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# FIXED MESH FINITE ELEMENT APPROXIMATIONS TO A FREE BOUNDARY PROBLEM FOR AN ELLIPTIC EQUATION WITH AN OBLIQUE DERIVATIVE BOUNDARY CONDITION

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**Abstract**—A method for approximating the solution of an elliptic equation with an oblique derivative on a curved boundary using an unfitted finite element mesh is presented and analysed. It is shown that the method retains the order of accuracy of the fitted mesh finite element method. A similar result is obtained for a variational inequality. The usefulness of this approach is then demonstrated by using it to approximate the solution of a free boundary problem on a fixed mesh.

## 1. INTRODUCTION

Free boundary problems for Poisson equations, in particular those amenable to variational inequality techniques, have been widely studied in recent years; see [2, 8, 9, 10]. Frequently an integral transformation of the dependent variable in the original problem is required in order to obtain a variational inequality formulation. This integration transforms Dirichlet boundary conditions into oblique derivative conditions for the transformed variable. A typical problem, arising in the mathematical modelling of an electrochemical machining process[7], is to find a curve $\Gamma$ defined by $y = d(x)$, $x \in [-L, L]$, such that

$$d(-L) = d(L) = Y_3, \quad d(x) > c(x) \tag{1.1a}$$

where $x \in [-L, L]$, $d(x) \in C[-L, L] \cap C^\infty(-L, L)$ and where $c(x) \in C^3[-L, L]$ is given satisfying

$$\begin{aligned} c(0) = c'(0) = 0, \quad c(-L) = Y_1 < Y_3, \\ c(L) = Y_2 < Y_3, \quad c''(x) > 0, \quad x \in (-L, L), \end{aligned} \tag{1.1b}$$

and a function $u(x, y) \in H^2(\Omega) \cap C^1(\bar{D})$, where

$$\Omega \equiv \{(x, y){:}\ -L < x < L, \quad c(x) < y < d(x)\}$$

and

$$D \equiv \{(x, y){:}\ -L < x < L, \quad c(x) < y < Y_3\},$$

such that

$$\nabla^2 u = \gamma > 0 \quad \text{and} \quad u > 0 \quad \text{in} \quad \Omega, \tag{1.1c}$$

$$u_y(x, c(x)) = -1 \quad \text{and} \quad u(x, Y_3) = 0, \quad x \in (-L, L), \tag{1.1d}$$

$$u(-L, y) = U_1(y) > 0, \quad y \in (Y_1, Y_3), \tag{1.1e}$$

$$u(L, y) = U_2(y) > 0, \quad y \in (Y_2, Y_3), \tag{1.1f}$$

$$u = u_x = u_y = 0 \text{ on } \Gamma, \quad u \equiv 0 \text{ in } \bar{D}\backslash\Omega. \tag{1.1g}$$

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Here $U_1(\cdot)$ and $U_2(\cdot)$ are given non-negative functions, which are continuously differentiable, satisfying $U_1(Y_3) = U_2(Y_3) = 0$ and $U_1'(Y_1) = U_2'(Y_2) = -1$; and $\gamma$ is a given positive constant. The region is depicted in Figure 1 which also defines the open sets $\Gamma_i$.

The first equation of (1.1d) is an oblique derivative boundary condition which can be written as

$$\frac{\partial u}{\partial \nu} + c'(x)\frac{\partial u}{\partial \sigma} = -(1 + c'(x)^2)^{1/2} \text{ on } \Gamma_0, \tag{1.2}$$

where $\nu$ and $\sigma$ are, respectively, the unit inward pointing normal and anticlockwise tangential vectors to the curve $\Gamma_0$ at $(x, y)$. Using the fact that $u$ solves a variational inequality, it was shown[7] that, when the problem is symmetric about $x = 0$, there exists a unique solution to this free boundary problem such that $u \in W^{2,p}(D) \cap C^{1,\lambda}(\bar{D})$ for all $p \in [1, \infty)$ and $\lambda \in (0, 1)$. Such free boundary problems for Poisson equations with oblique derivative conditions on fixed curved boundaries occur also in the theory of flow in porous media[2, 3, 14] where they are usually formulated as quasivariational inequalities.

