Coordinates on $\mathcal{ML}(S)$ (I)

[2018-09-12, Fields]

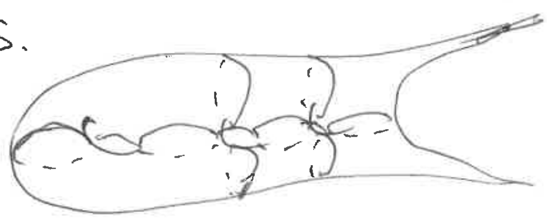
(I) $\mathcal{PML}$, Thurston's compactification, and the Nielsen-Thurston classification of $g \in \mathcal{MCG}(S)$ [and the Brouwer fixed pt thm!]

~~Here is the first goal of these talks. Notation:~~

Notation: $S = S_{g,p}$ is the surface of genus g with p punctures.

$S_{3,2}$

Define: $\xi(S) = 3g-3+p$ the complexity of S. Exercise: this is the number of cuffs in a pants decomposition of S.



$8 = 9-3+2$ ✓

Last time: defined $\mathcal{ML}(S)$.

Note there is a natural action of $\mathbb{R}_{>0}$ on $\mathcal{ML}(S)$. Define

$$\mathcal{PML}(S) = \mathcal{ML}(S) \Big/ \mu \sim r\mu \ \forall r>0.$$

Theorem [Thurston]: (A)

$$\mathcal{PML}(S) \cong S^{2\xi-1}$$
$$\mathcal{ML}(S) \cong \mathbb{R}^{2\xi}$$

This should be compared to

Thm [Fricke-Klein]

$$\mathrm{Teich}(S) \cong \mathbb{R}^{2\xi}.$$

[The history here is not clear. see a'Campo, Ji, Papadopoulos]

Theorem [Thurston] There is a compactification $\overline{T}$ of $\mathrm{Teich}(S)$ so that

(i) $\overline{T} = \mathrm{Teich}(S) \sqcup \mathcal{PML}(S)$

(ii) $\overline{T} \cong B^{2\xi}$, $\partial \overline{T} = \mathcal{PML}(S)$, $(\overline{T})^\circ = \mathrm{Teich}(S)$

(iii) The $\mathcal{MCG}(S)$ actions extend to a continuous action on $\overline{T}$.

Application:

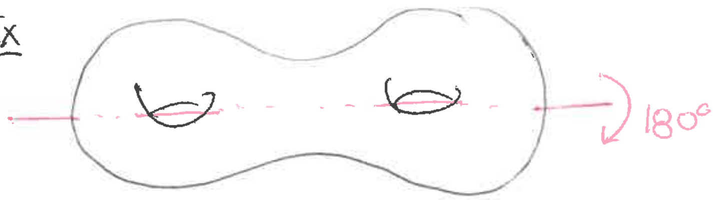
Theorem [Nielsen-Thurston classification].

By the Brouwer fixed point theorem for any $g \in \mathcal{MCG}(S)$, $\mathrm{Fix}_{\overline{T}}(g) \neq \emptyset$. There are several cases.

Case (1): Suppose $\exists X \in \mathrm{Teich} \cap \mathrm{Fix}(g)$

we call g elliptic or periodic

Ex



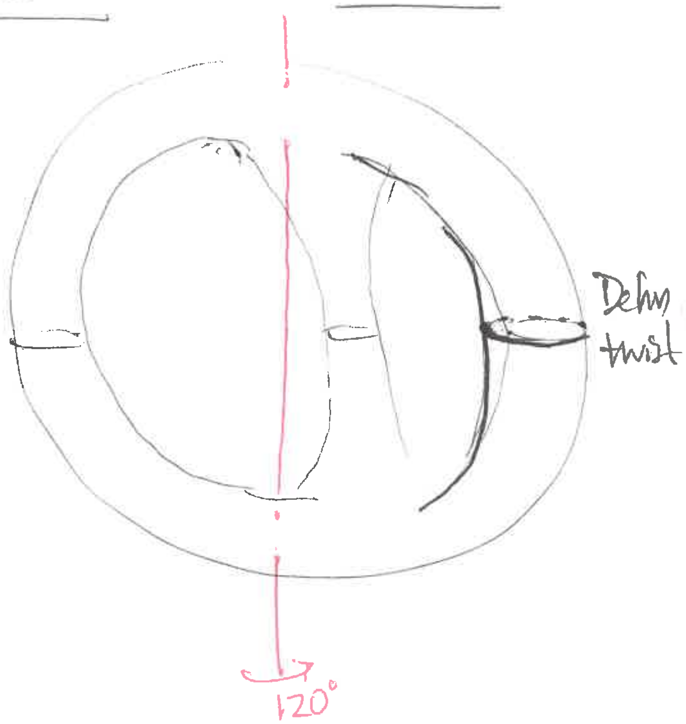
For the remaining cases we assume that $\mathrm{Teich} \cap \mathrm{Fix}(g) = \emptyset$.

