

Conical and Spherical Graphs

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Abstract

We introduce and study the notions of conical and spherical graphs. We show that these mutually exclusive properties, which have a geometric interpretation, provide links between apparently unrelated classical concepts such as dominating sets, independent dominating sets, edge covers, and the homotopy type of an associated simplicial complex. In particular, we solve the problem of characterizing the forests whose dominating sets of minimum cardinality are also independent. To establish these connections, we prove a formula to compute the Euler characteristic of an arbitrary simplicial complex from a set of generators of its Stanley-Reisner ideal.

Keywords: independence complex, domination number, Euler characteristic

Dedicated to Tony Machí

1. Introduction

Let G be a graph and let D be a set of vertices of G . The set D is called independent if it contains no edge of G , and it is called dominating if every vertex not in D is adjacent to a vertex in D . Several authors have considered classes of graphs for which independent and dominating sets

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have various properties. One of the first papers in this direction is [AL], whose main result is that, for a claw-free graph, there are dominating sets of minimum cardinality that are also independent (see also [ALH, BC] for quantitative extensions of this result). Intuitively, graphs having dominating sets of minimum cardinality that are also independent correspond to networks that can be efficiently controlled with well-spaced (i.e. non-adjacent) controlling nodes. Several variations and generalizations have been studied, motivated also by applications in computer science, graph theory, network analysis, and statistical mechanics (see, for example, [HHS, HY, LW, My]). A natural class of graphs that arises from such considerations is the class of graphs for which every dominating set of minimum cardinality is automatically independent; we call such graphs *starred*. Intuitively, the property of being starred corresponds to networks that not only can be efficiently controlled, but for which also the process of determining an optimal controlling system of nodes is straightforward. The possibility of recognizing starred graphs efficiently is therefore crucial. In this paper we give several complete characterizations of starred forests in Theorem 4.10: we show that the property of being starred is related to graph theoretic parameters defined in terms of independent sets, dominating sets and edge covers, and that it also admits a topological interpretation in terms of independence complexes.

More precisely, we partition (finite, simple, loopless) graphs in two classes: conical and spherical graphs. These concepts are purely graph theoretical, but are inspired by the approach of [MT1], dealing with monomial ideals and simplicial complexes. In fact, they have geometric implications for the associated independence complex: a conical graph has independence complex homotopic to a point, a (solvable) spherical graph has independence complex homotopic to a sphere (whose dimension, in the case of forests, equals the domination number, decreased by one). The characterization of Theorem 4.9 states that the non-trivial starred forests are precisely the spherical ones. From a computational point of view, combining the previous result with Theorem 3.3, we find that if a forest is starred, then it can be recognized by the computation of a single resolution (see Definition 3.1). In general, it is much more difficult to determine the dominating sets of minimum cardinality of a graph than to find its resolutions. Nonetheless, we show that, for spherical forests, the dominating sets of minimum cardinality are in bijection with the resolutions.

The other characterizations of conical and spherical forests given in the paper are in terms of dominating sets, independent dominating sets and edge

covers, and hence establish relationships between all these graph theoretic invariants, as well as links to the homotopy type of the associated independence complex. To prove some of the connections, we need a combinatorial formula to compute the Euler characteristic of a simplicial complex from a set of generators of its Stanley-Reisner ideal (Theorem 4.2). This formula holds in complete generality for an arbitrary simplicial complex.

The paper is organized as follows. Section 2 establishes the notation of the paper. In Section 3, we define the notions of spherical and conical graphs, show that they are mutually exclusive and prove some other basic properties. Section 4 contains the main results of the paper. We prove that the Euler characteristic of an arbitrary simplicial complex can be easily computed from a set of generators of its Stanley-Reisner ideal. Then, we give several characterizations of being a conical or spherical forest, thus obtaining, among others, results about dominating sets of minimum cardinality and cross-matchings.

2. Notation and Background

If $r \in \mathbb{Z}$, $r \geq 0$, we let $[r] := \{1, \dots, r\}$. The cardinality of a set A will be denoted by $|A|$.

We consider finite undirected graphs $G = (V, E)$ with no loops or multiple edges. For all $S \subset V$, let $N[S] := \{w \in V \mid \exists s \in S, \{s, w\} \in E\} \cup S$ be the *closed neighborhood of S* ; when $S = \{v\}$, then we let $N[v] := N[\{v\}]$. A *leaf* is a vertex $v \in V$ such that $|N[v]| = 2$. A set $D \subset V$ is called *dominating* if, for all $v \in V$, $N[v] \cap D \neq \emptyset$. A set $D \subset V$ is called *independent* if no two vertices in D are adjacent, i.e. $\{v, v'\} \notin E$ for all $v, v' \in D$. An *edge cover of G* is a subset $S \subset E$ such that the union of all the endpoints of the edges in S is V . We consider the following classical invariants of a graph G :

- $\gamma(G) := \min\{|D|, D \text{ is a dominating set of } G\}$ be the *domination number of G* ;
- $i(G) := \min\{|D|, D \text{ is an independent dominating set of } G\}$ be the *independent domination number of G* ;
- $cov_G(t) := \sum_{S \text{ edge cover}} t^{|S|}$ be the generating function for the edge covers of G .

We refer the reader to [B] or [D] for all undefined notation on graph theory.

We let $X := \{x_1, \dots, x_n\}$ and $\mathbb{Z}[X]$ be the polynomial ring with variables x_1, \dots, x_n over the integers. Let $m, m' \in \mathbb{Z}[X]$; we write $m'|m$ if m' divides m . A *simplicial complex* Δ on X is a set of subsets of X such that, if $\sigma \in \Delta$ and $\sigma' \subset \sigma$, then $\sigma' \in \Delta$. The subsets in Δ are the *faces of Δ* . The faces of cardinality one are called *vertices*. Equivalently, identifying a set $S \subset X$ with the monomial $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where $\varepsilon_i = \begin{cases} 1, & \text{if } x_i \in S, \\ 0, & \text{if } x_i \notin S; \end{cases}$ a simplicial complex Δ on $\mathbb{Z}[X]$ is a finite set of monic square-free monomials of $\mathbb{Z}[X]$ such that, if $m \in \Delta$ and $m'|m$, then $m' \in \Delta$.

