspond to the singular divisor in $|-K_X|$. These in turn are parameterized by the tangent lines to R. $\square$

2.3. The Picard Group and the Orbits of the Weyl Group. In this section we prove some results on the divisor classes of del Pezzo surfaces. In particular, we determine the orbits of pairs of conics under the Weyl group.

Let X_δ be a del Pezzo surface of degree $9-\delta$.

Definition 2.6. A divisor C on X_δ is a *conic* if $-K_{X_\delta} \cdot C = 2$ and $C^2 = 0$.

Suppose that $\{\ell, e_1, \ldots, e_\delta\}$ is a standard basis of $\mathrm{Pic}(X_\delta)$. If $C = a\ell - b_1e_1 - \ldots - b_\delta e_\delta$ is a divisor on X_δ, we sometimes write it as $(a\,; b_1, \ldots, b_\delta)$.

Proposition 2.7. *The conics on X_8 are given, up to permutation of the e_i's, by the following table:*

(2.1)

Type	ℓ	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
A	1	1	0	0	0	0	0	0	0
B	2	1	1	1	1	0	0	0	0
C	3	2	1	1	1	1	1	0	0
D	4	2	2	2	1	1	1	1	0
E	5	2	2	2	2	2	2	1	0
D'	4	3	1	1	1	1	1	1	1
F	5	3	2	2	2	1	1	1	1
G	6	3	3	2	2	2	2	1	1
H	7	3	3	3	3	2	2	2	1
H'	7	4	3	2	2	2	2	2	2
I	8	4	3	3	3	3	2	2	2
I'	8	3	3	3	3	3	3	3	1
J	9	4	4	3	3	3	3	3	2
K	10	4	4	4	4	3	3	3	3
L	11	4	4	4	4	4	4	4	3

Their numbers are given by the table:

δ	8	7	6	5	4	3	2
conics	2160	126	27	10	5	3	2

Proof. We proceed just like in [Ma] IV, §25. The condition of being a conic translates to the equations

$$a^2 - \sum_{i=1}^{8} b_i^2 = 0 \quad \text{and} \quad 3a - \sum_{i=1}^{8} b_i = 2$$

and we may equivalently rewrite these as

$$\sum_{i=1}^{8} \big(a - 2b_i - 2\big)^2 = 16 \quad \text{and} \quad 3a - \sum_{i=1}^{8} b_i = 2$$

It is now easy (but somewhat long) to check that (2.1) is the complete list of solutions up to permutations. □

Remark 2.8. The classes of conics on X_δ for $\delta \leq 7$ are obtained from the ones in list (2.1) by erasing $8-\delta$ zeros and permuting the remaining coordinates. Thus (up to permutations) the first five rows and seven columns describe conics on X_7, the first three rows and six columns are the conics on X_6 and so on.

Denote by $\cdot$ the intersection form on the lattice $\mathrm{Pic}(X_\delta)$. From now on by an automorphism of $\mathrm{Pic}(X_\delta)$ we will always mean a group automorphism of the lattice which preserves the intersection form and the canonical class; we let $W_\delta := \mathrm{Aut}\big(\mathrm{Pic}(X_\delta), K_{X_\delta}, \cdot\big)$, and we refer to W_δ as the Weyl group. It will be useful later to know what are the orbits of pairs of conics under the automorphism group W_δ of $\mathrm{Pic}(X_\delta)$.

Lemma 2.9. *The group W_δ, $2 \leq \delta \leq 8$, acts transitively on the conics.*

Proof. We only prove this in the case $\delta = 8$ and it will be clear from the proof that the same argument applies to the other cases.

Choose a standard basis $\{\ell, e_1, \ldots, e_8\}$ of $\mathrm{Pic}(X)$; it is enough to prove that the elements in the list (2.1) are in the same orbit, since any permutation of the indices is in W_8. Let T_{123} be the involution of $\mathrm{Pic}(X_8)$ such that

$$\big(a\,;\, b_1, \ldots, b_8\big) \xrightarrow{T_{123}} \big(2a-b_1-b_2-b_3\,;\, a-b_2-b_3, a-b_1-b_3, a-b_1-b_2, b_4, \ldots, b_8\big).$$