This paper has two objects. First, in Sec. 2 we propose and analyse a finite element approximation of a Poisson equation holding on $D$ with an oblique derivative condition on the curved boundary $\Gamma_0$, using a mesh which is not fitted to $D$. This extends the method of Barrett and Elliott[4] who considered a Neumann boundary condition. The technique is then applied to the variational inequality formulation of (1.1). It is shown that there is no loss of order of accuracy when compared with the use of fitted meshes. The motivation for using unfitted meshes, as proposed in [4] is the possibility of their use in solving free or moving boundary problems where the same equation has to be solved on a large number of changing domains. The advantage of unfitted meshes over fitted meshes lies in the avoidance of the need to triangulate the region. The second object of the paper is then to explore this possibility in the context of the trial free boundary method (TFBM)[1, 6, 16], as applied to (1.1). Given a guess $\Gamma^{(k)}$ to $\Gamma$, the elliptic equation is solved using just one of the boundary conditions on $\Gamma^{(k)}$, say

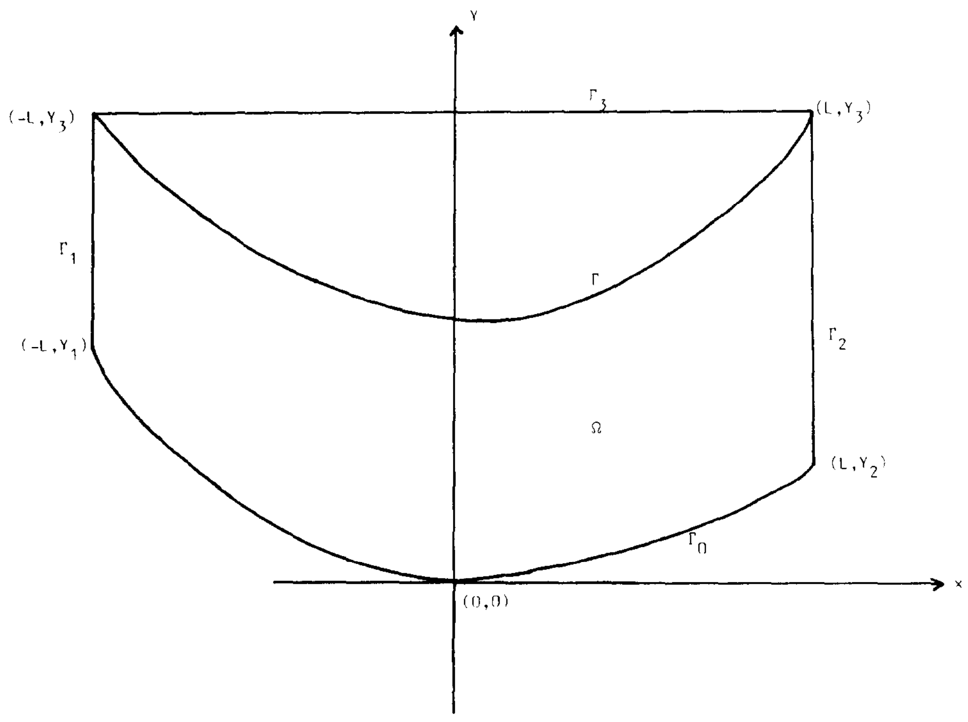


Fig. 1.

$u_n = 0$, where $n$ is the unit outward pointing normal. The resulting solution is then used to obtain an updated guess to $\Gamma$. Thus a sequence of elliptic equations with derivative boundary condition are required to be solved. In Sec. 3 we report on the results of some numerical experiments.

## 2. ERROR ESTIMATES FOR A FINITE ELEMENT APPROXIMATION

### 2.1. *Approximation of an elliptic equation*

To illustrate the numerical method to cope with an oblique derivative condition on a curved boundary we consider the following Poisson equation with mixed boundary data, using the notation for $D$ and its boundary introduced in Sec. 1:

$$\begin{aligned} -\nabla^2 u &= f \quad \text{in} \quad D, \\ u &= g \quad \text{on} \quad \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \end{aligned} \tag{2.1}$$

and

$$u_\nu = g_0 \quad \text{on} \quad \Gamma_0,$$

where the data is such that $f \in L^2(D)$, $g \in H^2(D)$ and $g_0$ is Lipschitz continuous on $D$. The weak formulation associated with (2.1) is to find

$$u - g \in V_0 = \{w \in H^1(D){:}\ w = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3\}$$

such that

$$a(u, v) = l(v), \quad \forall v \in V_0, \tag{2.2}$$

where

$$a(u, v) \equiv \int_D \underline{\nabla} u \cdot \underline{\nabla} v \,\mathrm{d}x\,\mathrm{d}y - \int_{\Gamma_0} c'(x) \frac{\partial u}{\partial \sigma} v \,\mathrm{d}\sigma \tag{2.3a}$$

and

$$l(v) \equiv \int_D fv \,\mathrm{d}x\,\mathrm{d}y - \int_{\Gamma_0} (1 + c'(x)^2)^{1/2} g_0 v \,\mathrm{d}\sigma. \tag{2.3b}$$

