FIRST LECTURE: RATIONAL POINTS ON RATIONAL SURFACES

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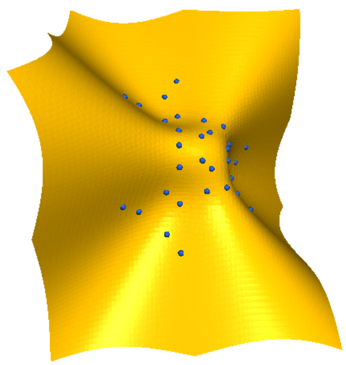
Abstract. By a result of Iskovskikh, a rational surface over a field is birational to either a del Pezzo surface or a conic bundle over a conic. Heuristically, over number fields, if a rational surface has points, they tend to be dense and often can be completely parameterized. I will talk about the classical Segre-Manin Theorem [Man86] on unirationality of del Pezzo surfaces of degree at least two and report on recent results obtained in collaboration with Cecília Salgado and Tony Várilly-Alvarado. I will also talk about Cox rings of rational surfaces and mention joint work with Antonio Laface [LT11], Tony Várilly-Alvarado and Mauricio Velasco [TVAV11] as well as work of Michela Artebani and Antonio Laface [AL11].



Notation. Throughout this course, we denote by k a field that is not necessarily assumed to be algebraically closed. Usual choices for the field k are finite fields and number fields. By *surface* I normally mean a smooth projective geometrically integral scheme of dimension two defined over the field k.

Let X be a surface defined over a field k. The overall goal of this mini-course is to give an introduction to some of the methods that are used to study the set of rational points on the surface X. Here are a few motivating questions.

- Is the set of rational points of X is empty? Is it finite?
- Is it Zariski-dense?
- Is it Zariski-dense possibly after a finite extension of the ground field?
- Can we give an explicit description of all the rational points of X?

- If the set of rational points is dense, is there a natural notion of height that allows us to study the asymptotic growth of the set of rational points of height bounded by B as B tends to infinity?

Ideally, we would like the answers to these questions to depend on *geometric* properties of the variety X. In this context, a property is geometric if it can be tested after a field extension (such as smoothness or having negative Kodaira dimension) and it is arithmetic if it may depend on the field of definition (such as having a rational point or containing a smooth curve to $\mathbb{P}^1$ defined over k).

I will not go into the details of the classification of surfaces over algebraically closed fields. Instead, I will select a few representative examples to illustrate the overall features of the classification that lead to questions of an arithmetic nature. The three lectures will cover three very different behaviours (at least conjecturally and heuristically) for the sets of rational points on surfaces: rational surfaces, K3 surfaces and surfaces of general type. We excluded ruled, elliptic and bielliptic surfaces, since these cases appear to reduce to the case of curves, at least superficially: in the first instance answering the motivating questions above for the natural curves arising in this context could yield an answer for the surface itself. We also excluded Enriques surfaces, since they are intimately related to K3 surfaces and understanding the set of rational points on K3 surfaces seems a natural first step in the study of Enriques surfaces. Finally, we excluded abelian surfaces since in this case the group structure can be effectively exploited and very few methods that work well for elliptic curves do not easily generalize to abelian surfaces and even abelian varieties.

1. Rational surfaces

A *rational surface* is a surface defined over the field k that is birational to the projective plane $\mathbb{P}^2$ over an algebraic closure of the field k.

Trivially, the projective plane $\mathbb{P}^2$ is itself a rational surface; in this case, the surface $\mathbb{P}^2$ is birational (and in fact isomorphic) to $\mathbb{P}^2$ over the ground field, with no need for any extension of the ground field. We shall see in Subsection 1.1 that already quadric surfaces exhibit a much wider range of possibilities.

Lemma 1 (Lang-Nishimura)**.** *Let X, Y be two varieties defined over a field k and suppose that X is smooth, Y is proper, and that $\varphi\colon X \dashrightarrow Y$ is a rational map defined over the field k. If the variety X has a rational point, then the variety Y also has a rational point.*

Proof. The result is clear if the dimension of X is zero: in this case, the rational map φ is a morphism and the image of a rational point on X is a rational point on Y. Suppose that the dimension of X is at least one. Let $p \in X$ be a rational point, and let $U \subset X$ be a dense open subset such that the map φ is defined over U. Choose any affine open subset A containing p, intersect $\dim X - 1 \geq 0$ sufficiently general hypersurfaces in A to obtain a *smooth* curve C in A containing p and not contained in the complement of U. Restricting the rational map φ to the curve C we obtain a rational map from the smooth curve C to the proper variety Y: such a rational map extends to a morphism $\varphi_C \colon C \to Y$. The image of the point p under φ_C is a rational point on Y, as required. □