Case (2) $\exists \lambda \in \mathrm{Fix}(g)$ which is not filling.

[Def: $\lambda \in \mathcal{ML}(S)$ is filling if for all $\alpha \in \mathcal{S}(S) = \{\text{ess. simple closed curves}\}$ $i(\lambda, \alpha) \neq 0$.]

we call g parabolic or pseudo-hyperbolic or reducible

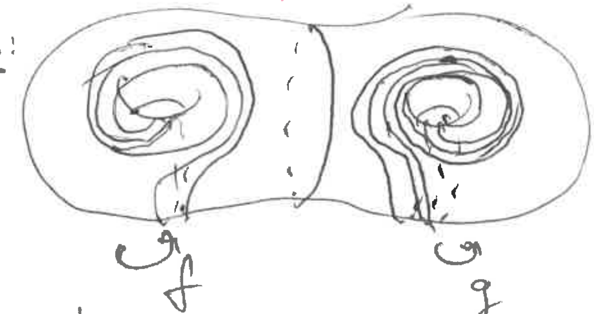
Ex



[Discussion: There are examples where $\mathrm{Fix}(g)$ is finite or infinite $\rightsquigarrow$ To ~~[illegible]~~ understand structure of $\mathrm{Fix}(g)$ break it into "eigenspaces"]

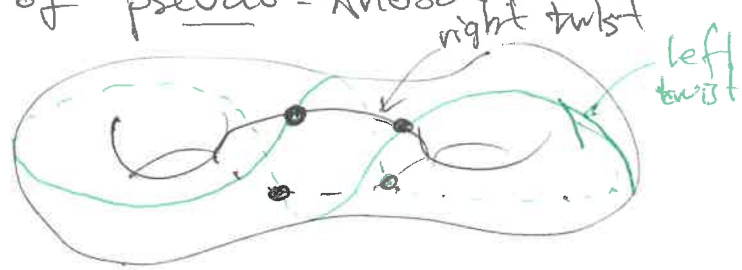
1B

Ex:



Topology on $\mathrm{Fix}(\langle f, g \rangle)$ is sensitive to dilatations of f, g.

Case (3): $\exists \lambda \in \mathrm{Fix}(g)$ which is filling. We call g hyperbolic or pseudo-Anosov



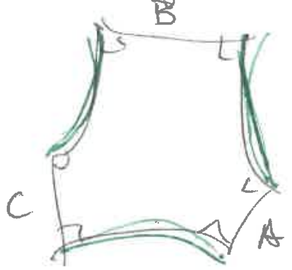
In case (3) $\mathrm{Fix}(g) = \{\lambda^{\pm}(g)\}$ is a pair.

II Fenchel-Nielsen coord's on Teich(S)

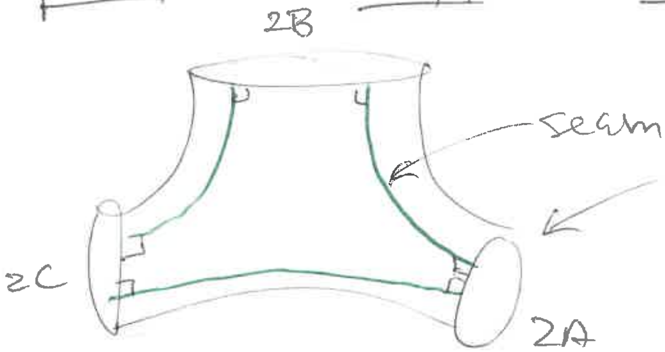
Theorem: $\mathrm{Teich}(S) \cong \underbrace{\mathbb{R}^{3}_{>0}}_{\text{lengths}} \times \underbrace{\mathbb{R}^{3}}_{\text{twists}}$

Rmk: The action of $\mathcal{MCG}(S)$ on these coordinates is real analytic (but pretty complicated!)