Every simplicial complex Δ on $\mathbb{Z}[X]$ different from $\{1\}$ has a standard geometric realization. Let e_1, \dots, e_n be the standard basis of \mathbb{R}^n . The realization of Δ is the union of the convex hulls of the sets $\{e_i \text{ such that } x_i|m\}$, for each monomial $m \in \Delta$. Whenever we mention a topological property of Δ , we implicitly refer to the geometric realization of Δ .

Let $I \subset \mathbb{Z}[X]$ be a monomial ideal (i.e. an ideal generated by monomials) containing x_1^2, \dots, x_n^2 . The set of monomials of $\mathbb{Z}[X] \setminus I$ is a simplicial complex on $\mathbb{Z}[X]$ that we denote by $R(I)$. Conversely, given a simplicial complex Δ on $\mathbb{Z}[X]$, let $I_\Delta \subset \mathbb{Z}[X]$ be the ideal generated by the monomials not in Δ . Clearly $\Delta = R(I_\Delta)$ and $I = I_{R(I)}$. Note that I_Δ is strictly related to the Stanley-Reisner ideal of the simplicial complex Δ , which is the ideal generated by the square-free monomials not in Δ (see [S]).

As examples, for $n \geq 1$, consider the monomial ideals $I_n = (x_1^2, \dots, x_n^2)$, $J_n = (x_1 \cdots x_n, x_1^2, \dots, x_n^2)$, and $K_n = (x_1 x_2, x_3 x_4, \dots, x_{2n-1} x_{2n}, x_1^2, \dots, x_{2n}^2)$. The simplicial complex $R(I_n)$ is the $(n-1)$ -dimensional simplex and $R(J_n)$ is its boundary; $R(K_n)$ is the boundary of the n -dimensional cross-polytope, which is the dual of the n -dimensional cube. Note that the cube, its boundary and the cross-polytope are not simplicial complexes.

Let M be a finitely generated graded $\mathbb{Z}[X]$ -module M ; we denote the (multi-graded) Hilbert series of M by $H(M; x_1, \dots, x_n)$. The *multi-graded face polynomial* $\mathcal{F}_\Delta(x_1, \dots, x_n)$ of a simplicial complex Δ is the polynomial of $\mathbb{Z}[X]$

$$\mathcal{F}_\Delta(x_1, \dots, x_n) := \sum_{m \in \Delta} m = H(\mathbb{Z}[X]/I_\Delta; x_1, \dots, x_n).$$

The *face polynomial* $F_\Delta(t)$ of Δ is the polynomial

$$F_\Delta(t) := \sum_{m \in \Delta} t^{\deg m} = \mathcal{F}_\Delta(t, \dots, t).$$

The *reduced Euler characteristic* of Δ is $\tilde{e}(\Delta) := -F_\Delta(-1)$.

We refer the reader to [MS, K, Mu] for all undefined concepts from commutative algebra and algebraic topology.

3. Conical and Spherical Graphs

In this section, we define spherical and conical graphs and prove some of their basic properties.

Let (a_1, \dots, a_r) be a sequence of vertices of G and, for $i \in [r + 1]$, let $G_i := G \setminus N[\{a_1, \dots, a_{i-1}\}]$.

Definition 3.1. *A resolution of G is a sequence $A = (a_1, \dots, a_r)$ such that, for every $i \in [r]$, either*

1. a_i is an isolated vertex of G_i , or
2. a_i is a vertex of G_i adjacent to a leaf of G_i .

We call $c(A) := G_{r+1}$ the core of A , $d(A) := r$ the depth of A , and

$$\begin{aligned} c(G) &:= \{c(A) \mid A \text{ is maximal}\} \\ d(G) &:= \min\{d(A) \mid A \text{ is maximal}\} \end{aligned}$$

respectively the core and the depth of G .

The resolution A is spherical if (1) never occurs, conical otherwise; the graph G is spherical if it admits a maximal resolution which is spherical, conical if it admits a (maximal) resolution which is conical, solvable if at least one graph in $c(G)$ is the empty graph.

Note that a forest is always solvable. For the reader's convenience, we prove that the properties of being spherical and conical are mutually exclusive. We give a direct proof of this fact; the argument provided here is simpler than the more general argument in [MT1, Theorem 4.10]. We need the following lemma.

Lemma 3.2. *Let $A = (a_1, \dots, a_r)$ be a resolution of G and suppose that, for some $i \in [r]$, the vertex a_i is adjacent to a leaf of G . Then the sequence $A' = (a_i, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r)$ is a resolution of G of the same type as A , and $c(A') = c(A)$.*

Proof. By induction on i , we immediately reduce to the case $i = 2$. If a_1 is an isolated vertex of G , then the result is clear. Thus a_1 is adjacent to a leaf of G . Let b_k be a leaf of G adjacent to a_k , for $k \in \{1, 2\}$.

If $a_1 \in N[a_2]$, then symmetrically $a_2 \in N[a_1]$ and (a_1, a_2) could not be a resolution. Thus $a_1 \notin N[a_2]$ and (a_2, a_1) is a resolution of G since b_1 is a vertex of $G \setminus N[a_2]$. Hence $A' = (a_2, a_1, a_3, \dots, a_r)$ is a resolution. Moreover, since a_1 is not isolated in $G \setminus N[a_2]$, the resolutions A and A' are either both spherical or both conical.

The assertion about the core is trivial since the core of a resolution does not depend on the sequence, but just on the underlying set. \square

Theorem 3.3. *If a maximal resolution of a graph G is spherical, then all maximal resolutions of G are spherical and have the same depth. Moreover, in this case, the core of G has a unique element.*

Proof. Let $A = (a_1, \dots, a_r)$ and $A' = (a'_1, \dots, a'_s)$ be maximal resolutions of a spherical graph G and suppose that A is spherical. We proceed by induction on r . If $r = 0$, then the only maximal resolution of G is the empty resolution; thus G is the unique element of its core, and we are done.

