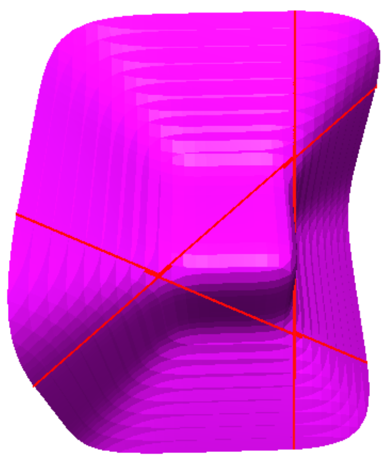
THIRD LECTURE: RATIONAL POINTS ON SURFACES OF GENERAL TYPE

DAMIANO TESTA

Abstract. Surfaces of general type are very mysterious from the point of view of their rational points, especially when there is no direct connection with abelian varieties; for instance, there is no simply connected surface of general type over a number field having non-empty and completely explicit set of rational points. I will give some examples of surfaces arising from classical number theoretic questions, and will then focus on moduli spaces of abelian surfaces (see Klaus Hulek and Greg Sankaran's notes [HS02] on Siegel modular threefolds for an introduction) and mention their relationship to the surface of cuboids (see Ronald van Luijk's undergraduate thesis [vL00] and a more recent joint work with Michael Stoll [ST10]).



In the classification of surfaces, the analogue of curves of genus at least two are the surfaces of general type. Unlike in the case of curves, there are very few general results on rational points on surfaces of general type, and even fewer fully explicit examples. In the case of curves over number fields there are three distinct behaviours.

- **Genus** 0**:** the set of rational could be empty; if it is not empty, then the curve is isomorphic to $\mathbb{P}^1$.
- **Genus** 1**:** the set of rational could be empty; if it is not empty, then the set of points forms a finitely generated abelian group.
- **Genus at least** 2**:** the set of rational is always finite.

A common property of curves of genus at least 2 over number fields is that the set of their rational points is never dense. The analogue of this property for higher dimensional algebraic varieties is called the weak Bombieri-Lang Conjecture.

Conjecture 1 (Weak Bombieri-Lang Conjecture). *If X is a smooth projective variety of general type defined over a number field, then the set of rational points of X is not dense.*

There is also a more precise version, describing geometrically the accumulation points of the set of rational points; for simplicity, we only state it in the case of surfaces.

Conjecture 2 (Strong Bombieri-Lang Conjecture). *If X is a smooth projective surface of general type defined over a number field, then there are only finitely many curves of geometric genus at most one contained in X and the set of rational points of X not contained in such curves is finite.*

The most general result known in this direction is due to Faltings' (it is also his second proof of the Mordell Conjecture).

Theorem 3 (Faltings). *If X is a smooth projective variety of general type defined over a number field contained in an abelian variety, then the set of rational points of X is finite.*

This theorem is the most substantial evidence in favour of the Bombieri-Lang Conjectures. Nevertheless, it applies to varieties containing no curves of genus at most one, nor abelian subvarieties of positive dimension. Moreover, the fundamental group of a subvariety of an abelian variety A surjects onto the fundamental group of its span: Faltings' Theorem does not apply to any simply connected variety. Starting from the case of surfaces, there are plenty of simply connected varieties of general type and among such varieties there is essentially no evidence for the Bombieri-Lang Conjectures.

For instance, there are no points known on the surface with equation

$$x^5 + y^5 + z^5 + w^5 = 0$$

that are not contained in a line on the surface. Such a surface exhibits the main features of what would be an "interesting evidence" in favour of the Bombieri-Lang Conjectures for simply connected varieties: a simply connected, smooth rational surface with non-empty set of rational points.

Another example is a question going back to Euler: the existence of a perfect cuboid. A *cuboid* is a parallelepiped with rectangular faces; it is *perfect* if the lengths of the three edges, the three face diagonals and the long diagonal are all integers. No such a cuboid is known to exist, but there are parametrized families of degenerate cuboids where one of the lengths vanishes. The set of perfect cuboids is naturally parameterized by a projective surface of general type. In joint work with M. Stoll [ST10], we computed the Picard group of the surface of cuboids.

The following exercise is not about surfaces, but it is about integral curves of general type. I did not try to solve this exercise, but I have been told that it is not easy: assume it is quite challenging! I would like to find a reference in English, and the closest I came is an expository paper in Dutch by G. Cornelissen [Cor06]: if you find a more accessible reference, let me know!

Exercise 4. Find the positive integers m and n such that the sum of the first m positive integers equals the sum of the first n positive squares.

References

[Cor06] Gunther Cornelissen, *Diophantische vergelijkingen: Kunnen we het echt?*, Nieuw Arch. Wiskd. **5** (2006), no. 7, 39-43, available at http://www.staff.science.uu.nl/~corne102/publications/kunnen_we_het.pdf. ↑3

[HS02] Klaus Hulek and Gregory K. Sankaran, *The geometry of Siegel modular varieties*, Higher dimensional birational geometry (Kyoto, 1997), 2002, pp. 89–156, available at http://arxiv.org/pdf/math/9810153v2.pdf. ↑1

[ST10] Michael Stoll and Damiano Testa, *The surface parametrizing cuboids*, arXiv:1009.0388 (2010), available at http://arxiv.org/pdf/1009.0388v1. ↑1, 2

[vL00] Ronald van Luijk, *On perfect cuboids*, Universiteit Utrecht, undergraduate thesis, 2000, available at http://www.math.leidenuniv.nl/~rvl/ps/cuboids.ps. ↑1