These lecture notes cover Hilbert's Third Problem. They are intended for the students taking the module *MA3J9-Historical Challenges in Mathematics* at the University of Warwick. In preparing these lecture notes, I have been inspired by the presentations, drawings, arguments, discussions and statements of the following references.

(1) Hilbert's third problem and Dehn's invariant, slides of a UMN Math Club talk.

(2) Hilbert's Third Problem (A Story of Threes), by Lydia Krasilnikova (available here as a pdf).

(3) Hilbert's Third Problem as a Second Year Essay at the University of Warwick.

(4) Hilbert's third problem: decomposing polyhedra, in *Proofs from THE BOOK*, by Martin Aigner and Günter M. Ziegler.

(5) A New Approach to Hilbert's Third Problem, by David Benko.

(6) Scissor congruence, by Paul Zeitz.

(7) Hilbert's 3rd Problem and Invariants of 3–manifolds, by Walter D Neumann.

If you have any comments or find any mistakes, please let me know, either in person or by email (`d.testa` at `warwick.ac.uk`).

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Hilbert's Third Problem:
Scissor congruence

Given two polyhedra in $\mathbb{R}^3$, when can they be dissected with finitely many planar cuts so that the resulting pieces are congruent?

1 Introduction

This problem is often called Scissor congruence. A first obvious condition that needs to be satisfied is that the polyhedra must have the same volume.

We begin with the two-dimensional analogue of Hilbert's question.

Given two planar polygons, when can they be cut along straight line segments so that the resulting pieces are congruent?

The answer is the Wallace–Bolyai–Gerwien Theorem: two planar polygons are scissor congruent if and only if they have the same area. We give a proof of this fact.

After dealing with the case of planar polygons, we turn our attention to three-dimensional figures and address Hilbert's Third Problem. Given two polyhedra of equal volume, is it always possible to decompose them into congruent subpolyhedra? For instance, do a cube and a regular tetrahedron of equal volume admit congruent decompositions? We will see the answer, given by Dehn, along with the definition and properties of the *Dehn invariant*.

Recall the definition of congruence.

Definition 1.1. Two subsets $A, B \subset \mathbb{E}^n$ of a Euclidean space are *congruent* if there is an isometry $\sigma\colon \mathbb{E}^n \to \mathbb{E}^n$ such that $\sigma(A)$ coincides with B.

We will use this notion almost exclusively for planar polygons in the Euclidean plane $\mathbb{E}^2$ and for polyhedra in the Euclidean space $\mathbb{E}^3$.

2 Planar polygons

We begin with the case of the Euclidean plane $\mathbb{E}^2$. Recall that if T is a topological space, then a subset $A \subset T$ is a *regular closed subset* if A is the closure of its interior: $A = \overline{\operatorname{int} A}$.

Definition 2.1. A *planar polygon* is a bounded, regular closed subset in the Euclidean plane $\mathbb{E}^2$ whose boundary is the union of finitely many line segments.

For instance, all triangles, squares, rectangles, parallelograms are planar polygons. See Figure 1. We do not require planar polygons to be convex, connected or simply connected. In particular, the bounded polygons in Figure 1 could be viewed as four distinct planar polygons, or they could be a single planar polygon with four connected components.

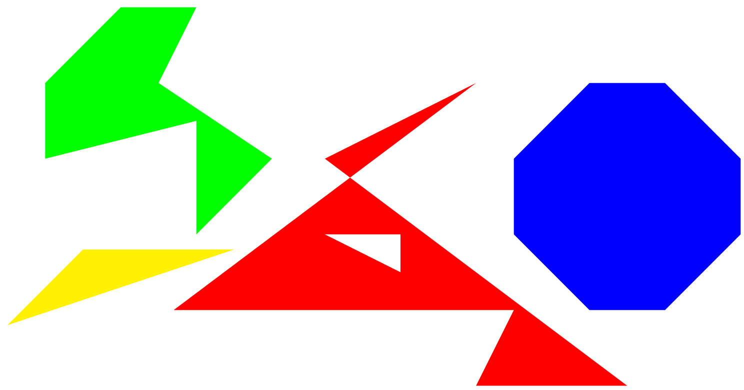


Figure 1: Examples of planar polygons

Let P be a planar polygon. Denote by ∂P the boundary of P, that is, $\partial P = P \setminus \operatorname{int} P$. By definition, the boundary of P is a finite union of line segments. Let $E \subset \partial P$ be a line segment contained in the boundary of P. If there is no larger line segment contained in P, then we call E a *maximal edge*. An *edge of* P is a line segment E, contained in ∂P, whose endpoints are points of intersections of maximal edges of P.

Example 2.2. Let T_{12} be the planar polygon formed by two adjacent triangles, shown in Figure 2.