By inspection, the quantity $2a-b_1-b_2-b_3$ for elements in list (2.1) is always strictly smaller than the initial value of a unless $a=1$. Permuting the indices so that b_1, b_2, b_3 are the three largest coefficients among the b_i's and iterating this strategy finishes the argument. Note that we are always "climbing up" list (2.1) and the conics on X_7 are the ones above line 5, and are hence preserved by the automorphism T_{123} and the permutations needed. Similar remarks are valid for X_δ, with $3 \leq \delta \leq 6$, and the result is obvious for X_2, where the automorphism T_{123} is not defined. $\square$

We now turn our attention to the action of the Weyl group on ordered pairs of conics $\big(Q_1, Q_2\big)$. The number $Q_1 \cdot Q_2$ is an invariant of this action. Looking at the list (2.1) we see that

$\delta =$	8	7	6	5	4	3	2
$Q_1 \cdot Q_2 \leq$	8	4	2	2	1	1	1

and that all the possible values between 0 and the number given above are attained; thus the action of W_8 on pairs of conics has at least 9 orbits.

If $\delta = 8$, there is one more "invariant" under W_8 of pairs of conics: define a pair $\big(Q_1, Q_2\big)$ to be *ample* if $Q_1 + Q_2$ is an ample divisor on X_8. Since the property of being ample is a numerical property, it follows that it is a property of the W_8-orbit of the pair.

The next Proposition proves that the lower bounds on the number of orbits obtained by considering the intersection product and ampleness (in case $\delta = 8$) of the pair are in fact the correct number of orbits.

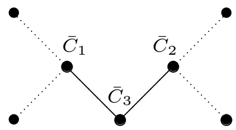
Proposition 2.10. *Let Q_1 and Q_2 be two conics in X_δ, $2 \leq \delta \leq 8$. The intersection product $Q_1 \cdot Q_2$ determines uniquely the orbit of the (ordered) pair $\big(Q_1, Q_2\big)$ under W_δ with the only exception of $\delta = 8$ and $Q_1 \cdot Q_2 = 4$ which has exactly two orbits, one ample and one not ample.*

Proof. The proof is similar to the proof of Lemma 2.9. First we may assume that $Q_1 = \ell - e_1$. Then we again climb up the list (2.1) using the automorphism T_{123} followed by a permutation of the indices $\{2, \ldots, 8\}$ so that the resulting b_2 and b_3 are the two largest b_i's, with $i \geq 2$. Note that the elements of W_δ described above do indeed fix Q_1. We leave the explicit checks to the reader. $\square$

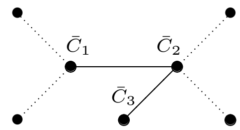
3. Realizing the Deformation: from Large to Small Degree

3.1. Breaking the Curve. In this section we construct deformations of a general point in every irreducible component of the space $\overline{\mathcal{M}}_{bir}\big(X, \beta\big)$ to morphisms with image containing only curves of small anticanonical degree.

Lemma 3.1. *Let $f : \bar{C}_1 \cup \ldots \cup \bar{C}_r \to X$ be a stable map of genus zero and suppose that f is a free morphism. If $f(\bar{C}_1) \cdot f(\bar{C}_2) > 0$ and $\bar{C}_3$ is the union of the components between $\bar{C}_1$ and $\bar{C}_2$, then in the same irreducible component of $\overline{\mathcal{M}}_{0,0}\big(X, f_*[\bar{C}]\big)$ containing $[f]$ there is a free morphism $g : \bar{C}_1 \cup \ldots \cup \bar{C}_r \to X$ with dual graph*

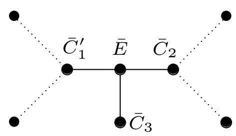


Dual graph of f

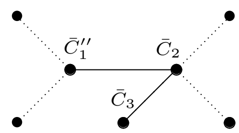


Dual graph of g

Proof. Assume first that $\bar{C}_3$ is irreducible. Slide $\bar{C}_1$ along $\bar{C}_3$ until it reaches $\bar{C}_3 \cap \bar{C}_2$ to obtain a morphism f_1 with dual graph



Dual graph of f_1



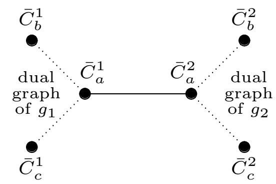
Dual graph of f_2

where $\bar{E}$ is a contracted component. The morphism f_1 is also obtained from a morphism f_2 with the dual graph shown above by sliding $\bar{C}_1''$ along $\bar{C}_2$ until it reaches $\bar{C}_2 \cap \bar{C}_3$. Finally, slide $f_2|_{\bar{C}_1''}$ until it reaches $f|_{\bar{C}_1}$.