Let $D_h^* \supset D$ be the union of a collection of elements $\{e\}$ with disjoint interiors and such that $e \cap D \neq \{\phi\}$. The elements, which we assume to be regular (Ciarlet, 1978, p. 124), are either triangles or rectangles whose sides are less than $h$ in length. The elements are assumed to fit the straight boundaries and also have $(0, 0)$, $(-L, Y_1)$ and $(L, Y_2)$ as element vertices. A polygonal approximation $D_h$ to $D$ is constructed in the following way. If for an element $e$, $\Gamma_0 \cap e \neq \{\phi\}$ then the arc of $\Gamma_0$ is approximated by its chord joining the points of intersection with the element boundary. The resulting piecewise linear approximation to $\Gamma_0$ is denoted by $\Gamma_0^h$ which is described by $y = c_h(x)$; $D_h$ is then defined to be the open region bounded by $\bar{\Gamma}_0^h \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$.

We define a finite element space $V^h(D_h^*)$ by

$$V^h(D_h^*) = \{w \in C(D_h^*){:}\ w \text{ is linear on triangular elements or bilinear on rectangular elements}\}$$

and set

$$V_0^h = \{w_h \in V^h(D_h^*){:}\quad w_h = 0 \text{ on } \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3\}$$

and

$$V_E^h = \{w_h \in V^h(D_h^*):\quad w_h(x_i, y_i) = g(x_i, y_i)$$
$$\text{for each vertex } (x_i, y_i) \text{ on } \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3\}.$$

Then $V^h(D_h^*) \subset H^1(D_h^*)$ and the following approximation property holds: for $w - g \in V_0 \cap H^2(D_h^*)$ there exists an interpolate $w_h^I \in V_E^h$ satisfying

$$|w - w_h^I|_{0,D_h^*} + h|w - w_h^I|_{1,D_h^*} \le C_1 h^2 |w|_{2,D_h^*}, \tag{2.4}$$

where $C_1$ is a constant independent of $w$ and $h$, Ciarlet (1978, p. 124).

The finite element approximation of (2.2) which we wish to analyse is to find $u_h \in V_E^h$ such that

$$a_h(u_h, v_h) = l_h(v_h), \quad \forall v_h \in V_0^h, \tag{2.5}$$

where

$$a_h(u_h, v_h) = \int_{D_h} \underline{\nabla} u_h \cdot \underline{\nabla} v_h \, dx\, dy - \int_{\Gamma_0^h} c_h'(x) \frac{\partial u_h}{\partial \sigma_h} v_h \, d\sigma_h, \tag{2.6a}$$

$$l_h(v_h) = \int_{D_h} f v_h \, dx\, dy - \int_{\Gamma_0^h} (1 + c_h'(x)^2)^{1/2} g_0 v_h \, d\sigma_h \tag{2.6b}$$

and $\sigma_h$ is the unit anticlockwise tangential vector to the curve $\Gamma_0^h$. This is then a finite element method with an unfitted mesh and extends the approach of Barrett and Elliott[4] for Neumann boundary conditions to oblique derivative conditions.

In the proofs that follow we shall make use of the fact that
(i) there exists a constant $C_2$ independent of $h$ and $w$ such that

$$|w|_{0,D_h} \le C_2 |w|_{1,D_h}, \quad \forall w \in V_0, \tag{2.7a}$$

and the following trace theorems[4, 11–13, 15]
(ii) there exist constants $C_3$, $C_4$ and $C_5$ depending only on the Lipschitz constant of $c_{(h)}(\cdot)$ and so independent of $h$ and $w$ such that

$$|w|_{0,\Gamma_0^{(h)}} \le C_3 |w|_{1,D_{(h)}}, \tag{2.7b}$$

$$\|w\|_{1/2,\Gamma_0^{(h)}} \le C_4 |w|_{1,D_{(h)}}, \tag{2.7c}$$

$$\left\| \frac{\partial w}{\partial \sigma_{(h)}} \right\|_{-1/2,\Gamma_0^{(h)}} \le C_5 |w|_{1,D_{(h)}}, \quad \forall w \in V_0, \tag{2.7d}$$

where $\|\cdot\|_{-1/2,\Gamma_0^{(h)}}$ is the norm on $H^{-1/2}(\Gamma_0^{(h)})$, the dual space of $H_{00}^{1/2}(\Gamma_0^{(h)})$.