Corollary 2. *Let X, Y be two smooth proper varieties defined over a field k and suppose that X and Y are birational over the field k; then the sets $X(k)$ and $Y(k)$ of k-valued points of X and Y are either both empty or both non-empty.*

Proof. The result is a direct consequence of the Lang-Nishimura Lemma (Lemma 1). □

An alternative formulation of Corollary 2 is that the property of having a rational point is a birational property among smooth projective varieties.

Let X, Y be smooth projective varieties defined over a field k that are also birational over the field k. Thus, the varieties X and Y admit two dense open subsets $U \subset X$ and $V \subset Y$ such that U and V are isomorphic over k; it follows that the sets of rational points of X and Y "differ" at most by the rational points contained in the proper closed subsets $X \setminus U$ and $Y \setminus V$. In particular, we reduced the problem of determining the set of rational points on X to the problem of determining the set of rational points on Y and on the subsets $X \setminus U$ and $Y \setminus V$ of strictly smaller dimension than $\dim X = \dim Y$. This justifies the "birational" approach to the study of the set of rational points on surfaces: we may replace a surface by a different birational model, provided we are willing to treat as easier the question of determining the set of rational points on curves.

Definition 3 (k-Minimal models)**.** A surface X is a k-minimal model (or a minimal model over the field k) if any birational k-morphism $X \to Y$ is an isomorphism.

Typically, when we study the set of rational points of a surface, we assume that the surface is a k-minimal model.

1.1. **Quadric surfaces.** Quadric surfaces are smooth surfaces in $\mathbb{P}^3$ defined by a homogeneous equation of degree two. First of all, quadric surfaces are rational surfaces since, over an algebraically closed field, they are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and are therefore birational to $\mathbb{P}^2$.

Over a general field, a quadric surface need not contain any point: for instance the quadric $Q \subset \mathbb{P}^3_{\mathbb{R}}$ with defining equation $x^2 + y^2 + z^2 + w^2 = 0$ admits no points, and it follows from the Lang-Nishimura Lemma that Q is not birational over $\mathbb{R}$ to $\mathbb{P}^2_{\mathbb{R}}$ since $Q(\mathbb{R})$ is empty, while $\mathbb{P}^2_{\mathbb{R}}(\mathbb{R})$ is not. On the other hand, as soon as a quadric contains a rational point, then it is birational over k to $\mathbb{P}^2_k$. Indeed, let Q be a quadric defined over a field k having a rational point p; project Q away from the point p to obtain the required birational map $Q \dashrightarrow \mathbb{P}^2_k$. The rational map constructed in this way establishes an isomorphism of the complement of the tangent plane to Q at p with $\mathbb{A}^2_k$; the intersection of Q with the tangent plane at p is a nodal plane conic and it is either irreducible over the field k, in which case it only has p as a rational point, or it is reducible over k, in which case it is the union of two copies of $\mathbb{P}^1_k$ joined at a rational point. In all cases, we found a complete description of the set of all rational points on the quadric Q.

Even if the quadric Q contains a point, it is not necessarily isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$. For instance, let Q be the quadric over $\mathbb{Q}$ with equation $x^2 + y^2 + z^2 - w^2 = 0$ admitting the point $p = [0, 0, 1, 1]$ as a rational point. If Q were isomorphic to $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$, then the tangent plane to Q at p would consist of the union of two smooth rational curves, corresponding to the two rulings of the quadric through the point p. The tangent plane T_p to the quadric Q at the point p is the plane with equation $z = w$ and the intersection of Q with the plane T_p is the curve Q_p with equations

$$Q_p = Q \cap T_p \colon \quad \left\{ \begin{array}{rcl} z & = & w, \\ x^2 + y^2 & = & 0. \end{array} \right.$$

Clearly, the curve Q_p is irreducible over $\mathbb{Q}$ and hence the quadric Q cannot be isomorphic to $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$.

Exercise 4. Let k be a field of characteristic different from two and let Q be a smooth quadric in $\mathbb{P}^3$ admitting a rational point. Let q denote the symmetric bilinear form corresponding to the quadratic form defining Q. Show that the quadric Q is isomorphic to the product $\mathbb{P}^1_k \times \mathbb{P}^1_k$ if and only if the discriminant of the form q is a square in k.