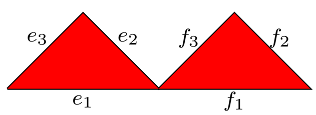


Figure 2: The edges of the planar polygon T_{12}

The maximal edges of T_{12} are e_2, e_3, f_2, f_3 and $e_1 \cup f_1$. Besides the five maximal edges, the polygon T_{12} has two more edges:

- e_1, whose endpoints are $(e_1 \cup f_1) \cap e_3$ and $(e_1 \cup f_1) \cap e_2$, and
- f_1, whose endpoints are $(e_1 \cup f_1) \cap f_3$ and $(e_1 \cup f_1) \cap f_2$.

Definition 2.3. Let P be a planar polygon. A *scissor decomposition of* P is a finite sequence $(P_1, \ldots, P_n)$ of planar polygons, such that

- $\cup_i P_i = P$, and
- for all distinct $i, j \in \{1, \ldots, n\}$, the polygons P_i and P_j have disjoint interior.

Let P be a planar polygon and let $(P_1, \ldots, P_n)$ be a scissor decomposition of P. To connect with the "scissor" intuition, we sometimes call *cuts* the edges of the planar polygons $P_1, \ldots, P_n$. We now introduce an equivalence relation among planar polygons.

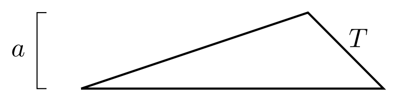
Definition 2.4. Two planar polygons P, Q are *scissor congruent* if there are scissor decompositions $(P_1, \ldots, P_n)$ of P and $(Q_1, \ldots, Q_n)$ of Q of the same length n such that for all $i \in \{1, \ldots, n\}$, the planar polygons P_i and Q_i are congruent.

Of course, the *area* of a planar polygon is invariant under scissor congruence: if two planar polygons are scissor congruent, then their areas are the same.

Before checking that scissor congruence is an equivalence relation, we give two examples of scissor congruence.

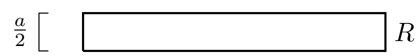
Example 2.5. We show that

a triangle T with height a with respect to its longest edge,



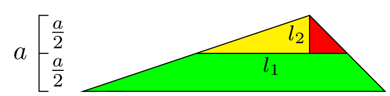
and

a rectangle R with the same area as T and an edge of length $\frac{a}{2}$,



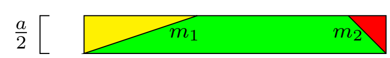
are scissor congruent. We do this with a sequence of drawings.

The cuts l_1, l_2 of T yield a scissor decomposition of T into a yellow, red and green connected component.



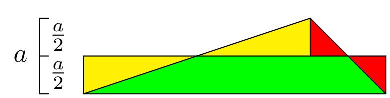
A scissor decomposition of the triangle T

The cuts m_1, m_2 of R yield a scissor decomposition of R into a yellow, red and green connected component.



A scissor decomposition of the rectangle R

It is now a matter of elementary Euclidean geometry to check that the pieces with the corresponding colour in the decomposition of the triangle T and of the rectangle R are congruent.



Example 2.6. We continue by showing that the $a \times b$ rectangle is scissor congruent to the $\sqrt{ab} \times \sqrt{ab}$ square. Again, we do this by a sequence of drawings in Figure 3. In the specific drawings, we used the 4×9 rectangle and the 6×6 square.

Lemma 2.7. *Scissor congruence is an equivalence relation.*

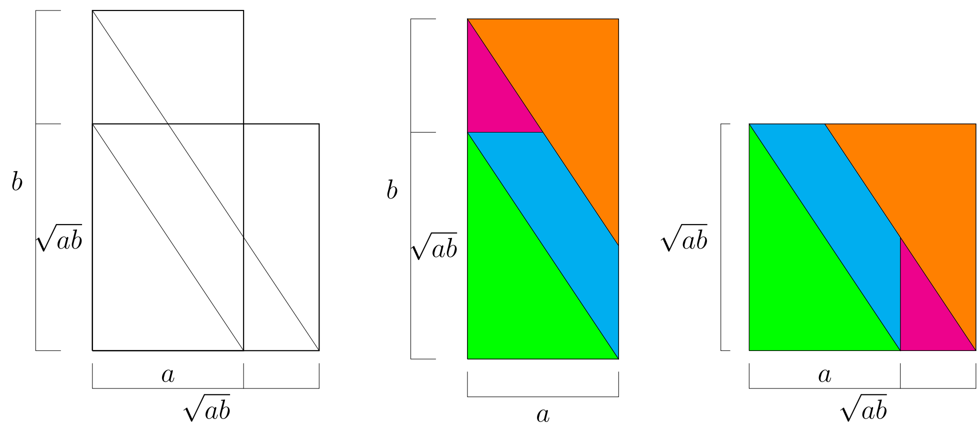


Figure 3: Every rectangle is scissor congruent to a square

Proof. We will check that scissor congruence is reflexive, symmetric and transitive. Reflexivity and symmetry are completely clear, so that we focus on transitivity.