For any $v \in C^\infty(D)$ such that $v = 0$ on $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, we see that

$$a(v, v) = \int_D |\underline{\nabla} v|^2 \, dx\, dy + \int_{\Gamma_0} \frac{c''(x)}{(1 + c'(x)^2)^{1/2}} \frac{v^2}{2} \, d\sigma. \tag{2.8}$$

The positivity of $c''(\cdot)$ immediately implies the coercivity of $a(\cdot, \cdot)$ on $V_0 \times V_0$ as $|\cdot|_{1,D}$ is a norm on $V_0$. Continuity of $a(\cdot, \cdot)$ on $V_0 \times V_0$ follows by noting that

$$|a(w, v)| \leq |w|_{1,D}|v|_{1,D} + |c'|_{\infty,\Gamma_0} \left\| \frac{\partial w}{\partial \sigma} \right\|_{-1/2,\Gamma_0} \|v\|_{1/2,\Gamma_0}, \quad \forall v, w \in V_0$$

and applying the inequalities (2.7c) and (2.7d). Thus there exists a unique solution to (2.2) by direct application of the Lax-Milgram theorem. We shall assume that the data is sufficiently regular and compatible at $(-L, Y_1)$, $(-L, Y_3)$, $(L, Y_2)$ and $(L, Y_3)$ so that $u \in H^2(D)$ and has Lipschitz continuous first derivatives.

PROPOSITION 2.1

There exists a unique solution to (2.5).

*Proof.* It is sufficient to show that $a_h(\cdot, \cdot)$ is coercive and continuous over $V_0^h \times V_0^h$ and $l_h(\cdot)$ is a continuous linear form over $V_0^h$. For $v_h \in V_0^h$ we have

$$a_h(v_h, v_h) = \int_{D_h} |\nabla v_h|^2 \, dx \, dy - \int_{\Gamma_0^h} c_h'(x) \frac{\partial v_h}{\partial \sigma_h} v_h \, d\sigma_h.$$

Ordering the intersection points $(x_i, c(x_i))$ of $\Gamma_0$ with the elements $e$ from left to right as $i = 0, 1, \ldots, N$ we find that

$$-\int_{\Gamma_0^h} c_h'(x) \frac{\partial v_h}{\partial \sigma_h} v_h \, d\sigma_h = \sum_{i=0}^{N-1} \frac{c(x_{i+1}) - c(x_i)}{2(x_{i+1} - x_i)} [v_h^2(x_i, c(x_i)) - v_h^2(x_{i+1}, c(x_{i+1}))] \geq 0,$$

because by the convexity of $c(\cdot)$ we have

$$\frac{c(x_{i+1}) - c(x_i)}{x_{i+1} - x_i} \geq \frac{c(x_i) - c(x_{i-1})}{x_i - x_{i-1}}$$

and that also $v_h(x_0, c(x_0)) \equiv v_h(-L, Y_1) = 0$ and $v_h(x_N, c(x_N)) \equiv v_h(L, Y_2) = 0$. Thus we have

$$a_h(v_h, v_h) \geq |v_h|_{1,D_h}^2, \tag{2.9}$$

which implies the coercivity of $a_h(\cdot, \cdot)$ as $|\cdot|_{1,D_h}$ is a norm on $V_0^h$. The continuity of $a_h(\cdot, \cdot)$ and $l_h(\cdot)$ with constants independent of $h$ follows from the Lipschitz bound on $c_h(\cdot)$ being independent of $h$ and the inequalities (2.7c) and (2.7d), and (2.7a) and (2.7b), respectively. ■

PROPOSITION 2.2

Let $u_h^* \in V_E^h$ be the unique solution of the projection:

$$a_h(u_h^* - u, v_h) = 0, \quad \forall v_h \in V_0^h. \tag{2.10}$$

Then the following estimates hold for $u$ and $u_h$, the solutions of (2.2) and (2.5), respectively,

$$|u_h - u_h^*|_{1,D_h} \leq Ch^2 \tag{2.11a}$$

and

$$|u - u_h^*|_{1,D_h} \leq Ch. \tag{2.11b}$$

*Proof.* For any $v_h \in V_0^h$ we find that

$$\begin{aligned} |a_h(u_h - u_h^*, v_h)| &= |a_h(u_h - u, v_h)| \\ &= \left| \int_{D_h} \{f v_h - \underline{\nabla} u \cdot \underline{\nabla} v_h\} \, dx \, dy \right. \\ &\quad \left. + \int_{\Gamma_0^h} \left\{ c_h'(x) \frac{\partial u}{\partial \sigma_h} - (1 + c_h'(x)^2)^{1/2} g_0 \right\} v_h \, d\sigma_h \right| \\ &= \left| \int_{\Gamma_0^h} (1 + c_h'(x)^2)^{1/2} \left( \frac{\partial u}{\partial y} - g_0 \right) v_h \, d\sigma_h \right| \\ &\leq \max_{x \in [0, L]} \{(1 + c_h'(x)^2)^{1/2}\} \left| \frac{\partial u}{\partial y} - g_0 \right|_{0, \Gamma_0^h} |v_h|_{0, \Gamma_0^h} \\ &\leq C h^2 |v_h|_{0, \Gamma_0^h}. \end{aligned}$$