Recall: Given three lengths A, B, C there is a unique right-angled hexagon in $\mathbb{H}^2$



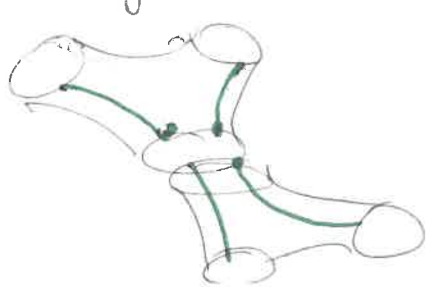
having A, B, C as non-adj. sides. Double across the other boundary edges to get a pants with cuffs and seams



Glue together $|\chi(S)|$ pants $(2g-2+p)$ to obtain S.

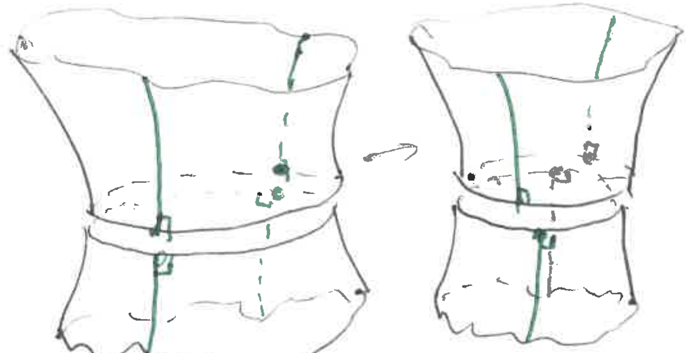
A pants decomposition.

we glue adjacent pants so that the feet of the seams agree. Call this surface X_0



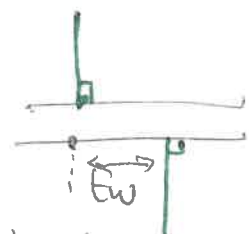
We call X_0 the surface with zero twist

we define the twist $tw_\alpha(X)$ for $X \in \mathrm{Teich}(X_0)$, α a cuff by "analytic continuation"



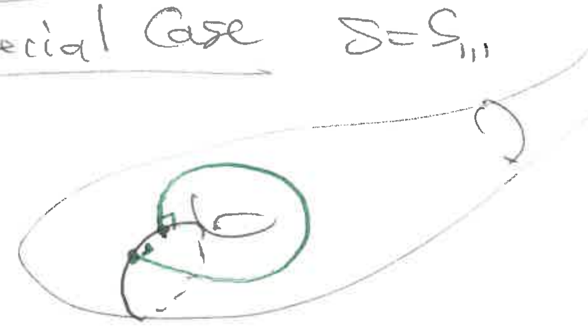
X_0

Local picture



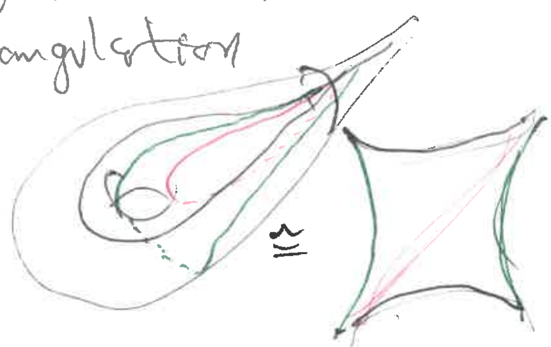
There are details here regarding the marking [by X_0] of X.