If $\bar{C}_3$ is not irreducible, it suffices to start the deformation smoothing out $\bar{C}_3$ fixing the remaining components, apply the Lemma, and then break again $\bar{C}_3$ into its components. Thanks to Propositions 1.6 and 1.7 all the points where the sliding moves ended are smooth. Thus all the deformations took place in the component of $\overline{\mathcal{M}}_{0,0}\big(X, f_*[\bar{C}]\big)$ containing f. $\square$

Lemma 3.2. *Let $f : \mathbb{P}^1 \to X$ be a free birational morphism to a del Pezzo surface. In the same irreducible component of $\overline{\mathcal{M}}_{bir}\big(X, f_*[\mathbb{P}^1]\big)$ as f there is a free morphism $g : \bar{C} \to X$ birational to its image such that each irreducible component of $\bar{C}$ has image of anticanonical degree two or three.*

Proof. We establish the Lemma by induction on $d := -K_X \cdot f_*[\mathbb{P}^1]$. There is nothing to prove if $d \leq 3$, since the image of a free morphism has anticanonical degree at least two. Suppose that $d \geq 4$. By Proposition 1.11, we may assume that the image of f contains $d-2 \geq 2$ independent points $p_1, \ldots, p_{d-2}$ of X; we know from (1.6) that $\dim_{[f]} \overline{\mathcal{M}}_{bir}\big(p_1, \ldots, p_{d-2}\big) = 1$. Thanks to the Bend and Break Theorem and Lemmas 1.14 and 1.15 we deduce that in the same irreducible component of $\overline{\mathcal{M}}_{bir}\big(X, f_*[\mathbb{P}^1]\big)$ as f we can find a free morphism $f_0 : \bar{C}_1 \cup \bar{C}_2 \to X$ birational to its image with $\bar{C}_i$ irreducible. Note that $[f_0]$ is a smooth point of the mapping space. Since $d_i := -K_X \cdot f_0(\bar{C}_i) \geq 2$, by induction on d the irreducible component of $\overline{\mathcal{M}}_{bir}\big(X, f_0(\bar{C}_i)\big)$ containing $f_0|_{\bar{C}_i}$ contains a morphism $g_i : \bar{C}_1^i \cup \ldots \cup \bar{C}_{r_i}^i \longrightarrow X$ with all the required properties. Sliding $\bar{C}_1$ until it reaches g_1 and then sliding $\bar{C}_2$ in the resulting morphism until it reaches g_2 we obtain a morphism f_1 with dual graph



Dual graph of f_1

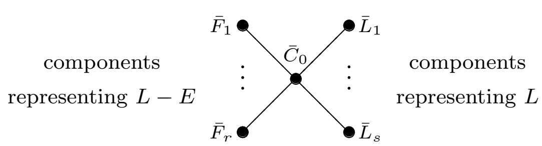
We need to show that the images $f_1(\bar{C}_a^1)$ and $f_1(\bar{C}_a^2)$ can be assumed to be distinct. This follows at once from Lemma 3.1 and the fact that if two non-zero nef classes have intersection product zero, then they must be multiples of the same conic class. $\square$

3.2. Easy Cases: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $Bl_p(\mathbb{P}^2)$. This section proves the irreducibility of the spaces $\overline{\mathcal{M}}_{bir}\big(Bl_p(\mathbb{P}^2), \alpha\big)$; a similar argument could be applied also to the case of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. In these last two cases though the result is obvious and it also follows also from [KP].

Theorem 3.3. *The spaces of stable maps $\overline{\mathcal{M}}_{bir}\big(Bl_p(\mathbb{P}^2), \alpha\big)$ are irreducible or empty for all divisor classes α.*

Proof. Let $L, E \in \operatorname{Pic}(Bl_p(\mathbb{P}^2))$ be the divisor classes of a line and the $(-1)-$curve respectively. Let $f : \mathbb{P}^1 \to Bl_p(\mathbb{P}^2)$ be a general morphism in an irreducible component of $\overline{\mathcal{M}}_{bir}\big(Bl_p(\mathbb{P}^2), \alpha\big)$ and let $d := -K_{Bl_p(\mathbb{P}^2)} \cdot \alpha$. Since the integral divisors of degree $d \leq 3$ on $Bl_p(\mathbb{P}^2)$ are E, $L - E$ and L and the Theorem is true for them, we may assume that $d \geq 4$.