Suppose that P, Q, R are planar polygons and that P, Q are scissor congruent and also Q, R are scissor congruent. Thus, there are positive integers m and n and lists $(P_1, \dots, P_m)$, $(Q_1, \dots, Q_m)$, $(Q'_1, \dots, Q'_n)$, $(R_1, \dots, R_n)$ of planar polygons such that

- no two distinct polygons in the same list share a common interior point;
- $\cup_i P_i = P, \qquad \cup_i Q_i = Q = \cup_j Q'_j, \qquad \cup_j R_j = R;$
- for all $i \in \{1, \dots, m\}$, the polygons P_i and Q_i are congruent via an isometry $\varphi^1_i \colon P_i \to Q_i$;
- for all $j \in \{1, \dots, n\}$, the polygons Q'_j and R_j are congruent via an isometry $\varphi^2_j \colon Q'_j \to R_j$.

For $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$, define planar polygons

$$Q_{ij} = \overline{\operatorname{int}(Q_i \cap Q'_j)}, \qquad P_{ij} = (\varphi^1_i)^{-1}(Q_{ij}), \qquad R_{ij} = \varphi^2_j(Q_{ij}).$$

We leave as an exercise to check that the lists $(P_{ij})_{(i,j)}$ and $(R_{ij})_{(i,j)}$, each consisting of mn planar polygons, are scissor decompositions of P and R respectively that satisfy the properties of Definition 2.4. We conclude that P and R are scissor congruent. □

We are now ready to prove the Wallace–Bolyai–Gerwien Theorem.

Theorem 2.8 (Wallace–Bolyai–Gerwien)**.** *Two planar polygons are scissor congruent if and only if they have the same area.*

Proof. It is clear that scissor congruent polygons have the same area. To prove the converse, we show, by a series of steps, that every planar polygon is scissor congruent to a rectangle with one edge of unit length. This is sufficient, since any two planar polygons with the same area are therefore scissor congruent to the same rectangle and hence are scissor congruent to each other.

Step 1: *every planar polygon P is scissor congruent to a union of convex planar polygons.*

Indeed, let $\ell = l_1 \cup \ldots \cup l_n$ be the union of all the lines containing an edge of P. Denote by $P^0 = (P \setminus \ell) \subset P$ the complement in P of ℓ. We leave as an exercise to show that every connected component of P^0 is also a connected component of $\mathbb{E}^2 \setminus \ell$. We deduce that the components of P^0 are convex and that P is scissor congruent to a disjoint union of convex planar polygons.

Step 2: *every convex planar polygon P is scissor congruent to a union of triangles.*

Indeed, if $p_0, \ldots, p_n$ are the vertices of P, numbered consecutively, then

$$(\operatorname{int} P) \setminus \bigcup_{i \in \{2,\ldots,n-1\}} \overline{p_0 p_i}$$

is the disjoint union of $n-1$ open triangles.

Step 3: *every triangle is scissor congruent to a rectangle with one edge of unit length.*

Indeed, by Example 2.5, every triangle is scissor congruent to a rectangle. Next, by Example 2.6, every rectangle is scissor congruent to a square. In particular, every square is congruent to a rectangle with an edge of unit length. Using that scissor congruence is an equivalence relation (Lemma 2.7), we conclude that every triangle is scissor congruent to a rectangle with a side of unit length.

Step 4: *every planar polygon P is scissor congruent to a rectangle with one edge of unit length.*

Indeed, by the previous steps, we find the following sequence of scissor congruences

$$\begin{aligned} P &\sim \text{(union of convex polygons)} \\ &\sim \text{(union of triangles)} \\ &\sim \text{(union of rectangles with an edge of unit length).} \end{aligned}$$

At this point, we can simply stack together all the rectangles with an edge of unit length to deduce that P is scissor congruent to a single rectangle with an edge of unit length. Since the area of P is preserved under these steps, we deduce that every planar polygon with area A is scissor congruent to a rectangle with edges of length 1 and A. We conclude that any two planar polygons with the same area are scissor congruent to one another, as needed. □

3 Aside on tensor products

Before moving on to the three-dimensional case, we introduce tensor products of abelian groups. We will see that they naturally arise in the definition of the Dehn invariant that will appear in the next section.

Definition 3.1 (Free abelian group on a set)**.** Let S be a set. The *free abelian group on* S is the abelian group $\langle S\rangle$ whose elements are formal, finite, integral linear combinations of the elements of S. That is, the elements of $\langle S\rangle$ are expressions of the form

$$\sum_{s\in S} n_s s,$$

where the coefficients n_s are integers and at most finitely many of them are non-zero. The identity element is $\sum_s 0\cdot s$. Sums and inverses are defined by

$$\sum_s m_s s + \sum_s n_s s = \sum_s (m_s+n_s)s \qquad \text{and} \qquad -\sum_s m_s s = \sum_s (-m_s)s.$$