Special Case $S = S_{1,1}$

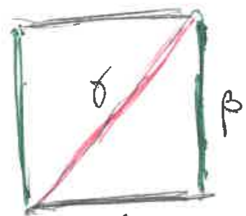


III Edge coordinates for $\mathcal{ML}(S_{1,1})$: Here is an important "digression" before we adapt FN coords to $\mathcal{ML}$.

Equip $S = S_{1,1}$ with an ideal triangulation



$\cong$



Fix $\lambda \in \mathcal{ML}(S)$ and set

$$r = i(\alpha, \lambda)$$
$$s = i(\beta, \lambda)$$
$$t = i(\gamma, \lambda)$$

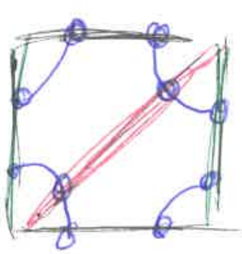
Better names would be:
$a = i(\alpha, \lambda)$
$b = i(\beta, \lambda)$
$c = i(\gamma, \lambda)$

Define $E : \mathcal{ML}(S) \longrightarrow \mathbb{R}^3_{\geq 0}$

$$\lambda \longmapsto (r, s, t)$$

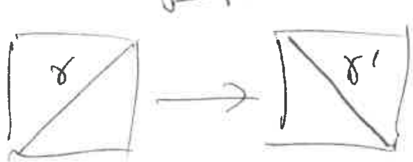
and note that this is cts.

It is not surjective. Eg $(2,2,2)$ is not in the image of E.



The peripheral curve has no geodesic rep.

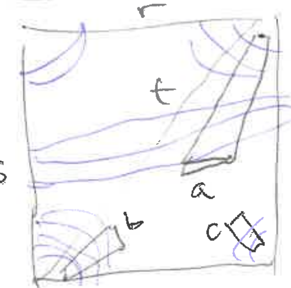
Here is a flip



II

Exercise 1. The change of coords under a flip is piecewise linear.

Corner coordinates



(aka normal coordinates)

Define

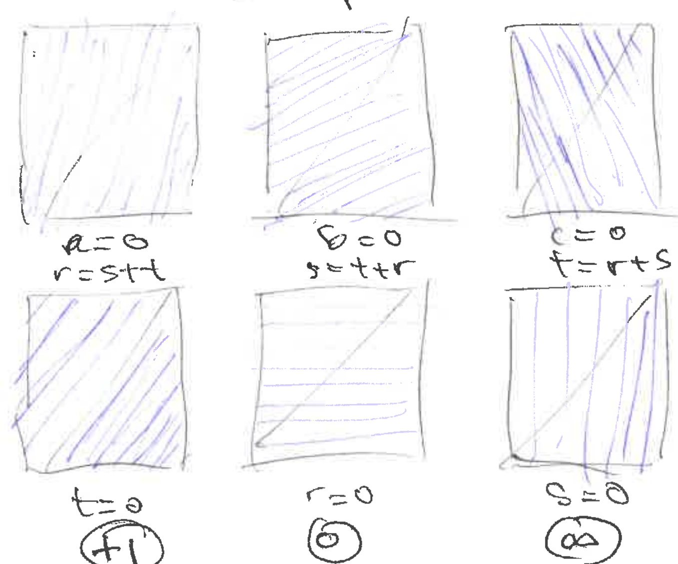
$$a = \frac{s+t-r}{2}$$
$$b = \frac{t+r-s}{2}$$
$$c = \frac{r+s-t}{2}$$

Better names would be
$A = \frac{b+c-a}{2}$
$B = \frac{c+a-b}{2}$
$C = \frac{a+b-c}{2}$

Exercise 2. for any $\lambda \in \mathcal{ML}(S)$

$$\min\{a, b, c\} = 0$$

There are three generic cases and three special cases



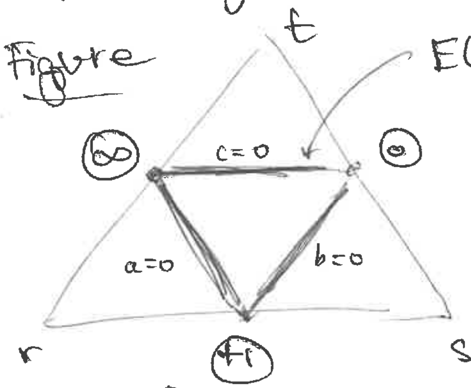
In every case we must

define the twist of λ about α. Again this is denoted $tw_\alpha(\lambda)$.