Using Lemma 3.2 we deform f to an immersion $f' : \bar{C} \to Bl_p(\mathbb{P}^2)$ birational to its image and such that the irreducible components of $f'(\bar{C})$ have anticanonical degree two or three. Since the divisors in $|L - E|$ are either disjoint or coincide, it follows that there must be a component $\bar{C}_0 \subset \bar{C}$ such that $f_*[\bar{C}_0] = L$. Using Lemma 3.1 we also assume that all the components of $\bar{C}$ different from $\bar{C}_0$ are adjacent to $\bar{C}_0$:



Dual graph of f'

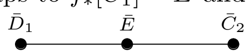
The locus of morphisms having this labeled dual graph is an open subset of $\big(\mathbb{P}^2\big)^{s+1} \times \big(\mathbb{P}^1\big)^r$. Thus the smooth locus of $\overline{\mathcal{M}}_{bir}\big(Bl_p(\mathbb{P}^2), \gamma\big)$ is connected and the Theorem follows. $\square$

4. Realizing the Deformation: from Small to Large Degree

4.1. Growing from the Conics. In this section we prove some results that allow us to deform unions of conics to divisors which are the anticanonical divisor on a del Pezzo surface dominated by X. These results are the main building blocks in the proof of Theorem 4.3.

Lemma 4.1. *Let $f : \bar{C} \to X$ be a free morphism from a connected, projective, nodal curve of arithmetic genus zero to a del Pezzo surface. Suppose that $\bar{C}_1$ and $\bar{C}_2$ are the irreducible components of $\bar{C}$ and that E is a $(-1)-$curve on X such that $f_*[\bar{C}_1] \cdot E = 0$ and $f_*[\bar{C}_2] \cdot E > 0$. Then in the irreducible component of $\overline{\mathcal{M}}_{0,0}\big(X, f_*[\bar{C}]\big)$ containing $[f]$ there is a morphism $g : \bar{D}_1 \cup \bar{D}_2 \to X$ representing a smooth point such that $\bar{D}_1$ and $\bar{D}_2$ are irreducible, $g_*[\bar{D}_1] = f_*[\bar{C}_1] - E$ and $g_*[\bar{D}_2] = f_*[\bar{C}_2] + E$.*

Proof. Since f is free, we may assume that $f(\bar{C}_2)$ does not contain any intersection point of $(-1)-$curves and that the intersection $f(\bar{C}_2) \cap E$ is transverse. Let $\bar{p} \in \bar{C}_2$ be a point such that $p := f(\bar{p}) \in E$; such a point exists since $f(\bar{C}_2) \cdot E > 0$. Slide $\bar{C}_1$ along $\bar{C}_2$ until it reaches $\bar{p}$ to obtain a morphism $f_1 : \bar{C}_1' \cup \bar{C}_2 \to X$. Since $f_*[\bar{C}_1] \cdot E = 0$ and $f_1(\bar{C}_1') \ni p$, it follows that $\bar{C}_1' = \bar{D}_1 \cup \bar{E}$, where $\bar{D}_1$ maps to $f_*[\bar{C}_1] - E$ and $\bar{E}$ maps to E:



Dual graph of f_1

Since $f(\bar{C}_2)$ does not contain the intersections of two $(-1)-$curves, there cannot be contracted components. Note that the node between $\bar{D}_1$ and $\bar{E}$ maps to a node, since $(f_1)_*[\bar{D}_1] \cdot E = 1$. The stable map $[f_1]$ represents a smooth point of its moduli space by an application of Proposition 1.6. Smoothing out $\bar{E} \cup \bar{C}_2$ concludes the proof. ☐