So far, we assumed the existence of a rational point, without worrying about how to check if such a point existed or not. We now address the issue of existence of a rational point on a quadric surface over finite fields and over number fields.

Over a finite field, a quadric surface always has a point: this is a consequence of the Chevalley-Warning Theorem.

Theorem 5 (Chevalley-Warning)**.** *Let k be a finite field and let n be a positive integer. If $X \subset \mathbb{P}^n_k$ is a hypersurface defined over k by an equation of degree $d \leq n$, then X has a rational point.* □

The proof of this result is elementary, but we will not reproduce it here.

Over number fields, the question of the existence of a rational point is more subtle than over finite fields, but can be algorithmically decided. The main ingredient is the Hasse-Minkowski Theorem.

Theorem 6 (Hasse-Minkowski)**.** *A smooth quadric over a number field k has a rational point if and only if it has a rational point over every completion of the field k.* □

Thus, on the one hand, if a quadric Q over a number field k has no rational points, then there is a completion of the field k over which the quadric still has no rational points and this is easy to test; on the other hand, if the quadric Q does have a rational point, then by enumerating the points of projective space and successively checking whether they lie on Q we are guaranteed to eventually find a point on Q. Running these two tests in parallel we are able to decide whether Q has a rational point or not. In fact, combining the Chevalley-Warning Theorem with Hensel's Lemma it is possible to determine a finite set $\mathscr{S}$ of completions of the field k, depending on the quadric Q, with the property that Q has a rational point if and only if it has a point over all the completions in $\mathscr{S}$.

Exercise 7. Show that the equation $x^2 + y^2 + z^2 + w^2 = 0$ has no solutions in $\mathbb{Q}$. Can you find all the completions of $\mathbb{Q}$ at which the quadric with the same equation has not rational points?

Obviously, if a variety defined over a number field has a rational point, then it has a rational point over every completion of the number field itself. The Hasse-Minkowski Theorem is the assertion that the converse to this statement holds for smooth quadrics. In general, a class of varieties $\mathscr{X}$ defined over a number field k *satisfies the Hasse principle* if the non-existence of a rational point on a variety X in $\mathscr{X}$ can be proved by exhibiting a completion of k over which the variety X has no points.

1.2. **Cubic surfaces.** Cubic surfaces are smooth surfaces in $\mathbb{P}^3$ defined by a homogeneous equation of degree three. As in the case of quadrics, cubic surfaces are rational surfaces since, over an algebraically closed field, they are isomorphic to the blow up of $\mathbb{P}^2$ in six points, and are therefore birational to $\mathbb{P}^2$ (this is a substantially harder result than the corresponding one for quadric surfaces).

Let $X \subset \mathbb{P}^3_k$ be a smooth cubic surface defined over a field k. The question of whether X is birational to $\mathbb{P}^2_k$ is quite subtle; as before, a necessary condition is that the surface X contains a rational point. Suppose that X contains a rational point p such that the intersection of the surface X with the tangent plane to X at p is a geometrically integral plane cubic X_p. The curve X_p is then a geometrically integral plane cubic curve with a singular point at p: it has genus zero and can be parameterized by $\mathbb{P}^1_k$ by projection away from the point p. We therefore obtained a non-constant morphism $\mathbb{P}^1_k \to X$ of points contained in the cubic surface X (more precisely, contained in the curve X_p). Repeating this construction starting from the general point in the image of $\mathbb{P}^1_k$ we obtain a dominant rational map $\mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow X$ of degree six. The surface X is not necessarily birational to the projective plane $\mathbb{P}^2_k$, but it is *unirational*: it admits a dominant rational map from projective space. In general a cubic surface with a point need not be birational to the projective plane; nevertheless elaborations on the construction described above prove the following result.

Theorem 8 (Segre, Manin, Kollár)**.** *Let k be a field and let X be a smooth cubic surface defined over k. If the surface X contains a rational point, then the surface X is unirational.*

In fact, Segre proved Theorem 8 over the rational numbers, Manin extended the argument to perfect fields with at least 35 elements, and finally Kollár proved it in general. Observe that we kept the hypothesis of the existence of a point (which is clearly necessary for proving that a surface is birational to the projective plane, or even simply unirational) and we used in the construction above a generality condition on the point: the tangent plane to the cubic at the point needed to be geometrically integral for our construction to work. Similar restrictions will be present also later.