Here we have used Green's formula, $D_h \subset D$, dist $(\Gamma_0, \Gamma_0^h) = O(h^2)$, the continuity of $c'$, and the regularity of $u$ and $g_0$. Taking $v_h = u_h - u_h^*$ and recalling (2.7b) and (2.9) yields the estimate (2.11a). Since we have

$$a_h(u_h^I - u_h^*, u_h^I - u_h^*) = a_h(u_h^I - u, u_h^I - u_h^*)$$

the interpolation estimate (2.4) and the continuity of $a_h(\cdot, \cdot)$ with the bound (2.9) immediately imply (2.11b). ■

THEOREM 2.1

The error in the approximation of (2.2) by (2.5) satisfies

$$|u - u_h|_{1, D_h} \leq Ch. \tag{2.12}$$

*Proof.* The bound follows directly from (2.11a) and (2.11b). ■

2.2. *Approximation of a variational inequality*

It is easy to see that a solution of problem (1.1) solves the elliptic variational inequality: find $u \in K$ such that

$$a(u, v - u) \geq l(v - u), \quad \forall v \in K, \tag{2.13}$$

where $K = \{w \in H^1(D): w = U_1 \text{ on } \Gamma_1, w = U_2 \text{ on } \Gamma_2, w = 0 \text{ on } \Gamma_3 \text{ and } w \geq 0 \text{ a.e. in } D\}$, $a(\cdot, \cdot)$ and $l(\cdot)$ are defined by (2.3) with $f = -\gamma$ and $g_0 = -1$. The unique solution of (2.13) is a member of $H^2(D)$ and $u_y$ is Lipschitz continuous in the neighbourhood of $\Gamma_0$[7]. Also $u$ satisfies the linear complementarity system:

$$\left.\begin{aligned} -\nabla^2 u + \gamma \geq 0, \quad u \geq 0, \\ (-\nabla^2 u + \gamma) u = 0. \end{aligned}\right\} \text{ a.e. in D} \tag{2.14}$$

The finite element approximation of (2.13) is to find $u_h \in K^h$ such that

$$a_h(u_h, v_h - u_h) \geq l_h(v_h - u_h), \quad \forall v_h \in K^h, \tag{2.15}$$

where $K^h = \{w_h \in V^h(D_h^*): w_h(-L, y_i) = U_1(y_i)$ for each vertex $(-L, y_i)$ on $\bar{\Gamma}_1$, $w_h(L, y_i) = U_2(y_i)$ for each vertex $(L, y_i)$ on $\bar{\Gamma}_2, w_h(x_i, Y_3) = 0$ for each vertex $(x_i, Y_3)$ on $\bar{\Gamma}_3$ and $w_h \geq 0\}$.

THEOREM 2.2

The error in the approximation of (2.13) by (2.15) satisfies

$$|u - u_h|_{1, D_h} \leq Ch. \tag{2.16}$$

*Proof.* We have for any $v_h \in K^h$ that

$$a_h(u - u_h, v_h - u_h) \le a_h(u, v_h - u_h) - l_h(v_h - u_h)$$
$$= \int_{D_h} (-\nabla^2 u + \gamma)(v_h - u_h)\,dx\,dy$$
$$+ \int_{\Gamma_0^h} (1 + c_h'(x)^2)^{1/2}\left(\frac{\partial u}{\partial y} + 1\right)(v_h - u_h)\,d\sigma_h.$$

Since $u_h \ge 0$, we have from (2.14) that

$$(-\nabla^2 u + \gamma)(v_h - u_h) = (-\nabla^2 u + \gamma)(v_h - u) - (-\nabla^2 u + \gamma)u_h$$
$$\le (-\nabla^2 u + \gamma)(v_h - u), \quad \forall v_h \in K^h.$$

Thus, combining the above results with $v_h = u_h^I$ and noting that $-\nabla^2 u + \gamma \in L^2(D)$, we obtain

$$a_h(u - u_h, u_h^I - u_h) \le |-\nabla^2 u + \gamma|_{0,D_h}|u_h^I - u|_{0,D_h}$$
$$+ \max_{x\in[0,L]}\{(1 + c_h'(x)^2)^{1/2}\}\left|\frac{\partial u}{\partial y} + 1\right|_{0,\Gamma_0^h}|u_h^I - u_h|_{0,\Gamma_0^h}.$$