Proposition 4.2. *Let X_δ be a del Pezzo surface of degree $9 - \delta$ such that the spaces $\overline{\mathcal{M}}_{bir}\big(X_\delta, \beta\big)$ are irreducible or empty if $-K_{X_\delta} \cdot \beta = 2, 3$. Let $f : \bar{Q} \to X_\delta$ be a morphism from a connected, projective, nodal curve of arithmetic genus zero. Suppose that $\bar{Q}_1$ and $\bar{Q}_2$ are the irreducible components of $\bar{Q}$ and that $f_*[\bar{Q}_1]$ and $f_*[\bar{Q}_2]$ are conics. If $f(\bar{Q}_1) \cdot f(\bar{Q}_2) \geq 2$, then in the irreducible component of $\overline{\mathcal{M}}_{0,0}\big(X_\delta, f_*[\bar{Q}]\big)$ containing $[f]$ there is a morphism $g : \bar{C} \to X_\delta$ such that*

- *all the irreducible components of $\bar{C}$ are immersed and represent nef divisor classes;*
- *there is a component $\bar{C}_1 \subset \bar{C}$ and a standard basis $\big\{\ell, e_1, \ldots, e_\delta\big\}$ of $\operatorname{Pic}(X_\delta)$ with $g_*[\bar{C}_1] = 3\ell - e_1 - \ldots - e_\alpha$ for some $\alpha \leq \delta$;*
- *the point $[g]$ is smooth.*

Proof. The cases $9 - \delta \leq 1$ are clear; suppose that $9 - \delta \geq 2$. Since $f^* T_X$ is globally generated, f represents a smooth point of $\overline{\mathcal{M}}_{0,0}\big(X_\delta, f_*[\bar{Q}]\big)$. Sliding

$f|_{\bar{Q}_1}$ fixing $\bar{Q}_2$, we may assume that $Q_1 := f(\bar{Q}_1)$ misses the intersection points between any two $(-1)-$curves.

If $Q_1 \cdot Q_2 = 2$, then by Proposition 2.10 we may write $Q_1 = \ell - e_1$ and $Q_2 = 2\ell - e_2 - e_3 - e_4 - e_5$ and it suffices to smooth $\bar{Q}_1 \cup \bar{Q}_2$ to conclude.

Suppose that $Q_1 \cdot Q_2 \geq 3$; by Proposition 2.10 we may choose a standard basis so that $Q_1 = \ell - e_1$ and $Q_2 = M_1 + M_2$ with

$Q_1 \cdot Q_2 = 3$	$Q_1 \cdot Q_2 = 4$
$Q_2 = \big(5\,;\,2,2,2,2,2,2,1\big)$	$Q_2 = \big(5\,;\,1,2,2,2,2,2,2\big)$
$M_1 = \big(2\,;\,0,1,1,1,1,1,0\big)$	$M_1 = \big(2\,;\,0,1,1,1,1,1,0\big)$
$M_2 = \big(3\,;\,2,1,1,1,1,1,1\big)$	$M_2 = \big(3\,;\,1,1,1,1,1,1,2\big)$

Apply Lemma 4.1 with $E = M_1$ to obtain a morphism $g' : \bar{Q}_1' \cup \bar{M}_2 \to X$, such that $g'_*[\bar{Q}_1'] = 3\ell - e_1 - \ldots - e_6$. Apply again the same argument in Lemma 4.1 with $E = E_7$ to conclude (we do not need g' to be nef, since we have some freedom in the choice of the limit of $\bar{Q}_1'$). □

4.2. Reduction of the Problem to Finitely Many Cases. This section gathers the information obtained in the previous sections to prove that the irreducibility of $\overline{\mathcal{M}}_{bir}\big(X,\beta\big)$ for all β can be checked by examining only finitely many cases. The proof involves several steps and is quite long.

Theorem 4.3. *Let X_δ be a del Pezzo surface of degree $9-\delta \geq 2$, such that the spaces $\overline{\mathcal{M}}_{bir}\big(X,\beta\big)$ are irreducible (or empty) for all nef divisors β with $2 \leq -K_X \cdot \beta \leq 3$. Then, for any nef divisor $D \subset X$ the space $\overline{\mathcal{M}}_{bir}\big(X,D\big)$ is irreducible or empty.*

Proof. We prove the Theorem by induction on $d := -K_X \cdot D$. By hypothesis the Theorem is true if $d \leq 3$ and by Theorem 3.3 it is true if $\delta \leq 2$.

If there is a $(-1)-$curve $L \subset X$ such that $L \cdot D = 0$, then $\overline{\mathcal{M}}_{bir}\big(X,D\big) \simeq \overline{\mathcal{M}}_{bir}\big(X', b_*D\big)$, where $b : X \to X'$ is the contraction of L. Hence we reduce to the case in which the divisor D intersects strictly positively every $(-1)-$curve, i.e. D is ample (Proposition 2.2).