Assume that $r \geq 1$. Let b'_1 be a leaf of G adjacent to a'_1 . Since A is maximal, $a'_1, b'_1 \in N[\{a_1, \dots, a_r\}]$; since the vertices of A are pairwise not adjacent and b'_1 is a leaf, exactly one of a'_1, b'_1 appears in A . If $b'_1 \in \{a_1, \dots, a_r\}$, say $b'_1 = a_i$, then a'_1 is a leaf of $G_i := G \setminus N[\{a_1, \dots, a_{i-1}\}]$ and the edge $\{b'_1, a'_1\}$ forms a connected component of G_i . Hence we may replace b'_1 with a'_1 .

Thus we reduce to the case $a'_1 \in \{a_1, \dots, a_r\}$, say $a'_1 = a_i$. By Lemma 3.2, the sequence $(a'_1, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r)$ is a spherical resolution of G with same core as A . By the inductive hypothesis, the maximal resolution (a'_2, \dots, a'_s) of $G \setminus N[a'_1]$ is spherical and has the same depth and the same core as the resolution $(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r)$. Hence A' is a spherical resolution of G of depth r and $c(A') = c(A)$. \square

Example 3.4. *Consider the graph G in Figure 1. It has two maximal resolutions, one of depth 2 and the other of depth 3, whose cores are, respectively, a cycle with four vertices and the empty graph. Therefore the graph G is conical and solvable.*

The property of being spherical or conical is a “global” property: for instance a path with n vertices is conical if and only if n is congruent to

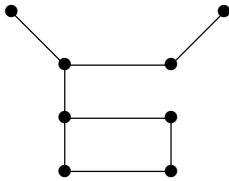


Figure 1: A graph with a two-element core.

1 modulo 3. Nevertheless, there are “local” obstructions for a graph to be spherical: for instance, if a graph has two leaves at distance three, then it is conical. It is therefore quite surprising that if one resolution produces an isolated vertex, then all maximal resolutions will necessarily produce one.

Remark 3.5. Define $qc(G) := \{c(A) \mid A \text{ is a maximal spherical resolution}\}$ to be the quasi-core of a graph G (namely isolated vertices are not cut away). Clearly the quasi-core of a spherical graph coincides with its core and hence it is a singleton by Theorem 3.3. For non-trivial trees the converse also holds.

Fact 3.6. The quasi-core of a conical tree T is a singleton only if T is trivial.

Proof. Let $A = (a_1, \dots, a_r)$ be a maximal spherical resolution of T and let x be an isolated vertex of $c(A)$. Suppose that x is an isolated vertex of $T \setminus N[\{a_1, \dots, a_i\}]$ but not of $T' = T \setminus N[\{a_1, \dots, a_{i-1}\}]$. This means that the neighbours of x in T' are all neighbours also of a_i . Since T is a tree, there is a unique such neighbour y . Hence x is a leaf of T' and $A = (a_1, \dots, a_{i-1}, y)$ is a spherical resolution of T whose core does not contain x . \square

Let m, n be positive integers and consider the tree $T_{m,n}$ in Figure 2. The maximal resolutions of $T_{m,n}$ have depth either $m + 1$ or $n + 1$; the quasi-core of $T_{m,n}$ is $\{\{l_0, \dots, l_m\}, \{r_0, \dots, r_n\}\}$.

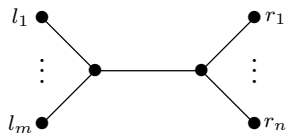


Figure 2: The conical tree $T_{m,n}$.

If a conical graph is not a tree, then its quasi-core may be a singleton (Figure 3).

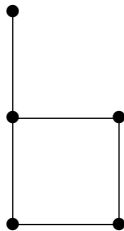


Figure 3: The quasi-core is a singleton.

The terminology introduced in this section is motivated by reasons concerning simplicial complexes associated to graphs. Let $G = (V, E)$ be a graph. The simplicial complex on V whose faces are the subsets of V containing no adjacent vertices (i.e. the independent sets) is denoted by $\text{Ind}(G)$ and is called the *independence complex* of G . The simplicial complex $\text{Ind}(G)$ is a cone of apex a if and only if a is an isolated vertex of G . The following result is a straightforward consequence of [MT1, Theorems 4.3 and 5.4].

Proposition 3.7. *Let G be a graph.*

1. *The independence complex $\text{Ind}(G)$ is homotopic to an iterated suspension of the independence complex of any element of the quasi-core $qc(G)$. Moreover, if G is conical, then $\text{Ind}(G)$ is contractible; if G is spherical and solvable, then $\text{Ind}(G)$ is homotopy equivalent to a sphere of dimension $d(G) - 1$.*
2. *If G is a forest, then $i(G) = d(G)$; if G is a spherical forest, then $\gamma(G) = d(G) = i(G)$. \square*

Remark 3.8. *Conical graphs are harder to handle than spherical graphs (see, for example, Theorem 3.3 and Theorem 4.9). This reflects the fact that the independence complex $\text{Ind}(G)$ has trivial homotopy type in the case G is conical, and hence it is less “rigid” than the independence complex associated to a spherical graph.*

As mentioned in the introduction, we want to study forests for which every dominating set of minimum cardinality is automatically independent. For this we define in general the class of graphs having this property and we call such graphs *starred*. In Theorem 4.9 we shall prove that spherical and starred forests without isolated vertices coincide.

A tree on r vertices is a *star* if it has a vertex c adjacent to the remaining $r - 1$ vertices, possibly $r = 1$ (isolated vertex). The vertex c is the *center* of the star (Figure 4).

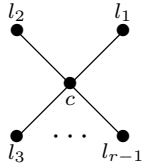


Figure 4: A star.

Definition 3.9. A graph G is *starred* if all its minimum dominating sets are independent. A graph G is *starred by n stars* if G is starred and $\gamma(G) = n$.