Recalling the interpolation estimate (2.4), the trace theorem (2.7b) and from the Lipschitz continuity of $u_y$ we see that

$$|u - u_h|_{1,D_h}^2 \le a_h(u - u_h, u - u_h^I) + a_h(u - u_h, u_h^I - u_h)$$
$$\le C\{h|u - u_h|_{1,D_h} + h^2\},$$

which yields the desired result (2.16). ■

## 3. NUMERICAL SOLUTION OF THE FREE BOUNDARY PROBLEM

First we report on some numerical computations with the finite element approximation (2.5) to the equation (2.2). For our finite element space $V^h(D_h^*)$ we chose piecewise bilinears on uniform squares with sides of size $h$. The parameters determining the shape of the domain $D$ were chosen: $L = Y_3 = 2, Y_1 = Y_2 = 1$ and $c(x) = x^2/4$. With the data $g = 0, g_0 = (x^2 - 4)$ and

$$f = (8 - 3x^2)\ln(3 - y) + [(4 - x^2)(12 - x^2)/4(3 - y)^2],$$

the solution of (2.1) is

$$u = (4 - x^2)(12 - x^2)\ln(3 - y)/4.$$

Owing to symmetry one can solve the problem on $\{(x, y): 0 < x < L,\ c(x) < y < Y_3\}$. We can see that the error between $u$ and the finite element approximation $u_h$, shown in Table 1 for various values of $h$, satisfies the rate of convergence given in Theorem 2.1.

Table 1. Results for the equation

| $h$ | $\lvert u - u_h\rvert_{1,D_h}$ | $\lvert u - u_h\rvert_{0,D_h}$ | $\max_{\text{nodes } \underline{x}_j\in D} \lvert u(\underline{x}_j) - u_h(\underline{x}_j)\rvert$ |
|---|---|---|---|
| $\frac{2}{4}$ | 1.404 | 0.217 | 0.061 |
| $\frac{2}{6}$ | 0.940 | 0.096 | 0.035 |
| $\frac{2}{8}$ | 0.706 | 0.055 | 0.023 |
| $\frac{2}{10}$ | 0.566 | 0.034 | 0.013 |
| $\frac{2}{12}$ | 0.472 | 0.024 | 0.009 |

Table 2. Position of the free boundary using the variational inequality approximation

| $x$ \ $h$ | $\frac{1}{5}$ | $\frac{1}{10}$ | $\frac{1}{15}$ |
|---|---|---|---|
| 0.0 | 1.381 | 1.377 | 1.375 |
| 0.2 | 1.393 | 1.391 | 1.390 |
| 0.4 | 1.425 | 1.424 | 1.424 |
| 0.6 | 1.479 | 1.473 | 1.479 |
| 0.8 | 1.530 | 1.548 | 1.549 |
| 1.0 | 1.624 | 1.628 | 1.629 |
| 1.2 | 1.712 | 1.711 | 1.708 |
| 1.4 | 1.787 | 1.792 | 1.795 |
| 1.6 | 1.871 | 1.868 | 1.875 |
| 1.8 | 1.930 | 1.943 | 1.945 |

### 3.1. *A variational inequality approximation*

Setting $\gamma = 1$ and $U_1(y) = U_2(y) = (2 - y)^2/2$, the problem is once again symmetric about $x = 0$. With $V^h(D_h^*)$ and $D$ chosen as above, the variational inequality approximation (2.15) was solved using the projected S.O.R. algorithm. As is well known, for $a_h(\cdot, \cdot)$ coercive and symmetric the projected S.O.R. procedure is convergent if the relaxation parameter $\omega \in (0, 2)$. However, in our case, due to the integral along $\Gamma_0^h$, $a_h(\cdot, \cdot)$ is not symmetric. With $\omega = 1$ the procedure in practice converged, but slowly. Attempts at trying to improve the speed of convergence by over-relaxing resulted in divergence for $\omega \geq \omega_0 \in (1, 2)$. It was observed that by setting $\omega = 1$ for those nodes whose associated basis function intersected $\Gamma_0^h$ and choosing $\omega \in (0, 2)$ for the remainder, the algorithm converged in all cases. This allowed the choice of $\omega$ to be optimised for the interior nodes which resulted in a vast improvement in the convergence rate over that achieved with Gauss–Seidel. Some calculated values of the free boundary along the $x = i \times 0.2$ lines $i = 0, 1, \ldots, 9$, are presented in Table 2 for various values of $h$. The position of the free boundary was obtained by quadratically extrapolating to zero the last two significantly positive $u_h$ values on a column of mesh points with fixed $x$ coordinate, using the fact that $u_y = 0$ on $\Gamma$, as described in Elliott and Ockendon (1982). That is, denoting $u_h(ih, jh) = u_h^j$ for fixed $ih$ and with $u_h^j$ the last nonzero mesh value along $x = ih$ one extrapolates using $u_h^j$ and $u_h^{j-1}$ and estimates the position of the free boundary along $x = ih$ to be $jh + h/[(u_h^{j-1}/u_h^j)^{1/2} - 1]$. To smooth out any irregularities caused by very small values of $u_h^j$, one then extrapolates using $u_h^{j-1}$ and $u_h^{j-2}$ if $u_h^j < 0.1\, u_h^{j-1}$.