We conclude our overview of cubic surfaces with a discussion of the existence of a rational point. Over finite fields, the Chevalley-Warning Theorem applies in this case and every smooth cubic surface over a finite field is therefore automatically unirational. Over number fields, the analogue of the Hasse-Minkowski Theorem for quadric surfaces fails: Cassels-Guy proved that the cubic surface X with equation

$$5x^3 + 12y^3 + 9z^3 + 10t^3 = 0$$

has a rational point over every completion of $\mathbb{Q}$, but has no point defined over $\mathbb{Q}$, and is therefore a violation of the Hasse principle. There is a more refined obstruction to the existence of rational points, the *Brauer-Manin obstruction*, that explains the failure of the Hasse principle in certain cases. The Brauer-Manin obstruction is expected to be the only obstruction to the validity of the Hasse principle for rational surfaces.

1.3. **Del Pezzo surfaces.** A del Pezzo surface is a smooth projective surface whose anticanonical divisor is ample. Over an algebraically closed field a del Pezzo surface is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or to a blow up of $\mathbb{P}^2$ at $r \in \{0, 1, \ldots, 8\}$ points in general position. The degree of a del Pezzo surface is the square of the canonical divisor class: the degree of $\mathbb{P}^1 \times \mathbb{P}^1$ is 8 and the degree of the blow up of $\mathbb{P}^2$ at r points is $9 - r$. As a heuristic rule, the smaller the degree of a del Pezzo surface, the more complicated the surface becomes.

- Some del Pezzo surfaces of degree 8 are quadric surfaces;
- del Pezzo surfaces of degree at least 6 over an algebraically closed field are toric varieties;
- del Pezzo surfaces of degree 5 are always birational to the projective plane (Enriques, Swinnerton-Dyer);
- del Pezzo surfaces of degree 4 are intersections of two quadrics in $\mathbb{P}^4$;
- all del Pezzo surfaces of degree 3 are smooth cubic surfaces;
- del Pezzo surfaces of degree 2 are double covers of $\mathbb{P}^2$ branched over a plane quartic, which is smooth if the characteristic of the field is not 2; equivalently, they are hypersurfaces of degree 4 in weighted projective space $\mathbb{P}(1, 1, 1, 2)$;
- del Pezzo surfaces of degree 1 are hypersurfaces of degree 6 in $\mathbb{P}(1, 1, 2, 3)$.

The exceptional curves on del Pezzo surface are one of the key features of these surfaces and are the main tool in many of the proofs of results on del Pezzo surfaces. For instance, the exceptional curves on a cubic surface are exactly the 27 lines it contains; the configuration of these lines is independent of the surface and has the Weyl group of E_6 as a symmetry group. The exceptional curves on a del Pezzo surface of degree 1 are 240 and naturally correspond to the roots of the Lie algebra E_8; the symmetry group of the resulting configuration of exceptional curves is the Weyl group of E_8.

Most of the results mentioned for cubic surfaces can be generalized to del Pezzo surfaces of degree at least 2.

Theorem 9 (Segre-Manin)**.** *Let X be a del Pezzo surface of degree 2 over a field k. If the surface X contains a sufficiently general point, then the surface X is unirational.*

Sketch of proof. We argue in the case in which the point p is not contained on any exceptional curve, nor on the ramification curve. Let X' be the blow up of the surface X at the point p; the conditions on the point p are equivalent to the statement that the surface X' is a del Pezzo surface of degree 1. It is a general fact that a del Pezzo surface of degree 1 has an involution, the *Bertini* involution, acting without fixed points on the set of exceptional curves. Denote by $\psi \colon X' \to X'$ the Bertini involution of the surface X'; the exceptional curve E on X' lying over the point p of X is transformed to a different exceptional curve $E' = \psi(E)$. The curve E' is therefore a smooth rational curve isomorphic to $\mathbb{P}^1_k$ and applying the blow up map $X' \to X$ we find a rational curve F on X singular at p. Iterating this construction with the general point of F we obtain a dominant rational map $\mathbb{P}^2_k \to X$, as required. □

In joint work with Cecília Salgado and Tony Várilly-Alvarado we extended the previous result, making the generality assumption on the point explicit. Let X be a del Pezzo surface of degree 2; the morphism associated to the anticanonical linear system on X presents the surface X as a double cover of $\mathbb{P}^2$ branched over a plane quartic. We call the ramification divisor of the anticanonical morphism the *ramification curve* of X.

