SECOND LECTURE: RATIONAL POINTS ON K3 SURFACES

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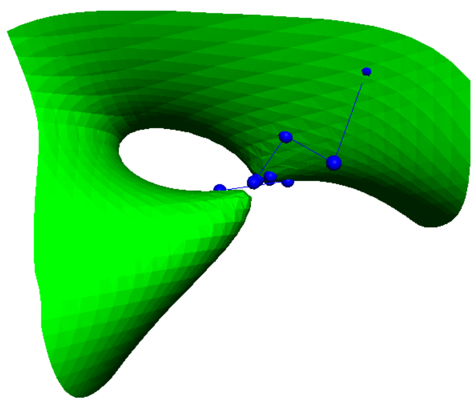
Abstract. Among the surfaces of vanishing Kodaira dimension, K3 surfaces play a very special role; many basic questions are known and many are still unknown. First, I will discuss the general method for computing Picard groups of surfaces with an emphasis on K3 surfaces appearing in the paper [vL07] of R. van Luijk. Then, I will talk about joint work with Michela Artebani and Antonio Laface on a specific K3 surface arising from a problem studied by Büchi.



1. K3 surfaces

A *K3 surface* is a surface X defined over the field k having trivial canonical line bundle and vanishing $\mathrm{H}^1(X, \mathscr{O}_X)$.

It is quite easy to write down explicit examples of K3 surfaces:

- double covers of $\mathbb{P}^2$ branched over a plane sextic;
- quartic surfaces in $\mathbb{P}^3$;
- intersections of a quadric and a cubic in $\mathbb{P}^4$;
- intersections of three quadrics in $\mathbb{P}^5$;

are all K3 surfaces. Moreover, the only K3 surfaces that are complete intersections in projective space are contained in the list above.

Exercise 1. Let X be a smooth projective surface and denote by K_X a canonical divisor on X. If D is a divisor on X such that $K_X \cdot D = 0$, show that D^2 is even.
[Hint: use the adjunction formula.]

It follows from Exercise 1 that the square of every ample line bundle on a K3 surface is even; for the class of a hyperplane in the examples mentioned above, the squares are

2, 4, 6 and 8. An ample divisor of degree 2 presents a K3 surface as a double cover of $\mathbb{P}^2$ branched over a plane sextic. A *polarization* of a K3 surface X is a choice of an ample line bundle A on X and the degree of the polarization is the square of the ample line bundle A. Every positive even integer is the degree of an ample divisor on some K3 surface. There are infinitely many irreducible components in the moduli space of K3 surfaces, each corresponding to a possible degree of a polarization: the families mentioned above correspond to the components of K3 surfaces with a polarization of degree 2, 4, 6 or 8.

1.1. **Hodge diamond.** Let X be a smooth projective variety over a field k and let Ω_X denote the sheaf of Kähler differentials on X (this is the sheaf of holomorphic 1-forms if X is defined over the complex numbers); for a non-negative integer q define the sheaf $\Omega_X^q = \bigwedge^q \Omega_X$ to be the sheaf of holomorphic q-forms on X. The Hodge diamond of X consists of the dimensions $h^{p,q}(X)$ of the cohomology groups $\mathrm{H}^q(X, \Omega_X^p)$; to simplify the notation, if $\mathscr{F}$ is a coherent sheaf on X and i is an integer, we denote the dimension of the cohomology group $\mathrm{H}^i(X, \mathscr{F})$ by $h^i(X, \mathscr{F})$ or even by $h^i(\mathscr{F})$ when the variety X is clear from the context.

If X is a surface, the sheaf Ω_X^2 is isomorphic to the dualizing sheaf ω_X and the Hodge diamond of X takes the form

$$\begin{array}{ccccc} & & h^2(\omega_X) & & \\ & h^1(\omega_X) & & h^2(\Omega_X) & \\ h^0(\omega_X) & & h^1(\Omega_X) & & h^2(\mathscr{O}_X) \\ & h^0(\Omega_X) & & h^1(\mathscr{O}_X) & \\ & & h^0(\mathscr{O}_X) & & \end{array}$$