### 3.2. *A trial free boundary method*

We wish to compare the variational inequality approximation with the TFBM. As stated previously in a trial free boundary approach, for a given guess $\Gamma^{(k)}$ to the unknown boundary $\Gamma$ the elliptic equation is solved by imposing just one of the boundary conditions on $\Gamma^{(k)}$. A new approximation to $\Gamma$ is then obtained, for example, by taking $\Gamma^{(k-1)}$ to be the curve on which the resulting solution satisfies the second boundary condition. This cycle is repeated in the hope that the successive approximations $\Gamma^{(k)}$ will converge. Thus the first point to be decided is which boundary condition should be imposed. From a computational viewpoint it is easier to impose weak rather than essential boundary conditions with the finite element method when using an unfitted mesh. That is, it is easier to impose the Neumann condition, $u_n = 0$, solve the elliptic equation and define the new boundary approximation to be where the resulting solution satisfies the Dirichlet condition, $u \equiv 0$.

With the finite element space chosen to be piecewise bilinears on uniform squares with sides of size $h$, the above procedure is then as follows: given a polygonal boundary $\Gamma^{(k)}$ and defining $\Omega_h^{(k)}$ to be the open polygonal region bounded by $\bar{\Gamma}_0^h \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}^{(k)}$ and $D_h^{(k),*}$ to be the union of elements $\{e\}$ such that $e \cap \Omega_h^{(k)} \neq \{\phi\}$, find

$$u_h^{(k)} \in V_E^{h,(k)} \equiv \{w_h \in V^h(D_h^{(k),*})\colon w_h(-L, y_i) = U_1(y_i)$$
$$\text{for each vertex } (-L, y_i) \text{ on } \bar{\Gamma}_1 \text{ and } w_h(L, y_i) = U_2(y_i)$$
$$\text{for each vertex } (L, y_i) \text{ on } \bar{\Gamma}_2\}$$

such that

$$\begin{aligned} &a_h^{(k)}(u_h^{(k)}, v_h) = l_h^{(k)}(v_h), \\ &\forall v_h \in V_0^{h,(k)} \equiv \{w_h \in V^h(D_h^{(k),*}): w_h = 0 \text{ on } \bar{\Gamma}_1 \cup \bar{\Gamma}_2\}, \end{aligned} \tag{3.1}$$

where

$$a_h^{(k)}(u_h, v_h) = \int_{\Omega_h^{(k)}} \nabla u_h \cdot \nabla v_h \, dx\, dy - \int_{\Gamma_0^h} c_h'(x) \frac{\partial u_h}{\partial \sigma_h} v_h \, d\sigma_h \tag{3.2a}$$

and

$$l_h^{(k)}(v_h) = -\int_{\Omega_h^{(k)}} \gamma v_h \, dx\, dy + \int_{\Gamma_0^h} (1 + c_h'(x)^2)^{1/2} v_h \, d\sigma_h. \tag{3.2b}$$

The new boundary approximation $\Gamma^{(k+1)}$, described by $y = d^{(k+1)}(x)$, can be defined in many ways. The most natural choice is to define it to be the curve on which $u_h^{(k+1)}$ satisfies the second boundary condition, that is by joining the points $\{(x_i, d^{(k+1)}(x_i))\}_{i=0}^{N}$ with straight lines where the lines $x = x_i$ are mesh lines and $d^{(k+1)}(x_i)$ is such $u_h^{(k)}(x_i, d^{(k+1)}(x_i)) = 0$. Unfortunately, starting with $\Gamma^{(0)} \equiv \Gamma_3$ the above TFBM diverges in practice. However, by moving the position of the free boundary using the following defect adjustment,

$$d^{(k+1)}(x_i) = d^{(k)}(x_i) + \alpha^{(k)} u^{(k)}(x_i, d^{(k)}(x_i)), \tag{3.3}$$

the trial free boundary procedure (3.2)–(3.3) converged, although slowly. We call this method TFBM1.