Sometimes, the free abelian group on a set S is suggestively denoted by $\bigoplus_{s\in S}\mathbb{Z}s$. This emphasizes the analogy of $\langle S\rangle$ with the vector space with basis S: instead of allowing the coefficients of the basis vectors to be elements from a field, we restrict them to be integers. We view each element s of S as an element of $\langle S\rangle$, using the identification

$$s = \sum_{t\in S}\delta_{st}t, \quad \text{where} \quad \delta_{st} = \begin{cases} 0, & \text{if } s\neq t; \\ 1, & \text{if } s = t. \end{cases}$$

Example 3.2. Let $S=\{s_1,s_2,s_3\}$ be a set with three elements. Each element of $\langle S\rangle$ is of the form

$$a_1s_1 + a_2s_2 + a_3s_3, \quad \text{where} \quad a_1,a_2,a_3\in\mathbb{Z}.$$

For instance, the three expressions

$$\sum_{i=1}^{3}(i^2-4)s_i, \qquad -3s_1+0s_2+5s_3, \qquad -3(s_1-s_3)+2s_3,$$

represent the same element of $\langle S\rangle$.

We will use the free abelian group on S construction in the case in which the set S is infinite. To clarify potential doubts, we emphasize that the elements of $\langle S\rangle$ are *finite* linear combinations of the elements of S. In particular, if an identity or an argument only involves finitely many elements of $\langle S\rangle$, then the same identity or argument essentially takes place in a free abelian group on a finite subset of S. Nevertheless, it is convenient for us to use an infinite set of generators, since it would be quite involved to explicitly describe which finite set would be enough.

In the definition of the free abelian group on a set S, whether or not the set S had a further structure (e.g. S itself is an abelian group, a vector space, a ring,...) plays no role. Indeed, if S and T are sets with the same cardinality, then the groups $\langle S \rangle$ and $\langle T \rangle$ are isomorphic. In the following definition, we mix the construction of $\langle S \rangle$ with extra structure coming from the underlying set S.

Definition 3.3 (Tensor product of abelian groups)**.** Let A, B be abelian groups. The *tensor product* $A \otimes B$ *of* A *and* B is the quotient of the free abelian group $\langle A \times B \rangle$ on the cartesian product $A \times B$ by the subgroup generated by the elements

$$(a + a', b) - (a, b) - (a', b) \qquad \text{and} \qquad (a, b + b') - (a, b) - (a, b'),$$

for all $a, a' \in A$ and all $b, b' \in B$. We denote the coset of (a, b) in $A \otimes B$ by $a \otimes b$, so that the identities

$$(a + a') \otimes b = a \otimes b + a' \otimes b, \qquad a \otimes (b + b') = a \otimes b + a \otimes b'$$

hold in $A \otimes B$.

We will see that the Dehn invariant associates to a polyhedron an element of the tensor product of two abelian groups.

The first group is the additive group of real numbers $\mathbb{R}$.

The second group is also familiar: it is the group of rotations in the plane with the origin as centre. We denote this group by S^1. We identify a rotation with the measure in radians $\theta \in [0, 2\pi)$ of the angle by which we rotate. The sum of two rotations is then addition of real numbers modulo 2π. Thus, two real numbers define the same rotation if and only if they differ by an integer multiple of 2π. Hence, we identify the two groups

$$S^1 \qquad \text{and} \qquad \mathbb{R}/(2\pi\mathbb{Z}).$$

We will see that the group $\mathscr{D} = \mathbb{R} \otimes S^1 \simeq \mathbb{R} \otimes \big(\mathbb{R}/(2\pi\mathbb{Z})\big)$ plays a fundamental role in the Scissor congruence problem.

Aside. We can view the identification $S^1 \simeq \mathbb{R}/(2\pi\mathbb{Z})$ also using linear algebra. The rotation with angle $\theta \in \mathbb{R}$ is the linear map

$$\begin{array}{rccl} R(\theta)\colon & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (a,b) & \longmapsto & (a\cos\theta - b\sin\theta, a\cos\theta + b\sin\theta). \end{array}$$

The matrix of the linear transformation $R(\theta)$ with respect to the standard basis of $\mathbb{R}^2$ is

$$R(\theta) \quad \rightsquigarrow \quad \rho_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The function R is actually a group homomorphism

$$\begin{array}{rccl} R\colon & \mathbb{R} & \longrightarrow & S^1 \\ & \theta & \longmapsto & \rho_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \end{array}$$

The homomorphism R is surjective. The kernel of R is the subgroup $2\pi\mathbb{Z}$ of $\mathbb{R}$ consisting of all the integer multiples of 2π. Thus, we identify the group of rotations S^1 with the quotient group $\mathbb{R}/(2\pi\mathbb{Z})$:

$$S^1 \simeq \mathbb{R}/(2\pi\mathbb{Z}).$$

As stated above, we are essentially interested in the finitely generated subgroups of S^1. For this reason, we concentrate on them now.

We will need to decide if elements of the group $\mathscr{D}$ are equal to 0 or not, and for this, we use the following proposition.