The Hodge diamond is symmetric with respect to a rotation of 180° by Serre duality; over a field of characteristic zero, the reflection across the vertical axis is also a symmetry of the Hodge diamond by Hodge theory. Since the surface X is connected, the identity $h^0(\mathscr{O}_X) = 1$ holds; classically the number $p_g = h^0(\omega_X)$ is called the *geometric genus* of X and the number $q = h^1(\mathscr{O}_X)$ is called the *irregularity* of X. Thus, over fields of characteristic zero, we can rewrite the Hodge diamond as

$$\begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1}(X) & & p_g \\ & q & & q & \\ & & 1 & & \end{array}$$

If X is a K3 surface, the irregularity q vanishes and the geometric genus p_g equals 1, since the canonical line bundle is trivial. To conclude the computation of the Hodge diamond, we still need to compute $h^{1,1}(X)$: from Noether's formula we can deduce that the Euler characteristic of a K3 surface is 24 and hence the Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

In order to give an upper bound for the Picard number of a K3 surface, I will use a few features of étale cohomology: I will use more étale cohomology later, and I will mention the relevant properties as I need them. Étale cohomology groups are replacements of singular cohomology groups for varieties defined over fields different from the complex numbers, including fields of positive characteristic; in fact, they were initially constructed with the goal of proving the Weil conjectures for varieties over finite fields. If the field k is the field of complex numbers, then there are comparison theorems between étale and singular cohomology showing that the correspoding étale and singular cohomology groups are vector spaces of the same dimensions; moreover, étale cohomology groups are functorial in the same way that singular cohomology is functorial: systematically replacing étale cohomology groups with the corresponding singular cohomology groups provides a good geometric intuition for étale cohomology groups. On the other hand, étale cohomology groups for varieties in positive characteristic have something more, that we will see and exploit later: they admit an action of the Frobenius endomorphism. If X is a variety defined over a finite field $\mathbb{F}$, the Frobenius endomorphism is the key to relate the number of points of X over the finite extensions of $\mathbb{F}$ with the (étale) Betti numbers of X.

Fix a prime number ℓ different from the characteristic of the ground field k: this is part of the construction of the étale cohomology groups, and the final output is essentially independent of the choice of the prime ℓ. The Picard group of a K3 surface over a field k is a sublattice of the étale cohomology group $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_\ell)$; since the latter has dimension 22, it follows that the Picard number of a K3 surface is at most 22. Over the complex numbers (and hence over fields of characteristic zero), the Hodge decomposition implies that the Picard group of X is a sublattice of the vector space $h^{1,1}(X)$ and hence we obtain a smaller bound of 20 for the Picard number of a K3 surface: this bound is in general incorrect for K3 surfaces over fields of positive characteristic, and a K3 surface with Picard number 22 is called *supersingular*.

1.1.1. *Kummer surfaces.* There is a classical construction of K3 surfaces starting with a principally polarized abelian variety of dimension two. Since I do not want to get into the details of what is a principal polarization of an abelian variety, I will simply start with a smooth projective curve C of genus two and let JC denote the Jacobian variety of C. The variety JC is an example of a principally polarized abelian surface (and, except for product of two elliptic curves, every principally polarized abelian surface is of this form). The surface JC admits an action of the group $\{\pm 1\}$ induced by taking the inverse ι in the group law of JC. The quotient $JC/\langle\iota\rangle$ is a projective surface and it has 16 double points corresponding to the 16 two-torsion points of JC. Each singular point is a singular point of type A_1 and a single blow up resolves them; the surface $\mathrm{Kum}(C)$ obtained by blowing up the 16 singular points of $JC/\langle\iota\rangle$ is the *Kummer surface* of the curve C.

Let C be a curve of genus two and let X be the corresponding Kummer surface; the Picard group of the surface X contains a sublattice spanned by the classes of the 16 pairwise disjoint exceptional curves lying above the singular points of the quotient $JC/\langle\iota\rangle$. In fact, it is a classical result that a K3 surface is a Kummer surface if and only if it contains 16 pairwise disjoint smooth rational curves. Since the 16 classes contacted in $JC/\langle\iota\rangle$ are independent in the Picard group, the rank of Picard group of X is at least 16; moreover, these classes are not the support of an ample divisor, since they are contracted in the morphism $X \to JC/\langle\iota\rangle$ and it follows that the rank of $\mathrm{Pic}(X)$ must be at least 17.

Exercise 2. Let C be a smooth projective curve of genus two. Denote by JC the Jacobian of C and by $\mathscr{O}$ the origin of JC. Let $\sigma\colon C \times C \to C \times C$ denote the exchange of the two factors and let $\mathrm{Sym}^2 C = C \times C/\langle\sigma\rangle$ be the quotient of $C \times C$ by the group generated by σ. Show that there is an isomorphism $\mathrm{Sym}^2 C \to \mathrm{Bl}_{\mathscr{O}}(JC)$.