In other words, G is starred by n stars if every dominating set D of cardinality n gives rise to a covering of the vertices of G by n stars with independent centers belonging to D . If G is starred by n stars, then $i(G) = \gamma(G) = n$. Note that an independent set of maximum cardinality is dominating, while a dominating set of minimum cardinality need not be independent. The graphs for which every dominating set of minimum cardinality is independent are the starred graphs. We characterize the starred forests in Theorem 4.9.

Proposition 3.10. Suppose that G is starred by n stars and that A is a resolution of G . Then $c(A)$ is starred by $n - d(A)$ stars.

Proof. By induction on $d(A)$ we reduce to the case $d(A) = 1$; let $A = (a)$. If a is an isolated vertex, then the result is clear. Suppose that a is adjacent to a leaf b of G , and let D be a dominating set of G of cardinality n . Either D contains a or D contains b , and, if D contains b , then $(D \setminus \{b\}) \cup \{a\}$ is a dominating set with n elements. It follows that D contains no neighbour of a except possibly b , and hence $D \setminus N[a]$ is a dominating set of $G' := G \setminus N[a]$ with $n - 1$ elements. Moreover, if D' is a dominating set of G' with $n - 1$ elements, then $D' \cup \{a\}$ is a dominating set of G with n elements; it follows that D is independent since G is starred with n stars, and therefore D' is also independent. Thus G' is starred by $n - 1$ stars and the lemma follows. \square

Clearly, the converse of Proposition 3.10 does not hold, as can be seen by taking any non-trivial conical tree.

Lemma 3.11. *If F is a conical tree with at least one edge, then F is not starred.*

Proof. If $\gamma(F) < i(F)$, then F cannot be starred. Suppose $\gamma(F) = i(F) = r$ and recall that, if F were starred, then it would be starred by r stars. Let (a_1, \dots, a_r) be a maximal resolution of F of depth r (such a resolution exists by Proposition 3.7, part (2)). Note that $D := \{a_1, \dots, a_r\}$ is an independent dominating set of F of cardinality $\gamma(F)$. Since F is conical, there exists an index $i \in [r]$ such that removing $N[a_i]$ from $F \setminus N[\{a_1, \dots, a_{i-1}\}]$ creates an isolated vertex v ; note that therefore $v \in \{a_{i+1}, \dots, a_r\}$. Let w be the unique vertex adjacent to both a_i and v . From the definitions, it is clear that $D' := (D \setminus \{v\}) \cup \{w\}$ is a non-independent dominating set of cardinality $\gamma(F)$. \square

We conclude this section with some results on the complexity of the properties of being conical and spherical. These results will not be needed elsewhere in the paper. We refer the reader to [GGL, Chapter 23, Section 4.5] and [LY] for definitions and notations about evasiveness of properties and simplicial complexes.

Proposition 3.12. *The properties of being a conical graph, a spherical graph, a conical forest, a spherical forest and a spherical tree are evasive. The property of being a conical tree is evasive for graphs on $n \geq 4$ vertices.*

Proof. We give a strategy which lets Hider decide whether to construct a conical or a spherical forest answering the last question of Seeker. Hence, this is a winning strategy for the first four properties in the statement.

When asked for the existence of an edge, Hider answers “no” as long as this does not produce an isolated vertex, and “yes” otherwise. Let G_i be the graph we obtain by considering the edges Seeker knows that occur after the i -th question, and let $G = G_{\binom{n}{2}}$ be the final graph. For all i , the graph G_i has connected components consisting of stars. In fact, if Hider answers that the edge $\{a, b\}$ occurs, then she has already answered “no” about either all the edges containing a , or all the edges containing b . Thus, at least one among a and b is a leaf of G .

Let us consider the very last question. It must be on an edge connecting two centers of $G_{\binom{n}{2}-1}$, two isolated vertices of $G_{\binom{n}{2}-1}$, or a center and an isolated vertex of $G_{\binom{n}{2}-1}$. In the first case, if Hider answers “yes”, then G is a conical forest; if she answers “no”, then G is a spherical forest. In the other

two cases, if Hider answers “no”, then G is a conical forest; if she answers “yes”, then G is a spherical forest.

The property of being a spherical (resp. conical) tree is evasive since it suffices to choose a spherical (resp. conical) tree on n vertices and always answer truthfully the questions of Seeker. There are no conical trees on two or three vertices. \square

Theorem 3.13. *Let G be a graph. If G is conical, then the complex $\text{Ind}(G)$ is non-evasive; if G is spherical and solvable, then the complex $\text{Ind}(G)$ is evasive.*

Proof. Let G be a graph and let $A = (a_1, \dots, a_r)$ be a conical resolution of G . Let $k = k(A) \in [r]$ be the index such that the vertex a_k is isolated in $G \setminus N[\{a_1, \dots, a_{k-1}\}]$, but for $i \in [k-1]$ the vertex a_i is not isolated in $G \setminus N[\{a_1, \dots, a_{i-1}\}]$. Proceed by induction on k . If $k = 1$, then Seeker never asks about a_1 . Suppose that $k \geq 2$ and in particular the vertex a_1 is not isolated; let b be a leaf adjacent to a_1 . In this case Seeker asks about a_1 . If Hider answers “no”, then Seeker does not need to ever ask about b and we are done. If Hider answers “yes”, then Seeker asks about all vertices in $N[a_1] \setminus \{a_1\}$ and Hider must answer “no”, since otherwise we are again done. We are left with the graph $G' = G \setminus N[a_1]$ and the conical resolution $A' = (a_2, \dots, a_r)$ of G' . Since we have $k(A') = k-1$, we conclude by induction on k .