Suppose that $d \geq 4$. Let $f : \mathbb{P}^1 \to X$ be a general morphism in an irreducible component of $\overline{\mathcal{M}}_{bir}\big(X,D\big)$. Thanks to Lemma 3.2 we may deform f to a morphism $g : \bar{C} \to X$ such that each component $\bar{C}_0 \subset \bar{C}$ is immersed to a curve of anticanonical degree two or three. We want to show that we may specialize g to a morphism in which one component is mapped to a curve with class $-K_X$. We prove this in a series of steps.

Step 1. There is a standard basis $\{\ell, e_1, \ldots, e_\delta\}$ of $\mathrm{Pic}(X)$ and a component $\bar{C}_1$ of $\bar{C}$ mapped birationally to a curve with class $3\ell - e_1 - \ldots - e_\alpha$, for some $\alpha \in \{1, \ldots, 7\}$. The morphism is free on all the components of $\bar{C}$.

The only nef divisors of anticanonical degree at most three on X that are not of the required form are the conics and the divisor ℓ in some standard basis. Hence we reduce to the case in which each component of $\bar{C}$ is mapped to a curve whose class is either a conic or ℓ, for some choice of standard basis. We reduce further to the following case:

($\star$) There is a standard basis $\{\ell, e_1, \ldots, e_\delta\}$ of $\mathrm{Pic}(X)$ such that all curves of degree three in the image of g have divisor class ℓ.

This is easily accomplished. Suppose that $\bar{C}_1$ and $\bar{C}_2$ are components of $\bar{C}$ such that $g_*[\bar{C}_1] = \ell_1$ and $g_*[\bar{C}_2] = \ell_2$, where $\{\ell_i, e_1^i, \ldots, e_\delta^i\}$ are two standard basis of $\mathrm{Pic}(X)$ and $\ell_1 \neq \ell_2$. We may first of all apply Lemma 3.1 to assume that $\bar{C}_1$ and $\bar{C}_2$ are adjacent. Since ℓ_2 is not proportional to ℓ_1, it cannot be orthogonal to $e_1^1, \ldots, e_\delta^1$. Thus we may assume that $\ell_2 \cdot e_1^1 > 0$. Applying Lemma 4.1 with $E = e_1^1$ and using Lemma 3.2 we decrease by two the number of components mapping to curves of degree 3. Iterating this argument we assume that condition ($\star$) holds.

If there are three conics in the image of g with pairwise intersection products equal to one, smoothing them concludes the proof of Step 1. This remark and Proposition 4.2 allow us further to reduce to the case in which there are curves representing the divisor class ℓ in the image of g and all conics represent the same two divisors classes Q_1 or Q_2, with $Q_1 \cdot Q_2 = 1$. The divisor D is ample and we may assume that the classes of the components of the image of g are ℓ or conics. Since no multiple of ℓ is ample, it follows that there must be components mapped to conics.

Suppose that $\bar{C}_1$ is mapped to a curve with class ℓ and $\bar{C}_2$ is mapped to a curve with class a conic Q. Thanks to Lemma 3.1 we may assume that $\bar{C}_1$ and $\bar{C}_2$ are adjacent. Referring to table (2.1), if Q is of type B, then smoothing out $\bar{C}_1 \cup \bar{C}_2$ concludes the first step; if Q is of type C, D or E, then applying Lemma 4.1 with $E = e_1$ concludes (with a different choice of standard basis, when $Q = D$ or E). The only remaining case is the one in which all the conics are of type A. Since the image of g is an ample divisor and $\delta \geq 2$, it follows that there must be two components $\bar{Q}_1, \bar{Q}_2$ mapped to $\ell - e_1, \ell - e_2$ respectively. Applying Lemma 3.1 and smoothing out $\bar{C}_1 \cup \bar{Q}_1 \cup \bar{Q}_2$ concludes the first step.

Step 2. There is a component $\bar{C}_1 \subset \bar{C}$ mapped to a curve with class $-K_X$.

Let $\bar{C}_1$ be the component mapped to a curve with class $3\ell - e_1 - \ldots - e_\alpha$ and apply Lemma 3.1 to assume that all remaining components are adjacent to $\bar{C}_1$. Apply successively Lemma 4.1 with $E = E_{\alpha+1}, \ldots, E_\delta$ to conclude.