To gain some insight into which boundary condition should be imposed and how the boundary should be adjusted in order to obtain a convergent process one can consider the following model one-dimensional free boundary problem: find $u(x)$ and $s$ such that

$$u_{xx} = 1 \text{ on } (0, s), \quad u(0) = \frac{1}{2} \tag{3.4a}$$

and

$$u(s) = u_x(s) = 0. \tag{3.4b}$$

The above problem has the unique solution $u(x) = \frac{1}{2}(1 - x)^2$ with $s = 1$. Let us consider a trial free boundary procedure applied directly to (3.4), without numerical discretisation, in which we impose the Neumann condition. Given a guess to $s$, denoted by $s^{(k)}$, and solving for $u^{(k)}(x)$ such that

$$u_{xx}^{(k)} = 1 \text{ on } (0, s^{(k)}), \quad u^{(k)}(0) = \frac{1}{2}$$

and

$$u_x^{(k)}(s^{(k)}) = 0.$$

we obtain $u^{(k)}(x) = 1/2x^2 - s^{(k)}x + 1/2$. Updating our approximation to $s$ to be $s^{(k+1)}$, where $u^{(k)}(s^{(k+1)}) = 0$ we obtain the following sequence of approximations to $s$:

$$s^{(k+1)} = s^{(k)} \pm (s^{(k)2} - 1)^{1/2}, \quad k = 0, 1, \ldots$$

Table 3. Position of the free boundary using the trial free boundary approximation

| $x$ \ $h$ | Unfitted mesh $\frac{1}{5}$ | $\frac{1}{10}$ | $\frac{1}{15}$ | Fitted mesh $NX = 40$ $NY = 20$ |
|---|---|---|---|---|
| 0.0 | 1.382 | 1.380 | 1.380 | 1.384 |
| 0.2 | 1.391 | 1.392 | 1.392 | 1.391 |
| 0.4 | 1.425 | 1.427 | 1.427 | 1.426 |
| 0.6 | 1.485 | 1.483 | 1.482 | 1.482 |
| 0.8 | 1.557 | 1.554 | 1.552 | 1.552 |
| 1.0 | 1.628 | 1.633 | 1.633 | 1.632 |
| 1.2 | 1.727 | 1.715 | 1.718 | 1.717 |
| 1.4 | 1.794 | 1.798 | 1.799 | 1.800 |
| 1.6 | 1.885 | 1.881 | 1.878 | 1.879 |
| 1.8 | 1.955 | 1.950 | 1.947 | 1.948 |

assuming $s^{(k)} \geq 1$ and $s^{(0)}$ given. Clearly this sequence is divergent. Adjusting the free boundary using the procedure (3.3) one obtains the following sequence of approximations to $s$,

$$s^{(k+1)} = s^{(k)} + \frac{a^{(k)}}{2}(1 - s^{(k)^2}),$$

which is convergent to 1 provided $0 < a^{(k)}s^{(k)} \leq 2 - \delta$. Thus the conclusions drawn from examining this model problem agree with what was observed in practice for the two-dimensional problem.

The problem was also solved with the same TFBM but using a fitted triangular mesh and piecewise linear basis functions. That is, at each iteration on $\Gamma^{(k)}$ the polygonal region $\Omega_h^{(k)}$ was covered exactly by a union of triangles. The mesh was defined by placing $NY + 1$ equally spaced points on each of $2*NX + 1$ equally spaced vertical lines whose end points lay on $\Gamma_0^{(k)}$ and $\bar{\Gamma}^{(k)}$ between $x = -L$ and $x = L$. Then each row of points was joined to form a union of quadrilaterals covering $\Omega_h^{(k)}$. The triangulation was completed by inserting the diagonal joining the lower left-hand vertex to the upper right-hand vertex of each quadrilateral. We call this procedure TFBM2. Note that at every iteration $(k)$ one is required to triangulate a new region and then calculate a new "stiffness" matrix. This was one of the motivations for introducing unfitted meshes. In the TFBM1 only the equations near the free boundary change at each iteration.

In each of the TFBM's, successive over relaxation was used to solve the equations since a good estimate was available from the previous iteration. The stopping criterion for the SOR iteration was successively refined in order to save computer time.

The iteration was said to have "converged" when the values of $|u^{(k)}|$ on $\Gamma^{(k)}$ were reduced to below $10^{-4}$. Indeed upon subsequent iterations it was found that figures in Table 3 did not change and the values of $u^{(k)}$ on $\Gamma^{(k)}$ could not all be reduced to zero simultaneously. The value $a^{(k)} = 1$ was found to be sufficient for convergence.

We expect that the values in the last two columns of Table 3 are more accurate than those obtained by the variational inequality approach. However, for a given mesh size solving the approximation of the variational inequality involves as much work as solving one elliptic equation. Thus it is the cheapest method and, although the numerical results suggest it is slowly and erratically converging to the solution for the boundary, fairly good accuracy is achieved with a modest amount of computing time. In comparison the TFBM is very expensive. However, for a given mesh size, these numerical experiments suggest that TFBM is likely to be more accurate. The fitted mesh method TFBM2 is more expensive than the unfitted method because of the extra work involved in calculating the new matrix coefficients at each iteration. However, the accuracy of the numerical results in Table 3 suggests that it is unnecessary to use a fitted mesh. We feel that these numerical results together with the analysis of Sec. 2 justify the use of the technique proposed in this paper.

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