Theorem 10 (Salgado, Várilly-Alvarado, T)**.** *Let X be a del Pezzo surface of degree 2 over a field k. If the surface X contains a point not lying on the ramification curve, nor on four exceptional curves, then the surface X is unirational.* □

We further specialized our analysis to del Pezzo surfaces of degree 2 over finite fields. An easy extension of the Chevalley-Warning argument shows that a del Pezzo surface over a finite field always has a rational point. In view of the Segre-Manin Theorem, we wondered whether the points that are guaranteed to exist on del Pezzo surfaces of degree 2 over finite fields were enough to show unirationality of such surfaces.

Theorem 11 (Salgado, Várilly-Alvarado, T)**.** *Let X be a del Pezzo surface of degree 2 over a finite field $\mathbb{F}$. The surface X is unirational except possibly if X is isomorphic to one of the following three surfaces:*

$$\begin{array}{lrcl} X/\mathbb{F}_9\colon & \nu w^2 & = & x^4 + y^2 + z^4, \\ X/\mathbb{F}_3\colon & -w^2 & = & x^4 + y^3 z - y z^3, \\ X/\mathbb{F}_3\colon & -w^2 & = & (x^2+y^2)^2 + y^3 z - y z^3, \end{array}$$

where $\nu \in \mathbb{F}_9$ is a fixed non-square. □

Note that to prove unirationality of any of the possible exceptions mentioned in Theorem 11 it is sufficient to find a single rational curve contained on the surface.

1.4. **Cox rings.** This section is intended to be a quick overview of Cox rings; for an in-depth study of Cox rings we refer to reader to the book [ADHL12] and to the references given below.

The projective space $\mathbb{P}^n$ is obtained from the affine space $\mathbb{A}^{n+1}$ by removing the origin and forming the quotient by the rescaling action of $\mathbb{G}_m$. This standard construction of projective space leads to the description of the points of $\mathbb{P}^n$ as $(n+1)$-tuples of elements of the field k subject to a familiar equivalence relation. In particular, every point in $\mathbb{P}^n(\mathbb{Q})$ can be represented by an $(n+1)$-tuple $(x_0, \dots, x_n)$ of integers with no non-trivial common factor; such a representative is unique up to a global sign change. This description can then be used to define the notion of *height* of a point, namely the maximum of the absolute values of the coordinates of one of these integral representatives. In turn, once a notion of height is defined, we can analyze the asymptotic growth of the number of rational points of projective space of height bounded by $B \in \mathbb{R}$ as B tends to infinity. In particular, for projective space it is easy to see that a positive proportion of the $(n+1)$-tuples of integers of absolute values at most B has no common factor and we conclude that the asymptotic growth of the set of rational points of bounded height on the projective space $\mathbb{P}^n$ is B^{n+1}.

We would like to generalize the construction mentioned above with the aim of obtaining similar asymptotic formulas for larger classes of varieties. The initial input in the case of projective space is the affine variety $\mathbb{A}^{n+1}$ with the action of the torus $\mathbb{G}_m$ and we want to find a good replacement for this. The affine variety $\mathbb{A}^{n+1}$ is the spectrum of the polynomial ring $k[x_0, \dots, x_n]$, and the action of $\mathbb{G}_m$ is determined by the natural grading on the polynomial ring. In turn, the homogeneous component of the ring $k[x_0, \dots, x_n]$ in degree d can be interpreted as the space of global sections of the line bundle $\mathscr{O}_{\mathbb{P}^n}(d)$ on $\mathbb{P}^n$, and the grading is then by the Picard group $\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$ (the homogeneous components of negative degree all vanish). The various homogeneous components can be multiplied together by using the tensor product of sections as multiplication.

To generalize this construction to a variety X, we are therefore naturally led to form the direct sum of the global sections of all the line bundles on X and to define a ring structure by tensor product on global sections. The resulting ring is then naturally graded by the Picard group. This construction requires some care and can be performed meaningfully only in certain special cases and varying degrees of effectiveness.

D. Cox was the first to introduce this construction systematically for toric varieties in [Cox95]; we highly recommend the recent book [CLS11] for an introduction to toric varieties. We recall the definition of Cox rings in a context that is somewhat more general than just for toric varieties. Let X be a smooth projective variety whose Picard group is free and finitely generated. Denote by r the rank of the Picard group of X. Choose divisors $D_1, \dots, D_r$ freely generating the Picard group, so that $\mathrm{Pic}(X) \simeq \mathbb{Z}^r$, and define a vector space

$$\mathrm{Cox}(X) = \bigoplus_{(n_1,\dots,n_r)\in\mathbb{Z}^r} \mathrm{H}^0\big(X, \mathscr{O}_X(n_1D_1 + \cdots + n_rD_r)\big).$$

Using the isomorphisms

$$\mathscr{O}_X(m_1D_1+\cdots+m_rD_r)\otimes\mathscr{O}_X(n_1D_1+\cdots+n_rD_r) \simeq \mathscr{O}_X((m_1+n_1)D_1+\cdots+(m_r+n_r)D_r)$$

induced by tensor product of sections allows us to introduce a multiplication on the vector space $\mathrm{Cox}(X)$. The resulting ring is the *total coordinate ring*, or *Cox ring*, of X.