Step 3. If $D + K_X$ is not a multiple of a conic, then we may deform g to a free morphism $h : \bar{C}_1 \cup \bar{C}_2 \to X$ where $\bar{C}_1$ and $\bar{C}_2$ are irreducible, $h_*[\bar{C}_1] = -K_X$, $h(\bar{C}_1) \neq h(\bar{C}_2)$. If $D + K_X$ is a multiple of a conic Q, then we may deform $g : \bar{C} \to X$ to a free morphism $h : \bar{C}_1 \cup \bar{Q}_1 \cup \ldots \cup \bar{Q}_r \to X$ where $\bar{C}_1$ and all $\bar{Q}_i$'s are irreducible, $h_*[\bar{C}_1] = -K_X$, $h_*[\bar{Q}_i] = Q$ and $\bar{C}_1$ is adjacent to all $\bar{Q}_i$.

Thanks to the previous steps, we may assume that $g_*[\bar{C}_1] = -K_X$. If the divisor $D + K_X$ is not a multiple of a conic, then any divisor representing it is connected (this follows at once from Corollary 2.3). Let $\bar{C}_2' \subset \bar{C}$ be a component different from $\bar{C}_1$; using Lemma 3.1 we may assume that every component of $\bar{C}$ is adjacent to $\bar{C}_2'$ and then we conclude by smoothing out all the components of $\bar{C} \setminus \bar{C}_1$ to a single irreducible component $\bar{C}_2$.

If $D + K_X = rQ$, where Q is a conic, use Lemma 3.2 to break every component different from $\bar{C}_1$ to a union of curves representing Q and then use Lemma 3.1 to make sure that all the components of the resulting curve are adjacent to $\bar{C}_1$. This concludes the third step.

Write $D = n_\delta(-K_{X_\delta}) + \ldots + n_2(-K_{X_2}) + D'$ (Corollary 2.3) and let S_D be the locus of free morphisms $k : \bar{Z} \cup \bar{C}_1 \cup \ldots \cup \bar{C}_r \to X$, where $\bar{Z}$ and the $\bar{C}_i$'s are all irreducible, $\bar{Z}$ is adjacent to all the $\bar{C}_i$'s and represents $-K_{X_\delta}$, $n_\delta - 1$ of the $\bar{C}_i$ represent $-K_{X_\delta}$, $n_{\delta-1}$ of the $\bar{C}_i$ represent $-K_{X_{\delta-1}}$, and so on. Applying repeatedly the reduction of Step 3 and Lemma 3.1, we deduce that every irreducible component of $\overline{\mathcal{M}}_{bir}\big(X, D\big)$ contains points in S_D.

Step 4. The locus S_D is connected.

Let $k_0 : \mathbb{P}^1 \to X_\delta$ be a morphism with $(k_0)_*[\mathbb{P}^1] = -K_{X_\delta}$ and let $S \subset \overline{\mathcal{M}}_{0,0}\big(X, -K_{X_\delta} - K_{X_\alpha}\big)$ be the locus of free morphisms $k : \mathbb{P}^1 \cup \mathbb{P}^1 \to X_\delta$ birational to their image, where k agrees with k_0 on the "first" component and sends the "second" component to $-K_{X_\alpha}$. It is clearly enough to show that S is connected for general k_0 and all α. Unless $\alpha = \delta = 7$, the statement is clear. If $\alpha = \delta = 7$, then S consists of at most two irreducible components: restricting a morphism in S to its second component is a dominant morphism to $\overline{\mathcal{M}}_{bir}\big(X, -K_{X_7}\big)$ with fibers of length $(-K_{X_7})^2 = 2$. If S is reducible, the closures of its two irreducible components meet at the morphisms $k : \mathbb{P}^1 \cup \bar{C} \to X$ for which $k(\mathbb{P}^1 \cap \bar{C})$ is a ramification point for the anticanonical map $X_7 \to \mathbb{P}^2$. For general k_0 some such points are in S and S is connected.

The Theorem follows from Step 3 and Step 4, since S_D is contained in the smooth locus of $\overline{\mathcal{M}}_{bir}\big(X, D\big)$. $\square$

Remark 4.4. It follows easily from Corollary 2.3 that if D is a nef divisor which is not a multiple of a conic, then the space $\overline{\mathcal{M}}_{bir}\big(X, D\big)$ is not empty.