[The notation should mention β and γ as well!]

Before we do that we use Exercise 2 to draw a picture of Image(E).

Figure



Exercise ③: E is a homeo onto its image, so $PML(S) \cong S^1$.

We may now define the twist

$$tw_\alpha(\lambda) = \begin{cases} +s & \text{if } a=0 \text{ or } b=0 \\ -s & \text{if } c=0 \end{cases}$$

Important remark: if $r=0$ then $b=c=0$

so the sign is not well-defined! This happens when $r = i(\alpha,\lambda)=0$ ...

So define

$$DT : ML(S_{1,1}) \longrightarrow (\mathbb{R}_{\geq 0}, \mathbb{R}) \big/ (0,tw)\sim(0,-tw)$$

$$\lambda \longmapsto [i(\alpha,\lambda), tw_\alpha(\lambda)]$$

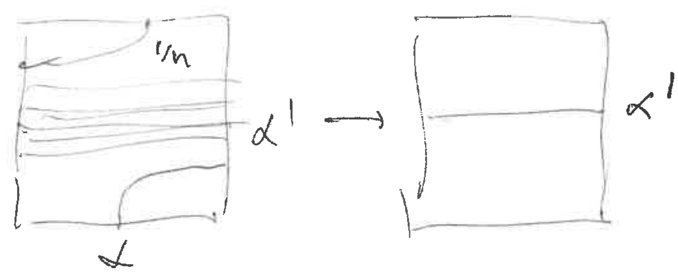
[Really a theorem!]

Exercise ④: This is a homeo.

Exercise ⑤: Compute the change of coords under a flip $\gamma \to \gamma'$.

[What happens if you flip α or β?]

Note that as we twist "about α" and rescale we converge to α' in $ML(S_{1,1})$.

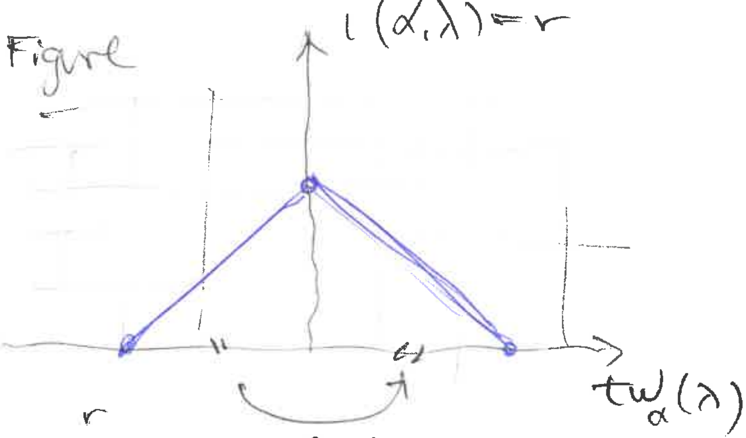


Final remarks: ① DT is homogeneous of degree one

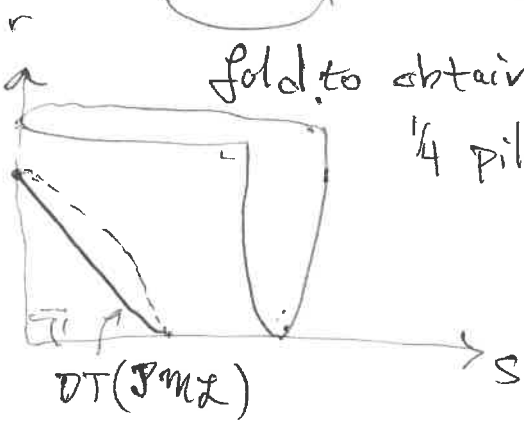
$$DT(a\cdot\lambda) = a\, DT(\lambda)$$

② We can draw $PML(S)$ in these coordinates as follows:

Figure



fold to obtain the $\frac{1}{4}$ pillowcase.