Proposition 3.4. *Let $L \subset S^1$ be a finitely generated abelian subgroup and let $d \in L$ be an element. If d is not a torsion element, then $1 \otimes d \in \mathbb{R} \otimes L$ is a non-zero element.*

Proof. Let (n, F, σ) be a triple consisting of a non-negative integer n, a finite abelian group F and an isomorphism

$$\sigma\colon \quad L \xrightarrow{\simeq} \mathbb{Z}^n \oplus F,$$

whose existence is guaranteed by the Structure Theorem of finitely generated abelian groups. Write $\sigma(d) = \big((a_1, \ldots, a_n), f\big)$, with $a_1, \ldots, a_n \in \mathbb{Z}$ and $f \in F$. Since d is non-torsion, not all integers $a_1, \ldots, a_n$ vanish.

Forming the tensor product of $\mathbb{R}$ with L and with $\mathbb{Z}^n \oplus F$, we find the isomorphisms

$$\begin{aligned} \mathbb{R} \otimes L &\simeq \mathbb{R} \otimes (\mathbb{Z}^n \oplus F) \\ &\simeq (\mathbb{R} \otimes \mathbb{Z}^n) \oplus (\mathbb{R} \otimes F) \\ &\simeq \mathbb{R} \otimes \mathbb{Z}^n \\ &\simeq \mathbb{R}^n. \end{aligned}$$

Under these isomorphisms, the element d maps to $(a_1, \ldots, a_n) \in \mathbb{R}^n$. As we saw, this is not the zero element and we are done. □

While we will not require the explicit structure of finitely generated subgroups of S^1, we state it here as a fact.

Fact 3.5. *Let $L \subset S^1$ be a finitely generated subgroup. There are non-negative integers n, a and an isomorphism*

$$\sigma\colon \qquad L \xrightarrow{\simeq} \mathbb{Z}^n \oplus \mathbb{Z}/a\mathbb{Z}.$$

4 Polyhedra

We now turn our attention to Scissor congruence in $\mathbb{E}^3$.

Definition 4.1. A *polyhedron* is a closed, bounded region in the Euclidean space $\mathbb{E}^3$ whose boundary is the union of finitely many planar polygons.

Often, the definition of polyhedron includes the property of being *convex*. We do not impose this condition, since we are again interested in decomposing polyhedra and we can always refine a decomposition to contain only convex pieces.

As before, we define scissor decompositions and scissor congruence also for polyhedra.

Definition 4.2. A *scissor decomposition of a polyhedron P* is finite sequence $(P_1, \ldots, P_n)$ of polyhedra such that

- $\cup_i P_i = P$;
- for all distinct $i, j \in \{1, \ldots, n\}$, the polyhedra P_i and P_j have no common interior point.

Definition 4.3. Two polyhedra P, Q are *scissor congruent* if there are scissor decompositions $(P_1, \ldots, P_n)$ of P and $(Q_1, \ldots, Q_n)$ of Q of the same length n, such that, for all $i \in \{1, \ldots, n\}$, the polyhedra P_i and Q_i are congruent.

Just as in the case of scissor congruence for planar polygons, there is a clear invariant that is preserved by scissor congruence for polyhedra: the *volume*.

In contrast with the case of planar polygons, there is one further quantity that is preserved under scissor congruence for polyhedra: the *Dehn invariant*.

4.1 The Dehn invariant

Max Dehn was a student of David Hilbert. Dehn solved Hilbert's Third problem within a year of Hilbert's 1900 address to the International Congress of Mathematicians. It was later discovered that the problem had already been asked and solved before Hilbert stated it, but we will go over Dehn's proof.

The Dehn invariant D is a function

$$D\colon \quad \{\text{polyhedra in } \mathbb{E}^3\} \longrightarrow \mathscr{D} = \mathbb{R} \otimes S^1$$

with two fundamental properties.

(1) Congruent polyhedra have equal Dehn invariant:
if P is a polyhedron and σ is an isometry of $\mathbb{E}^3$, then

$$D(\sigma(P)) = D(P).$$

(2) The Dehn invariant is additive for polyhedra with disjoint interiors:
if P, Q are polyhedra and $(\operatorname{int} P) \cap (\operatorname{int} Q) = \emptyset$, then

$$D(P \cup Q) = D(P) + D(Q).$$

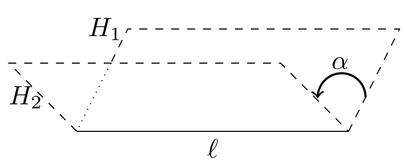
An easy induction shows that the additivity property actually holds more generally for unions of finitely many polyhedra no two of which share a common interior point.

The volume function, with values in $\mathbb{R}$, has the same properties as the Dehn invariant. Thus, we can view the Dehn invariant as a variation on the notion of volume.