If G is spherical and solvable, then by Theorem 3.7 the complex $\text{Ind}(G)$ is homotopic to a sphere and hence must be evasive, since a non-evasive complex is contractible (see [LY, Lemma 3.2]). \square

In particular the independence complex of a spherical forest is evasive. Let us give a strategy for Hider in this case. We use the fact that, if F is a spherical forest, then for every vertex v of F exactly one of $F \setminus \{v\}$ and $F \setminus N[v]$ is spherical (this can be easily proved using Proposition 3.7). Informally, the strategy that Hider follows is to answer the questions of Seeker in such a way to always return a spherical forest pretending that answering “no” to the question about vertex v removes just v , while answering “yes” removes v and its neighbors. More precisely, let F be a spherical forest; at the i -th stage Seeker asked about the i vertices in V_i of F and Hider answered “yes” about the vertices in $V_i^Y \subset V_i$ and “no” about the vertices in $V_i^N := V_i \setminus V_i^Y$. Hider defines a subset Y_i of the set of vertices of $F \setminus V_i$ as follows: vertex w lies in Y_i if and only if $w \notin N[V_i^Y]$ and the graph obtained from F by removing

$V_i^N \cup N[V_i^Y \cup \{w\}]$ is spherical. When at the $(i + 1)$ -st stage Seeker asks about the vertex $v \notin V_i$, Hider answers “yes” if $v \in Y_i$ and she answers “no” otherwise. It is immediate from the description of the strategy that Hider never answers “yes” to adjacent vertices. Thus Seeker need not ask about vertex v if and only if Hider answered “no” to all the neighbors of v . But this would make the vertex v isolated and this is excluded since a forest with an isolated vertex is not spherical.

If G is spherical but not solvable, both phenomena may occur. The two graphs in Figure 5 are spherical, since they have no vertex of valence at most one. The independence complex of the connected graph on the left is evasive, since it is not contractible. The independence complex of the connected graph on the right is non-evasive, as can be seen immediately by asking first for the vertex v .

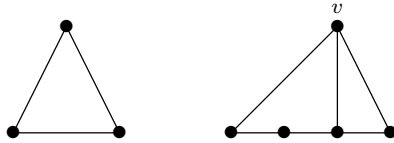


Figure 5: Evasive and non-evasive spherical graphs.

A conical graph G with vertex-transitive automorphism group contains no edge. Indeed, since G is conical, it contains vertices of valence less than two and, since it has a vertex-transitive automorphism group, every vertex of G has the same valence; since the disjoint union of edges is not conical, the statement follows. This proves the AKR conjecture in the case of conical graphs. On the other hand, if the AKR conjecture were false, then there would be non-evasive complexes Δ such that for all vertices v of Δ the face deletion and the link of v are non-evasive. Such complexes exist, an example being $\text{Ind}(T_{m,n})$ with $m, n \geq 2$ (see Figure 2).

4. Main Results

In this section we give several characterizations of conical and spherical forests: in particular, we show that spherical trees are precisely the non-trivial starred ones. The characterizations involve edge covers, independent sets and dominating sets, providing links between these different concepts. Some of the equivalences are obtained via the associated independence com-

plex using Theorem 4.2, which gives a combinatorial way to compute the Euler characteristic of a simplicial complex Δ .

Let M be a finite set of monomials of $\mathbb{Z}[X]$ and let $p \in \mathbb{Z}[X]$ be a monomial.

Definition 4.1. A cover S of p by M , denoted by $S \triangleright_M p$, is a subset $S \subset M$ such that $p = \text{lcm}(S)$, the least common multiple of the monomials in S . The covering polynomial $C_M(x_1, \dots, x_n) \in \mathbb{Z}[X]$ of M is the polynomial

$$C_M(x_1, \dots, x_n) := \sum_{S \subset M} (-1)^{|S|} \text{lcm}(S) = \sum c_M(p)p$$

where the sum is over all monomials p and $c_M(p) := \sum_{S \triangleright_M p} (-1)^{|S|}$.

An equivalent way to define the covering polynomial is as follows. If $M = \{m_1, \dots, m_r\}$, then $C_M(x_1, \dots, x_n) = (1 - m_1) \star \dots \star (1 - m_r)$, where $\star : \mathbb{Z}[X] \times \mathbb{Z}[X] \rightarrow \mathbb{Z}[X]$ is the \mathbb{Z} -bilinear distributive map defined on monomials m, m' by $m \star m' = \text{lcm}\{m, m'\}$. Note that if m, m' are monomials and $m \mid m'$, then $(1 - m) \star (1 - m') = 1 - m$. We deduce that, if M and M' are finite sets of monomials generating the same ideal, then the covering polynomials C_M and $C_{M'}$ coincide; equivalently, for every monomial p , we have $c_M(p) = c_{M'}(p)$.

The following result relates topological and combinatorial properties of a simplicial complex Δ , establishing a connection between the Euler characteristic of Δ and the leading coefficient of the covering polynomial of the Stanley-Reisner ideal of Δ .

Theorem 4.2. Let B be a finite set of monic monomials generating the Stanley-Reisner ideal of a simplicial complex Δ . Then the reduced Euler characteristic of Δ is

$$\tilde{e}(\Delta) = (-1)^{n-1} c_B(x_1 \cdots x_n).$$

Proof. Let M be a finite set of monomials in $\mathbb{Z}[X]$. The formula

$$H(\mathbb{Z}[X]/(M); x_1, \dots, x_n) = \frac{C_M(x_1, \dots, x_n)}{\prod_{s=1}^n (1 - x_s)} \quad (4.1)$$

holds: this can be proved either appealing to [MS, Theorem 4.11] applied to the Taylor resolution of the ideal generated by M , or by a direct argument using the Inclusion-Exclusion Principle.