Remark 4.5. The spaces $\mathcal{M}_{0,0}\big(X, mC\big)$, where C is the class of a conic, are easily seen to be irreducible, for $m \geq 0$.

5. Conclusion

Theorem 5.1. *Let X_δ be a del Pezzo surface of degree $9-\delta \geq 2$. The spaces $\overline{\mathcal{M}}_{bir}\big(X_\delta, \beta\big)$ are irreducible or empty for every divisor $\beta \in \mathrm{Pic}(X_\delta)$.*

Proof. Suppose $\overline{\mathcal{M}}_{bir}\big(X_\delta, \beta\big)$ is not empty. Then β is represented by an effective integral curve on X_δ. If β is not nef, then it follows that $\beta^2 < 0$. We deduce that β is a positive multiple d of a $(-1)-$curve. If $d = 1$, then $\overline{\mathcal{M}}_{bir}\big(X_\delta, \beta\big)$ consists of a single point. If $d > 1$, then $\overline{\mathcal{M}}_{bir}\big(X_\delta, \beta\big) = \emptyset$.

Suppose now that β is a nef divisor. Thanks to Theorem 4.3, we simply need to check that on a del Pezzo surface of degree at least two, the spaces $\overline{\mathcal{M}}_{bir}\big(X_\delta, \beta\big)$ are irreducible for all effective integral divisor classes β such that $-K_{X_\delta} \cdot \beta$ equals two or three. These cases are easy and some of them have already been treated in Propositions 2.4 and 2.5 and in Theorem 3.3. $\square$

As a Corollary of the above Theorem, we deduce the irreducibility of the Severi varieties of rational curves on the del Pezzo surfaces (of degree $d \geq 2$). Let β be a divisor class in $\mathrm{Pic}(X_\delta)$ and let $V_{0,\beta} \subset |\beta|$ be the closure of the set of points corresponding to integral rational divisors. We call $V_{0,\beta}$ the *Severi variety of rational curves on X with divisor class β.*

Corollary 5.2. *Let X_δ be a del Pezzo surface of degree $9-\delta \geq 2$. The Severi varieties $V_{0,\beta}$ of rational curves on X_δ are either empty or irreducible for every divisor $\beta \in \mathrm{Pic}(X_\delta)$.*

References

BS. R. Beheshti, J. Starr, *Rational surfaces in index-one Fano hypersurfaces*, to appear in J. Algebraic Geom.

dJS1. A. J. de Jong, J. Starr, *Cubic fourfolds and spaces of rational curves*, Illinois J. Math. 48 (2004), no. 2, 415-450.

dJS2. A. J. de Jong, J. Starr, *Higher Fano manifolds and rational surfaces*, math.AG/0602647.

DM. P. Deligne, D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. No. 36 1969 75-109.

Fu. W. Fulton, *Hurwitz schemes and irreducibility of moduli of algebraic curves*, Ann. of Math. 90 (1969) 542-575.

GHS. T. Graber, J. Harris, J. Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. 16 (2003), no. 1, 57-67.

EGA4. A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. No. 32 1967.

H. J. Harris, *On the Severi problem*, Invent. Math 84, 445-461 (1986).

HRS. J. Harris, M. Roth, J. Starr, *Rational curves on hypersurfaces of low degree*, J. Reine Angew. Math. 571 (2004), 73-106.

HS. J. Harris, J. Starr, *Rational curves on hypersurfaces of low degree II*, Compos. Math. 141 (2005), no. 1, 35-92.

Ha. R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977.

KP. B. Kim, R. Pandharipande, *The connectedness of the moduli space of maps to homogeneous spaces*, Symplectic geometry and mirror symmetry (Seoul, 2000), 187-201, World Sci. Publ., River Edge, NJ, 2001.

Ko. J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 32, Springer-Verlag, Berlin, 1996.

La. R. Lazarsfeld, *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 48, Springer-Verlag, Berlin, 2004.

Ma. Iu. I. Manin, *Cubic Forms. Algebra, geometry, arithmetic*, second edition, North-Holland Mathematical Library, 4, North-Holland Publishing Co., Amsterdam, 1986.

Se. F. Severi, *Vorlesungen über Algebraische Geometrie, Anhang F*, Leipzig, Teubner 1921.