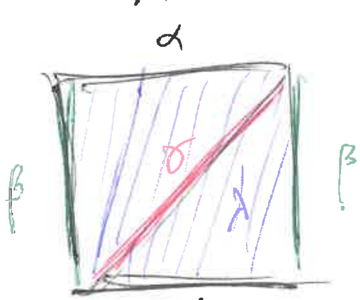
Coordinates on $\mathcal{MF}(S)$ II

[2018-09-17, Fields]

Lightning review: we defined

$$DT : \mathcal{ML}(S_{1,1}) \longrightarrow \frac{\mathbb{R}_{\geq 0} \times \mathbb{R}}{(0,s)\sim(0,-s)}$$

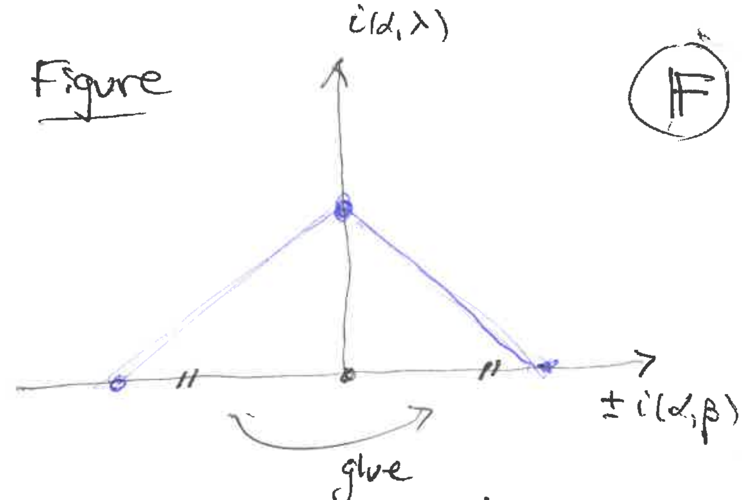
$$\lambda \longmapsto [i(\alpha,\lambda), tw_\alpha(\lambda)]$$



as slope is positive

$$DT(\lambda) = [i(\alpha,\lambda), i(\alpha,\beta)]$$

Figure F



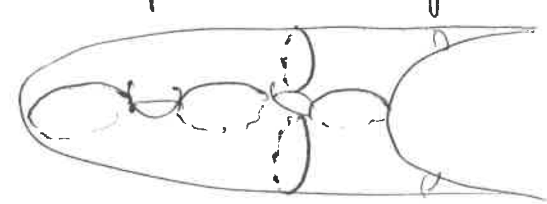
So $\mathbb{P}(DT(\mathcal{ML})) \cong S^1$.

[Various people asked about holomorphic coordinates on Teich(S) - this requires a ~~discussion of~~ lot of other work... see Bers embedding, for example]

IV Dehn-Thurston coordinates

[following FLP]

Fix $A = \{\alpha_i\}$ a pants decomposition of S



Lemma The map

$$\mathcal{ML}(S) \longrightarrow \mathbb{R}^{\xi(S)}_{\geq 0}$$

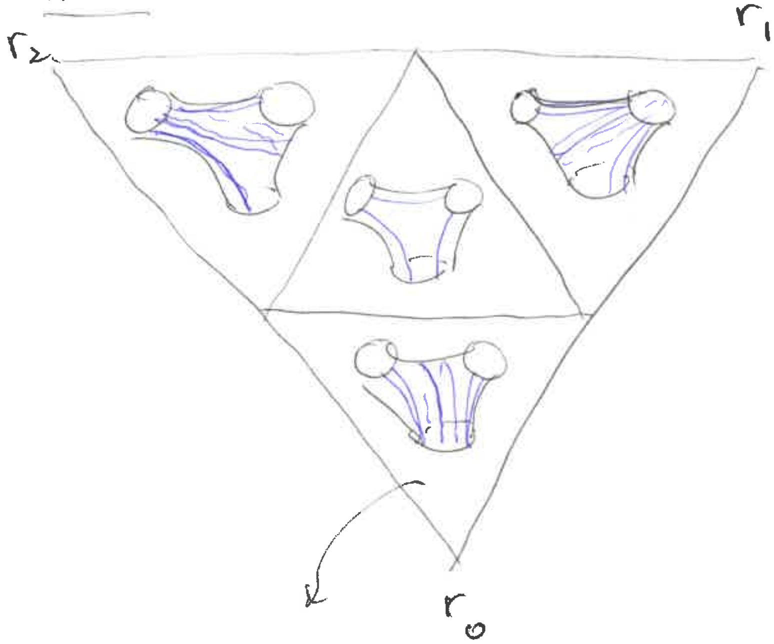
$$\lambda \longmapsto (i(\alpha_i,\lambda))_i$$