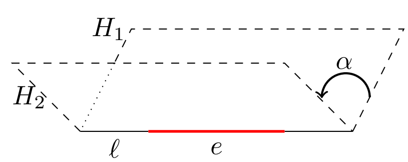
Aside. The volume function can be extended to much more varied sets of shapes than just polyhedra. This forms the foundation of measure theory. In this setting, there are subtle relations between finite additivity and measurability: see the Banach-Tarski paradox. Since we are only concerned with polyhedra, these possible extensions will not be relevant for our purposes.

The fundamental idea behind the definition of the Dehn invariant is to mix lengths of edges with the corresponding angles. Let us make this precise.

Definition 4.4. A *dihedral angle* α is a closed region in $\mathbb{E}^3$ whose boundary consists of two distinct half-planes $H_1, H_2 \subset \mathbb{E}^3$ originating from the same line ℓ. We call ℓ the *edge* of α and H_1, H_2 the *faces* of α.



The dihedral angle α

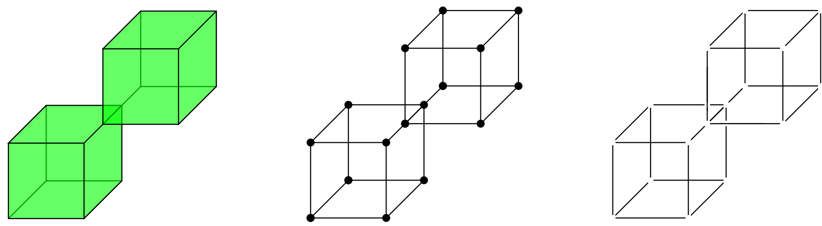


A dihedral angle with a line segment in red

Let α be a dihedral angle with edge $\ell \subset \mathbb{E}^3$ and let $e \subset \ell$ be an open segment. For us, pairs (e, α) arise in connection with dihedral angles of polyhedra along their edges. We assign to the pair (e, α) an element $\delta(e, \alpha) \in \mathscr{D}$ as follows. Denote by $|e| \in \mathbb{R}_{>0}$ the length of the segment e and denote by $\hat{\alpha} \in (0, 2\pi)$ the measure in radians of the dihedral angle α. We set

$$\delta(e, \alpha) = |e| \otimes \hat{\alpha} \quad \in \mathscr{D}. \tag{1}$$

For the next construction, a good example of a polyhedron to keep in mind is the union of two cubes meeting along part of an edge. An example C of such



The polyhedron C The sets $e(C)$ and $v(C)$ The segments in $\mathrm{ess}(C)$

Figure 4: A polyhedron and its essential segments

a polyhedron is drawn in Figure 4, along with the subsets $e(C)$, $v(C)$ and $\mathrm{ess}(C)$ that we now define.

Let P be a polyhedron. We denote by $e(P) \subset P$ the union of all the edges of P, that is, $e(P)$ is the union of all the points of P where at least two faces of P meet: the set $e(P)$ is the union of finitely many closed line segments. We also denote by $v(P) \subset P$ the union of all the vertices of all the faces of P: the set $e(P)$ consists of finitely many points. Finally, we observe that the complement $e(P) \setminus v(P)$ of the set $v(P)$ inside $e(P)$ is the union of finitely many, disjoint, open line segments. We denote by $\mathrm{ess}(P)$ the set of connected components of $e(P) \setminus v(P)$.

Definition 4.5. Let P be a polyhedron. An *essential segment of* P is an element of $\mathrm{ess}(P)$.

Thus, essential segments of a polyhedron P are open line segments contained in edges of P and $\mathrm{ess}(P)$ is the set of all finitely many essential segments of P.

We come to the definition of the fundamental building block for the Dehn invariant.

Definition 4.6. Let P be a polyhedron. An *essential pair for* P is a pair (e, α) consisting of an essential segment $e \in \mathrm{ess}(P)$ and a dihedral angle $\alpha \subset \mathbb{E}^3$ with edge containing e and faces H_1, H_2 such that there are two distinct faces f_1, f_2 of P satisfying

- H_1 contains f_1;
- H_2 contains f_2;
- the essential segment e is contained in the boundary of both f_1 and f_2;
- a neighbourhood of e in α is contained in the polyhedron P.

We denote by $\mathscr{E}(P)$ the set of all essential pairs of P.

We are ready to define the Dehn invariant of a polyhedron.

Definition 4.7 (Dehn invariant)**.** Let P be a polyhedron. The Dehn invariant $D(P)$ of P is the expression

$$\sum_{(e,\alpha)\in\mathscr{E}(P)} \delta(e,\alpha) \quad = \sum_{(e,\alpha)\in\mathscr{E}(P)} |e|\otimes\hat{\alpha} \quad \in \mathscr{D}.$$

Now that we have a definition of the Dehn invariant, we proceed to check that it really is invariant under Scissor congruence. By construction, if P and Q are congruent polyhedra, then $D(P)$ and $D(Q)$ coincide. Thus, to conclude it suffices to show that if we cut a polyhedron P using a plane into polyhedra P_+ and P_-, then the Dehn invariant of P is the sum of the Dehn invariants of P_+ and of P_-.