[Hint: Find a divisor of degree 0 associated to a pair of points on C.]

1.2. **Picard groups of K3 surfaces and étale cohomology.** In this section, we give a method that was used by Ronald van Luijk in [vL07] to construct explicit examples of K3 surfaces defined over $\mathbb{Q}$ and having Picard number one over an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. Whenever we talk about the rank of the Picard group of a variety Y defined over a field k, we always mean the rank of the Picard group of the base-change of Y to an algebraic closure of the field k.

The starting point is the observation that in a smooth family of algebraic varieties over a connected base, the generic Picard group injects in the Picard group of any smooth element of the family. Let X be a K3 surface defined over the rational numbers $\mathbb{Q}$. Clearing denominators, we obtain a scheme $\mathscr{X}$ over the integers, that we interpret as a family of K3 surfaces over the one-dimensional base $\mathrm{Spec}(\mathbb{Z})$: the generic fiber of the morphism $\mathscr{X} \to \mathrm{Spec}(\mathbb{Z})$ is the surface X over the rational numbers, and the fibers of the family over the closed points of $\mathrm{Spec}(\mathbb{Z})$ are the reductions of the equations of $\mathscr{X}$ modulo the prime numbers. We now apply the observation about the behaviour of Picard numbers under specialization to the family $\mathscr{X} \to \mathrm{Spec}(\mathbb{Z})$: the Picard number of the K3 surface X is at most equal to the Picard number of the reductions of $\mathscr{X}$ modulo the primes of good reduction of $\mathscr{X}$. Ideally, we would like to find a prime p such that the reduction $\mathscr{X}_p$ of $\mathscr{X}$ modulo p is smooth and the Picard number of $\mathscr{X}_p$ is one. In fact, a consequence of the Tate conjectures (that is known for K3 surfaces) is that the Picard number of a K3 surface defined over the algebraic closure of a finite field is always even: the strategy mentioned above cannot work without an extra input.

Suppose that there are two primes p, q of good reduction for $\mathscr{X}$ such that the Picard numbers of the reductions of $\mathscr{X}$ modulo p and modulo q are both equal to 2; it follows that the Picard number of X is at most 2. If the Picard number of X were 2, then the Picard group would have finite index in the Picard group of both surfaces $\mathscr{X}_p$ and $\mathscr{X}_q$, and in particular the ratios $\frac{\operatorname{disc}(\operatorname{Pic}(X))}{\operatorname{disc}(\operatorname{Pic}(\mathscr{X}_p))}$ and $\frac{\operatorname{disc}(\operatorname{Pic}(X))}{\operatorname{disc}(\operatorname{Pic}(\mathscr{X}_q))}$ of the discriminants of the various Picard groups would all be squares. We deduce that also the ratio $\frac{\operatorname{disc}(\operatorname{Pic}(\mathscr{X}_p))}{\operatorname{disc}(\operatorname{Pic}(\mathscr{X}_q))}$ is a square.

We can now state the overall strategy. First, find two smooth quartic surfaces X_2, X_3, with X_2 defined over the field $\mathbb{F}_2$ and X_3 defined over the field $\mathbb{F}_3$, both with Picard number 2. Show that the discriminants of the Picard groups of the two surfaces X_2 and X_3 do not differ by a square. Use the Chinese remainder theorem to find a quartic polynomial with integer coefficients reducing modulo 2 to an equation of X_2 and modulo 3 to an equation of X_3. By the behaviour of Picard numbers in families, the Picard number of X is at most equal to 2. Since the discriminants of the Picard groups of X_2 and of X_3 do not differ by a square, the Picard group of X cannot have finite index in both $\operatorname{Pic}(X_2)$ and $\operatorname{Pic}(X_3)$: the Picard number of X is at most one and we are done.

In order for this strategy to work we need to be able to prove that the Picard number of the K3 surfaces X_2 and X_3, defined over the finite fields $\mathbb{F}_2$ and $\mathbb{F}_3$ respectively, is equal to 2. It is in general easy to give lower bounds on Picard numbers of surfaces: the rank of the intersection matrix of any set of curves on a surface Y is a lower bound for the Picard number of Y. In particular, we may start with surfaces X_2 and X_3 each containing two curves whose intersection matrices have rank 2 and discriminants not differing by a square. But we still need to ensure that the Picard numbers of the surfaces are at most equal to 2. This is where étale cohomology and the action of the Frobenius endomorphism come into the argument.