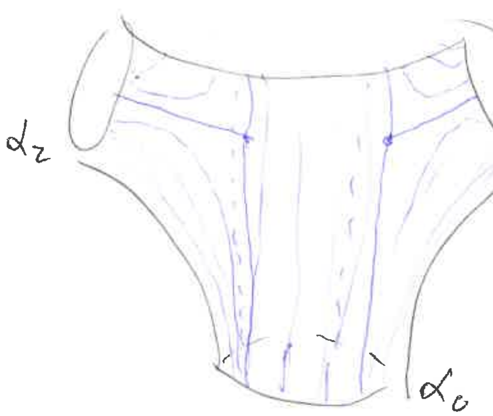
is surjective (and continuous).

Proof sketch space of foliations

we first draw the space of foliations of a pair of pants.

Picture:



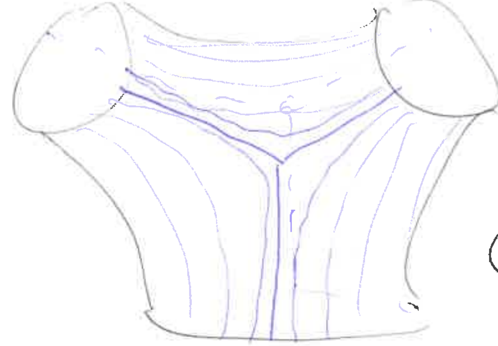


$r_0 > r_1 + r_2$

Generically there are a pair of three-pronged singularities in each pants. So if $p=0$ we find $4g-4$ singularities.

The foliation/lamination has no weight at cusps there we instead have once-pronged (or twice!) singularities.

Figure



all triangle inequalities (strictly!) satisfied.

Glue up: Glue P_i to P_j with any twist you like. This proves the lemma. //

Dual and double dual curves.

Now for the hard part! We record the twisting using curves β_i, γ_i as follows [Rest of talk]

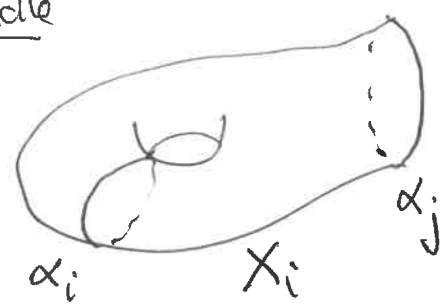
Define X_i to be the non-pants component of

$$S - \{\alpha_j\}_{j \neq i}$$

Better to define $N(\alpha)$ first and $X(\alpha)$ second

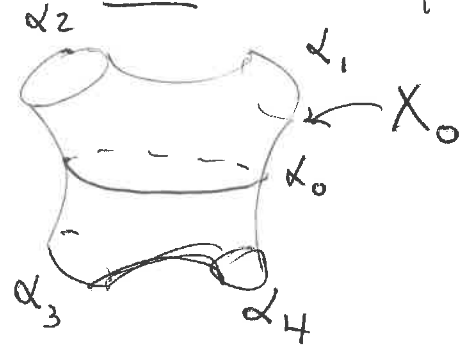
Call $X(\alpha)$ the "Support of α"

Handle



Notation: Also write X_α if suppressing subscripts.

Shirt [double pants?]