Let $X^2 := \{x_1^2, \dots, x_n^2\}$ and $I = I_\Delta = (B, x_1^2, \dots, x_n^2)$. Since the monomial $x_1 \cdots x_n$ is square-free, we reduce to the case in which every monomial in B

is square-free. Recall that $\tilde{e}(\Delta) = -H(\mathbb{Z}[X]/I; -1, \dots, -1)$. By (4.1), we deduce that

$$2^n \tilde{e}(\Delta) = - \sum_{K \subset B \cup X^2} (-1)^{|K| + \deg(\text{lcm}(K))}.$$

Since $B \cap X^2 = \emptyset$, we have

$$\begin{aligned} 2^n \tilde{e}(\Delta) &= - \sum_{S \subset B} (-1)^{|S|} \sum_{T \subset X^2} (-1)^{|T| + \deg(\text{lcm}(S \cup T))} = \\ &= - \sum_{S \subset B} (-1)^{|S|} \sum_{\substack{T_1 \cup T_2 \subset X^2 \\ \text{lcm}(T_1) | (\text{lcm}(S))^2 \\ (\text{lcm}(S), \text{lcm}(T_2)) = 1}} (-1)^{|T_1| + |T_2| + \deg(\text{lcm}(S \cup T_1 \cup T_2))}. \end{aligned}$$

Since $\deg(\text{lcm}(S \cup T_1 \cup T_2)) = |T_1| + 2|T_2| + \deg(\text{lcm}(S))$, we obtain

$$2^n \tilde{e}(\Delta) = - \sum_{S \subset B} (-1)^{|S| + \deg(\text{lcm}(S))} \sum_{\substack{T_1 \cup T_2 \subset X^2 \\ \text{lcm}(T_1) | (\text{lcm}(S))^2 \\ (\text{lcm}(S), \text{lcm}(T_2)) = 1}} (-1)^{|T_2|}.$$

The last sum is zero unless $\text{lcm}(S) = x_1 \cdots x_n$, and thus

$$2^n \tilde{e}(\Delta) = -2^n \sum_{\substack{S \subset B \\ \text{lcm}(S) = x_1 \cdots x_n}} (-1)^{|S| + n}.$$

Hence

$$\tilde{e}(\Delta) = (-1)^{n-1} \sum_{S \triangleright_B x_1 \cdots x_n} (-1)^{|S|}$$

and we are done. \square

In particular, $\tilde{e}(\Delta)$, $c_B(x_1 \cdots x_n)$ and $|\{S \subset B \mid S \triangleright_B x_1 \cdots x_n\}|$ have same parity.

Example 4.3. Let $I = (x_1, x_2, x_3)^2 \subset \mathbb{Z}[x_1, x_2, x_3]$. Thus $R(I)$ is the disjoint union of three points, $\tilde{e}(R(I)) = 2$, and $B = \{x_1x_2, x_1x_3, x_2x_3\}$ generates the Stanley-Reisner ideal of $R(I)$. There are four covers of $x_1x_2x_3$ by B and we have $c_B(x_1x_2x_3) = 2$, as predicted by Theorem 4.2.

Large classes of simplicial complexes are *a priori* known to be either contractible or homotopic to a wedge of spheres of equal dimension (such as in the case of pure shellable complexes and of most of the complexes studied in [MT2]). In these cases, Theorem 4.2 may be useful to understand their homotopy type.

Remark 4.4. *Let M and M' be finite sets of monomials. Since the Hilbert series of $\mathbb{Z}[X]/J$ uniquely determines the monomial ideal J , it follows from (4.1) that M and M' generate the same ideal if and only if $C_M(x_1, \dots, x_n) = C_{M'}(x_1, \dots, x_n)$.*

By (4.1), we can generate closed formulas for certain sums: we give the following identities (which can of course be proved also by direct use of the Inclusion-Exclusion Principle) as easy examples of the possible strategies. In the following sums, if $S \subset H$ is the empty set, then we define $\min S := \max H$ and $\max S := \min H$.

1. *Let M be a finite non-empty subset of \mathbb{N} . Then*

$$\sum_{S \subset M} (-1)^{|S|} x^{\max S} = 1 - x^{\min M}.$$

2. *Let $\lambda = \lambda_0, \lambda_1, \dots, \lambda_{r-1}$ be a partition of n and let $\lambda_r := 0$. Then*

(a)

$$\sum_{S \subset \{0, 1, \dots, r\}} (-1)^{|S|} \lambda_{\min S} \max S = n$$

and, in particular, we have

$$\sum_{S \subset \{0, 1, \dots, r\}} (-1)^{|S|+1} \min S \max S = r(r+1)/2,$$

(b)

$$\sum_{S \subset \{0, 1, \dots, r\}} (-1)^{|S|} \binom{\lambda_{\min S} + \max S}{2} = n.$$

- 3.

$$\lim_{n \rightarrow \infty} \left(\frac{4}{n^2} \sum_{S \subset \{0, 1, \dots, n\}} (-1)^{|S|} \max(S) \left\lfloor \sqrt{n^2 - \min(S)^2} \right\rfloor \right) = \pi.$$

The identity in (1) follows since the ideal generated by $\{x^m : m \in M\}$ in $\mathbb{Z}[x]$ is also generated by $x^{\min M}$.

For the identities in (2), let $M := \{x^i y^{\lambda_i} : i = 0, \dots, r\} \subset \mathbb{Z}[x, y]$ and J be the ideal generated by M and compute the Hilbert series using (4.1). Pass first to the limit $x \rightarrow 1$ and then $y \rightarrow 1$ using de L'Hôpital's rule to obtain (2a), the special case is obtained considering the partition $\lambda = r, r-1, \dots, 1$. Identity (2b) follows by first substituting $y = x$ in the Hilbert series of J and then passing to the limit $x \rightarrow 1$, using de L'Hôpital's rule twice.

Similarly, the identity in (3) is obtained considering the sequence of ideals $J_n \subset \mathbb{Z}[x, y]$ generated by $\{x^i y^{\lfloor \sqrt{n^2 - i^2} \rfloor} : 0 \leq i \leq n\}$ and noting that the staircase diagram of J_n (see [MS, Section 3.1]) approximates a quarter of the circle of radius n .

An ordered pair (a, b) of vertices of a graph G is a *directed edge* if $\{a, b\}$ is an edge of G ; the vertex a is called the *source* and the vertex b is called the *target* of the directed edge. If ℓ is a directed edge of G , then we denote by $\underline{\ell}$ the corresponding (undirected) edge of G . We say that a set of directed edges M is a *cross-matching* of G if, for all $\ell, m \in M$ with $\ell \neq m$, we have $N[\underline{\ell}] \cap \underline{m} = \emptyset$ and the set of sources of the elements of M is a dominating set of G . The definition of cross-matching is a specialization of cross-cycles introduced by Jonsson to independence complexes (see [J, Proposition 3.1]).