Let P be a polyhedron in $\mathbb{E}^3$ and let $L \subset \mathbb{E}^3$ be a plane. Denote by L_+ and L_- the two closed half-spaces into which L decomposes the vector space $\mathbb{E}^3$. Let $P_+ = \overline{\operatorname{int}(P \cap L_+)}$ and $P_- = \overline{\operatorname{int}(P \cap L_-)}$ be the two polyhedra obtained by splitting P with the plane L. We want to show that the identity

$$D(P) = D(P_+) + D(P_-) \tag{2}$$

holds. Observe that the edges of P_+ and P_- are contained either in edges of P or in the intersection of a face of P with the plane L. We begin by analyzing how essential pairs change under the scissor decomposition (P_+, P_-) of P.

Let $(e,\alpha) \in \mathscr{E}(P)$ be an essential pair of P. We consider three cases separately:

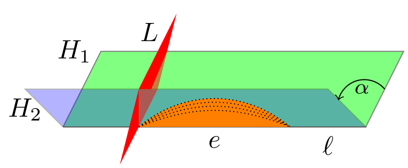
(1) a neighbourhood of e in α is entirely contained either in L_+ or in L_-;

(2) the plane L contains exactly one (interior) point of the essential segment e;

(3) the plane L contains the essential segment e and the angle α is not contained in either one of the half-spaces L_+ or L_-.

Note that there are no further possibilities, since if the plane L is not disjoint from e, then it either meets e at exactly one point or it contains e.

Case (1)**.** The plane L misses a neighbourhood of the essential segment e in α.

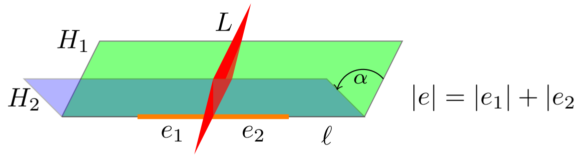


Case (1): the plane L, in red, misses the orange neighbourood of e

Thus, the essential pair (e,α) is an essential pair of exactly one among P_+ and P_-.

The contribution of (e,α) is $\begin{cases} \delta(e,\alpha) = |e|\otimes\alpha, & \text{to } D(P); \text{ and} \\ \delta(e,\alpha) + 0 = |e|\otimes\alpha, & \text{to } D(P_+) + D(P_-). \end{cases}$

Case (2)**.** The plane L slices the essential segment e at a unique interior point.

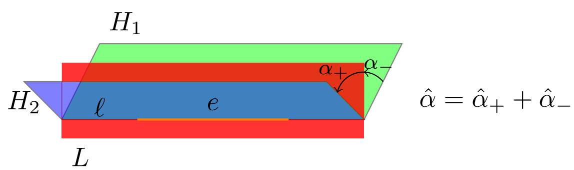


Case (2): the plane L, in red, slices the essential segment e, in orange

The measure $\hat{\alpha}$ of the dihedral angle α in radians is unaffected by L, but the edge e is split into two separate essential edges e_1, e_2, one in each polyhedron P_+ and P_-.

$$\text{The contribution of } (e,\alpha) \text{ is } \begin{cases} \delta(e,\alpha) = |e| \otimes \alpha, & \text{to } D(P); \\ & \text{and} \\ \delta(e_1,\alpha) + \delta(e_2,\alpha) & \\ = |e_1| \otimes \alpha + |e_2| \otimes \alpha & \text{to } D(P_+) + D(P_-). \\ = (|e_1| + |e_2|) \otimes \alpha, & \end{cases}$$

Case (3). The plane L splits the dihedral angle into two dihedral angles α_1 and α_2, each with the same edge e.



Case (3): the plane L, in red, slices the diehdral angle α

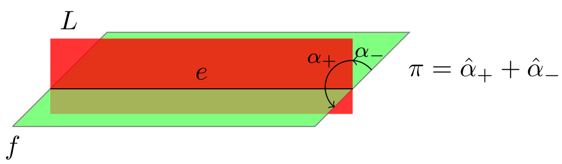
The essential pair (e,α) of P gives rise to the essential pairs (e,α_1) and (e,α_2) of P_+ and P_- and the relation $\hat{\alpha} = \hat{\alpha}_1 + \hat{\alpha}_2$ holds.

$$\text{The contribution of } (e,\alpha) \text{ is } \begin{cases} \delta(e,\alpha) = |e| \otimes \alpha, & \text{to } D(P); \\ & \text{and} \\ \delta(e,\alpha_+) + \delta(e,\alpha_-) & \\ = |e| \otimes \alpha_+ + |e| \otimes \alpha_- & \text{to } D(P_+) + D(P_-). \\ = |e| \otimes (\alpha_+ + \alpha_-), & \end{cases}$$

So far, we have seen how slicing with a plane affects the essential pairs of the polyhedron P. There is one more computation that we need.