Fix a prime ℓ different from the characteristic of the ground field k. For any integer n, the étale cohomology functor $\mathrm{H}^n_{\text{ét}}(-, \mathbb{Q}_\ell)$ is a contravariant functor from schemes over k to vector spaces over $\mathbb{Q}_\ell$, that are entirely analogous to singular cohomology with coefficients in a field for varieties defined over the complex numbers. In fact, if X is a smooth projective variety over $\mathbb{C}$, then there are comparison theorems showing that the dimensions of the vector spaces $\mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_\ell)$ and $\mathrm{H}^n(X(\mathbb{C}), \mathbb{C})$ are equal.

Let X be a scheme defined over a finite field $\mathbb{F}$ with q elements. Because of the functoriality of étale cohomology, for every integer n there is an action of the Frobenius automorphism of $\mathbb{F}$ on the $\mathbb{Q}_\ell$-vector spaces $\mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_\ell)$; denote by Fr the endomorphism of $\mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_\ell)$ induced by the Frobenius automorphism. The vector subspace A^n of $\mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_\ell)$ spanned by the classes of algebraic subvarieties of dimension $\frac{n}{2}$ is invariant under the endomorphism Fr; even more is true: the eigenvalues of the action of the Frobenius endomorphism on A all have absolute value $q^{\frac{n}{2}}$ and the characteristic polynomial of Fr has integral coefficients. This is the key part of the argument: the dimension of the subspace A^2 is an upper bound for the Picard number of X, and the dimension of the subspace of the vector space $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_\ell)$ where the Frobenius endomorphism acts with eigenvalues of absolute value q is in turn an upper bound for the dimension of A^2. Observe that, in concrete cases, by systematically trying divisors on X and checking the rank of the intersection matrix with curves on X, we can easily obtain lower bounds for the Picard number: what is missing is a criterion that allows us to conclude that the divisors we constructed form indeed a basis. This bound is the one described above in terms of the subspace A^2. In general, the vector space A^n spanned by the algebraic classes of codimension $\frac{n}{2}$ may not coincide with the vector subspace of $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_\ell)$ where the Frobenius endomorphism acts with eigenvalues of absolute value $q^{\frac{n}{2}}$: this is part of the *Tate conjectures*. Nevertheless, if we can compute the dimension of the subspace where the endomorphism Fr acts with eigenvalues of absolute value $q^{\frac{n}{2}}$, even without knowing that the Tate bound is achieved, whenever it is true in a concrete case, it can be verified. Thus to conclude our program, it suffices to give a procedure to compute the dimension of the eigenspaces of the Frobenius endomorphism: indeed, we will compute the characteristic polynomial of Fr.

On the one hand, the fixed points of the action of the Frobenius automorphism of the field on the variety X are precisely the points of X defined over the field $\mathbb{F}$. On the other hand, the Lefschetz Trace Formula admits an analogue for étale cohomology with an action of the Frobenius endomorphism and we obtain the identity

$$\#X(\mathbb{F}) = \sum_n (-1)^n \operatorname{Tr}\big(\mathrm{Fr}\colon \mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_\ell) \to \mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_\ell)\big).$$

It follows easily that by computing the number of points of X over successive extensions of the ground field $\mathbb{F}$, we obtain more and more relations on the traces of the Frobenius action on the étale cohomology vector spaces: with an upper bound on the dimension of the étale cohomology vector spaces, we eventually compute all the traces of the Frobenius elements. This is clearly enough information to deduce the characteristic polynomial χ of Fr acting on $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_\ell)$. To conclude, we extract the degree of the largest factor of χ whose roots all have absolute value equal to q and this number is the required bound: the surfaces X_2 and X_3 mentioned above have Picard number equal to 2 and the argument is complete.

In the paper [ALT12] we use a different method to compute the Picard group of a specific K3 surface arising from a problem of Büchi. The computation of the Picard group guides the study that we make of the rational points on the Büchi K3 surface allowing us to prove Zariski density of the set of rational points. The paper [HT08] of B. Hassett and Yu. Tschinkel contains an alternative geometric approach in the case of K3 surfaces over function fields.

References

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