It is straightforward to construct all the possible resolutions of a graph: it simply suffices to iteratively locate vertices adjacent to leaves and isolated vertices. On the other hand, it is a priori quite difficult to construct all the cross-matchings of a given graph or all the dominating sets of minimum cardinality. We shall see that, in the case of a spherical forest F , resolutions, cross-matchings, and dominating sets of minimum cardinality are essentially in bijection. To establish these bijections, as a starting point we prove that both cross-matchings and dominating sets of minimum cardinality of F need to contain a special element related to some leaf of F .

The following lemma is needed in the proof of Proposition 4.6 (and implies that the support of a cross-cycle of the independence complex of a forest contains at least as many leaves as the number of its connected components).

Lemma 4.5. *Let F be a forest and let M be a cross-matching of F . Then there is a directed edge of M whose target is a leaf of F .*

Proof. Let S_M and T_M denote the set of sources and targets of elements of M . Since there is a cross-matching of F , we deduce that F contains no

isolated vertices, and hence that all its connected components contain leaves. Let a_0 be a leaf of F .

If a_0 is contained in T_M , then we are done. Suppose a_0 is contained in S_M and let a_1 be the target of the directed edge with source a_0 . If a_1 is a leaf, then we are done; otherwise let a_2 be a vertex of F adjacent to a_1 and different from a_0 . The vertex a_2 is not contained in $S_M \cup T_M$, but there is an element a_3 of S_M adjacent to a_2 , since S_M is a dominating set. Let a_4 be the target of the directed edge of M with source a_3 . If a_4 is a leaf, then we are done. Otherwise we repeat this argument; eventually we find that there must be a leaf contained in T_M .

Finally, suppose that a_0 is not contained in $S_M \cup T_M$. Let a_1 be the vertex of S_M adjacent to a_0 and let a_2 be the target of the directed edge with source a_1 . If a_2 is a leaf, then we are done. Otherwise, there is a vertex a_3 adjacent to a_2 and different from a_1 . By the definition of cross-matching, the vertex a_3 cannot be contained in $S_M \cup T_M$. On the other hand, $a_3 \in N[S_M]$ and hence there is an element a_4 in S_M adjacent to a_3 and a directed edge (a_4, a_5) contained in M . As before, if a_5 is a leaf, then we are done. Otherwise we repeat this argument; eventually we find that there must be a leaf contained in T_M . \square

Proposition 4.6. *Let F be a forest. Then the graph F is spherical if and only if there is a cross-matching of F .*

Proof. Let (a_1, \dots, a_r) be a maximal resolution of F and let (b_1, \dots, b_r) be a sequence of vertices of F such that, for $i \in [r]$, in the graph $F \setminus N[\{a_1, \dots, a_{i-1}\}]$ the vertex b_i is a leaf adjacent to a_i . It follows from the definitions that $\{(a_1, b_1), \dots, (a_r, b_r)\}$ is a cross-matching of F .

Conversely, if M is a cross-matching of F , then by Lemma 4.5 there is a directed edge $(a, b) \in M$ where b is a leaf of F . Thus there is a resolution of F starting with the vertex a . Replacing F by $F \setminus N[a]$, we conclude by induction on the number of vertices of F . \square

Given a spherical resolution (a_1, \dots, a_r) of a graph G , and vertices (b_1, \dots, b_r) of G such that, for $i \in [r]$ in the graph $G \setminus N[\{a_1, \dots, a_r\}]$ the vertex b_i is a leaf adjacent to the vertex a_i , we obtain a set of directed edges $\{(a_1, b_1), \dots, (a_r, b_r)\}$. It follows from the proof Proposition 4.6 that the cross-matchings of a spherical forest F are in bijection with the sets of directed edges arising from the maximal resolutions of F constructed above. In particular there are cross-matchings M of F such that the set of sources

of directed edges in M contains all the vertices adjacent to leaves in F . Note that if M is a cross-matching of a spherical forest F , then M is a cross-matching of any forest F' obtained by removing vertices from the forest F not appearing in the directed edges of M . Hence, by Proposition 4.6, any such forest F' is also spherical.

Lemma 4.7. *Let F be a spherical forest. Then every dominating set D of minimum cardinality contains a vertex p adjacent to a leaf such that $N[p] \cap D = \{p\}$.*

Proof. We proceed by induction on the number of edges of F , so that we reduce to the case in which F is a tree. If the number of edges of F is at most three, then the result is clear. Suppose that the number of edges of F is at least four and let D be a dominating set of minimum cardinality of F .

Suppose that there is a path of length three in F with edges $\{a, b\}$, $\{b, c\}$ and $\{c, d\}$, where a is a leaf of F and the vertices b and c have valence two; define $F' := F \setminus N[b]$. Theorem 3.3 implies that the tree F' is spherical, note also that $\gamma(F') = \gamma(F) - 1 = |D| - 1$. The number of elements of $N[b]$ contained in D is either one or two. If there is exactly one element of D in $N[b]$, then this element cannot be c , so that $D' := D \setminus N[b]$ is a dominating set of minimum cardinality of F' . If there are two elements of D in $N[b]$, then $D' := (D \setminus N[b]) \cup \{d\}$ is a dominating set of F' of minimum cardinality. In either case, by the inductive hypothesis, there is a vertex p in D' adjacent to a leaf f of F' such that $N_{F'}[p] \cap D' = \{p\}$, where $N_{F'}$ denotes the neighborhood in the graph F' . If $d = p$, then, by minimality of D , the vertex c is not in D and hence d is the required vertex; if $d = f$, then, again by minimality of D , we have $D = D' \cup \{b\}$ and b is the required vertex; finally if $d \neq p$ and $d \neq f$, then p is the required vertex.