(4) The essential pairs of P_+ and P_- that are created by the slice. These "new" pairs arise when the plane L cuts through faces of P, creating new edges.

Case (4). The plane L intersects the interior of a face f, determining an edge e in both P_+ and P_-.



Case (4): the flat dihedral angle above f is split into two dihedral angles α_+ and α_-

In this situation, the length of the edge e is unknown, as are the two angles α_+ and α_-, but the relation $\hat{\alpha}_+ + \hat{\alpha}_- = \pi$ holds.

$$\text{The contribution of } (e, \alpha) \text{ is } \begin{cases} \qquad\qquad\qquad 0, & \text{to } D(P); \text{ and} \\ \delta(e, \alpha_+) + \delta(e, \alpha_-) & \\ = |e| \otimes (\alpha_+ + \alpha_-) & \text{to } D(P_+) + D(P_-). \\ = |e| \otimes \pi, & \end{cases}$$

In one of the exercises, there is a proof that $|e| \otimes \pi$ is the zero element of $\mathscr{D}$.

Summarizing, let P be a polyhedron, let L be a plane and let P_+ and P_- be the polyhedra "sliced up" by the plane L. Our previous computations show that the identity

$$D(P) = D(P_+) + D(P_-)$$

holds in $\mathscr{D}$. Since any scissor decomposition of a polyhedron can be refined to a scissor decomposition obtained as a succession of slices by planes, we deduce the following theorem.

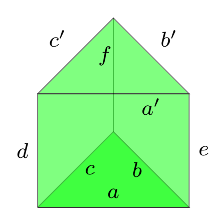
Theorem 4.8. *If P and Q are scissor congruent polyhedra, then the identity*

$$D(P) = D(Q)$$

holds.

Example 4.9 (Dehn invariant of a triangular prism)**.** Let T be the prism of height h over the equilateral triangle with edge length l. We denote by

- a, b, c the edges on the bottom face of T, each of length l;
- a', b', c' the edges on the top face of T, each of length l;
- d, e, f the edges joining vertices on opposite faces of T, each of length h;



The prism T

The essential segments of T are the 9 (open) line segments $a, b, c, a', b', c', d, e, f$. The dihedral angles on the edges a, b, c and a', b', c' are right angles, measuring $\frac{\pi}{2}$ radians. The dihedral angles on the edges d, e, f correspond to the internal angles of an equilateral triangle, measuring $\frac{\pi}{3}$ radians. We find

$$D(T) = 6\left(l \otimes \frac{\pi}{2}\right) + 3\left(h \otimes \frac{\pi}{3}\right) = 0.$$

We can now answer Hilbert's Third Problem.

Theorem 4.10. *The regular tetrahedron and the cube with the same volume are not scissor congruent.*

Proof. We compute the Dehn invariant of a cube C with edge length c. Let $(e, \alpha) \in \mathscr{E}(C)$ be an essential segment of the cube C. The dihedral angle α is a right angle, measuring $\frac{\pi}{2}$ radians. Thus, we compute

$$D(C) = \sum_{(e,\alpha)\in\mathscr{E}(C)} \delta(e, \alpha) = 12c \otimes \frac{\pi}{2} = 3c \otimes (2\pi) = 0.$$

Similarly, let T be a tetrahedron with edge length t and let $(e, \alpha) \in \mathscr{E}(T)$ be an essential segment of T. The dihedral angle α measures $\arccos \frac{1}{3}$ radians. Thus, the Dehn invariant of the tetrahedron T is

$$D(T) = \sum_{(e,\alpha)\in\mathscr{E}(T)} \delta(e, \alpha) = 4t \otimes \arccos \frac{1}{3}.$$

To conclude that C and T are not scissor congruent, it is enough to check, by Proposition 3.4, that π is not a rational multiple of $\arccos \frac{1}{3}$. Equivalently, we want to show that, for no rational number r, the real number $\cos(r\pi)$ equals $\frac{1}{3}$. We therefore conclude using Lemma 4.11. □

Lemma 4.11. *Let $r \in \mathbb{Q}$ be a rational number. The real number $2\cos(r\pi)$ is an algebraic integer. In particular, if $\cos(r\pi)$ is rational, then its denominator divides 2.*

Proof. Let $\alpha \in \mathbb{C}$ be the complex number

$$\alpha = \cos(r\pi) + i \sin(r\pi) = e^{i\pi r}.$$

Write $r = \frac{p}{q}$, with $p, q \in \mathbb{Z}$, $q \neq 0$, so that $\alpha^{2q} = (-1)^{2p} = 1$. Thus, the numbers α and its complex conjugate $\overline{\alpha}$ are roots of the polynomial $x^{2q} - 1$. We deduce that α and $\overline{\alpha}$ are algebraic integers and hence, so is $2\cos(r\pi) = \alpha + \overline{\alpha}$. □