By construction, the Cox ring of the variety X admits a grading by the Picard group of X and hence there is an action of the Picard torus $\widehat{\mathrm{Pic}(X)}$ of X on the affine variety $\mathrm{Spec}(X)$. If the variety X is a toric variety, then the Cox ring of X is a polynomial ring: this was the starting point of the initial construction of Cox. In fact, the converse also holds: if the Cox ring of a smooth projective variety with finitely generated free Picard group is a polynomial ring, then the variety is a toric variety (see [HK00, Corollary 2.10]).

If the Cox ring of X is finitely generated, then there is an open subset $\mathscr{U} \subset \mathrm{Spec}(\mathrm{Cox}(X))$ such that the action of the Picard torus on $\mathscr{U}$ is free and the quotient is the variety itself. The quotient morphism $\mathscr{U} \to X$ is a torsor for the Picard torus and it is an example of a *universal torsor*.

Universal torsors have been used to study the set of rational points: they were introduced by Colliot-Thélène and Sansuc to systematize the use of $\mathbb{G}_m$-torsors as a tool for proving the absence of rational points. Subsequently, Salberger used universal torsors to prove asymptotic bounds for the number of rational points of bounded height on Fano varieties. Later refinements of these techniques by, among others, de la Bretèche, Browning, Derenthal, Le Boudec, Loughran, Peyre, led to the proof of special cases of Manin's Conjecture for del Pezzo surfaces.

What we said motivates the search for varieties whose Cox ring is finitely generated; such varieties are called *Mori dream spaces* in [HK00]. It is often difficult to decide if a variety is a Mori dream space; even the case of rational surfaces is not settled. Nevertheless, there are a few general classes of Mori dream spaces:

- toric varieties [Cox95];
- del Pezzo surfaces [BP04];
- Fano varieties, and more generally log Fano varieties [BCHM10];
- rational surfaces with big anticanonical divisor [TVAV11].

Artebani and Laface give in [AL11] a complete classification of the Mori dream rational surfaces with Iitaka dimension of the anticanonical divisor equal to 1. In the joint work [LT11] with Laface we give partial results for the case of rational surfaces with Iitaka dimension of the anticanonical divisor equal to 0.

References

[ADHL12] Ivan Arzhantsev, Ulrich Derenthal, Juergen Hausen, and Antonio Laface, *Cox rings*, arXiv:1003.4229 [math.AG] (2012), available at http://arxiv.org/pdf/1003.4229v2.pdf. ↑7

[AL11] Michela Artebani and Antonio Laface, *Cox rings of surfaces and the anticanonical Iitaka dimension*, Adv. Math. **226** (2011), no. 6, 5252–5267, available at http://dx.doi.org/10.1016/j.aim.2011.01.007. ↑1, 9

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468. ↑9

[BP04] Victor V. Batyrev and Oleg N. Popov, *The Cox ring of a del Pezzo surface*, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), 2004, pp. 85–103. ↑9

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. ↑8

[Cox95] David A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4** (1995), no. 1, 17–50. ↑8, 9

[HK00] Yi Hu and Sean Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), 331–348. Dedicated to William Fulton on the occasion of his 60th birthday. ↑9

[LT11] Antonio Laface and Damiano Testa, *Nef and semiample divisors on rational surfaces*, arXiv:1104.4270 (2011), available at http://arxiv.org/pdf/1104.4270v2. ↑1, 9

[Man86] Yuri I. Manin, *Cubic forms*, Second, North-Holland Mathematical Library, vol. 4, North-Holland Publishing Co., Amsterdam, 1986. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel. ↑1

[TVAV11] Damiano Testa, Anthony Várilly-Alvarado, and Mauricio Velasco, *Big rational surfaces*, Math. Ann. **351** (2011), no. 1, 95–107, available at http://dx.doi.org/10.1007/s00208-010-0590-7. ↑1, 9