The case of paths follows from what we said. Thus we assume that F is not a path. Let G be the smallest subtree of F containing all vertices whose valence in F is at least three and let v be a vertex whose valence in G is at most one. Such a vertex exists since G is a tree and either it is a single vertex or it has at least one leaf. All the components of the forest $F \setminus \{v\}$, except for at most one, are paths with an endpoint adjacent to v (in F). By what we said above, we may assume that these paths consist of at most two vertices. For $i \in [2]$, we let s_i be the number of such paths with i vertices. Note that $s_1 + s_2 \geq 2$, since v has valence at least three; moreover s_1 and s_2 cannot be both non-zero, since otherwise F would not be spherical. Hence, we either have $s_1 \geq 2$ and $s_2 = 0$, or $s_1 = 0$ and $s_2 \geq 2$.

Suppose first that $s_1 \geq 2$, and note that, by minimality of D , the vertex v is in D and none of the leaves adjacent to v is in D . The graph F' obtained by removing from F one of the leaves adjacent to v is also spherical and D is also a dominating set of minimum cardinality for F' ; we therefore conclude by induction. Suppose now that $s_2 \geq 2$; if there is a leaf at distance two from v contained in D , then we let b be this leaf, otherwise, we let b be any leaf at distance two from v . We also let a be the vertex adjacent to b (so that a is also adjacent to v). Since D is a dominating set of minimum cardinality, exactly one among a and b is in D . Therefore, the graph F' obtained by removing from F the vertices a, b is also spherical and $D' := D \setminus \{a, b\}$ is also a dominating set of minimum cardinality for F' ; we therefore conclude by induction that there is a vertex p in D' adjacent to a leaf f in F' such that $N_{F'}[p] \cap D' = \emptyset$. Since v is not a leaf in F' , it follows that f is a leaf also in F ; thus p is the required vertex also for F . \square

As a consequence of Lemma 4.7, there is a bijection between the dominating sets of minimum cardinality of a spherical forest F and the sets underlying the maximal resolutions of F .

Corollary 4.8. *Let F be a spherical forest. If D is a dominating set of minimum cardinality of F , then there is a resolution (a_1, \dots, a_r) of F such that $D = \{a_1, \dots, a_r\}$.*

Proof. We proceed by induction on the cardinality of D . By Lemma 4.7, the set D contains a vertex a_1 adjacent to a leaf such that $N[a_1] \cap D = \{a_1\}$. The set $D \setminus N[a_1]$ contains one fewer element than D and hence, by Theorem 3.3 and Proposition 3.7, part (2), is a dominating set of minimum cardinality of the spherical forest $F \setminus N[a_1]$. The result follows by the inductive hypothesis. \square

Theorem 4.9. *Let F be a forest. Then the graph F is spherical if and only if the graph F is starred and contains no isolated vertices.*

Proof. If the graph F is starred, then it is spherical by Lemma 3.11.

Conversely, let D be a dominating set of F of minimum cardinality. We proceed by induction on the cardinality of D to show that D is independent. By Lemma 4.7, the set D contains a vertex p adjacent to a leaf such that $N[p] \cap D = \{p\}$. The set $D \setminus N[p]$ has cardinality one less than the cardinality of D and hence, by Theorem 3.3 and Proposition 3.7, part (2), is a dominating set of minimum cardinality of the spherical forest $F \setminus N[p]$. By

the inductive hypothesis, the set $D \setminus N[p]$ is independent and hence also D is also independent. \square

The following result summarizes the characterizations of conical and spherical forests that we have already proved, and establishes a few more equivalences. Recall that $cov_G(t) := \sum_{\substack{S \text{ edge} \\ \text{cover of } G}} t^{|S|}$ is the generating function for the edge covers of the graph G .

Theorem 4.10. *Let F be a forest. The following are equivalent:*

1. *the graph F is spherical;*
2. *there is a cross-matching of F ;*
3. *the graph F is starred and contains no isolated vertices.*
4. *the number $cov_F(-1)$ is either 1 or -1 ;*
5. *the number $cov_F(1)$ of edge covers of F is odd;*
6. *the number of independent sets of F is odd;*

Proof. The equivalence of (1), (2) and (3) follows from Proposition 4.6 and Theorem 4.9.

Let x_1, \dots, x_n be the vertices of F and E the set of the edges. The minimal non-faces of the independence complex $\text{Ind}(F)$ of F are the edges of F ; thus the set of minimal square-free generators of the Stanley-Reisner ideal of $\text{Ind}(F)$ is $B := \{x_i x_j \mid \{x_i, x_j\} \in E\}$. Hence it follows from the definitions that $c_B(x_1 \cdots x_n) = cov_F(-1)$. By Proposition 3.7, part (1), F is spherical if and only if $\text{Ind}(F)$ is homotopy equivalent to a sphere and also if and only if the reduced Euler characteristic $\tilde{e}(\text{Ind}(F))$ is 1 or -1 ; by Theorem 4.2, the equivalence of (1) and (4) follows. By Proposition 3.7, part (1), $\text{Ind}(F)$ can be either contractible or homotopic to a sphere; in the first case $\tilde{e}(\text{Ind}(F))$ is zero, in the second it is odd. Hence (4) \Leftrightarrow (5) follows. Finally, to prove (1) \Leftrightarrow (6) note that

$$\tilde{e}(\text{Ind}(F)) = - \sum_{S \in \text{Ind}(F)} (-1)^{|S|} \equiv \sum_{S \in \text{Ind}(F)} 1 = |\text{Ind}(F)|$$

where the symbol \equiv denotes equality modulo 2. \square

Some of the implications in the previous theorem hold more generally and not simply for forests.

Remark 4.11. *The problem of characterizing when the domination number equals the independent domination number has been studied by several authors (see for example [LW, HHS]); the trees for which this equality holds are analyzed in [My]. The assertion that every minimal dominating set of a tree F is independent if and only if it is spherical (Theorem 4.9) is related to these questions. Note that this characterization is stronger than the equality between the domination and the independent domination numbers.*

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