Pick $\beta_i \subset X_i$ ess. simp closed curve s.t.

$$(*) \quad i(\alpha_i, \beta_i) = \begin{cases} 1 & X_i \cong S_{1,1} \\ 2 & X_i \cong S_{0,4} \end{cases}$$

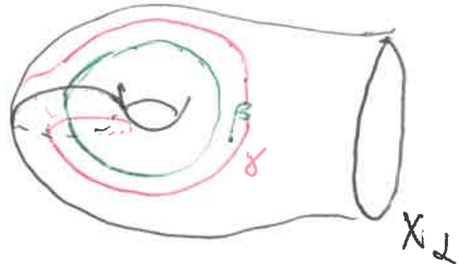
Set

$$(**) \quad \gamma_i = T_{\alpha_i}(\beta_i)$$

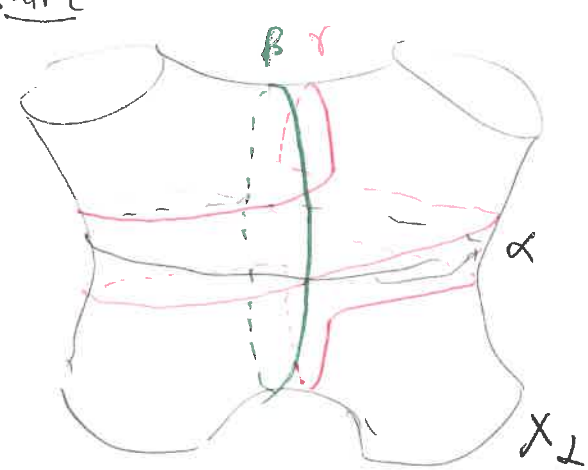
= right Dehn twist of β_i about α_i

Pictures:

Handle



Shirt



~~The annulus of interest~~ (#)

~~Fix an annular neighborhood of $\alpha = \alpha_0$ $N = N(\alpha)$.~~

The annuli of interest

Let $N_i = N(\alpha_i)$ be a regular product neighborhood of α_i

Let

$$\sqcup P_j = S - \sqcup \mathring{N}_i$$

[shrunken pants, shorts?]

Fix attention on $N(\alpha)$ where $X(\alpha)$ is a shirt. [we leave the case of $X(\alpha)$ a handle as an exercise]

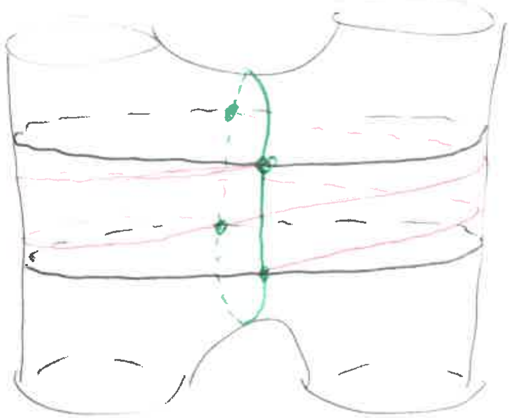
Let $P^\pm$ be the pants in $X(\alpha)$ above and below $N(\alpha)$

[important point: How do we tell P^+ apart from P^-? We must worry about the "hyper-elliptic case. This point seems to be omitted from [FLP] There will be a similar issue later when discussing "front" versus "back"].

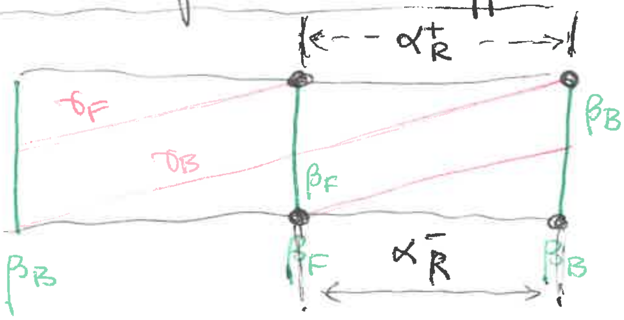
Isotope β and γ in X so that they agree in $P^{\pm}$. Set $\beta^{\pm} = \beta \cap P^{\pm}$.

Picture:

N



Picture of N, unwrapped



Let $\beta \cap N = \beta_F \sqcup \beta_B$

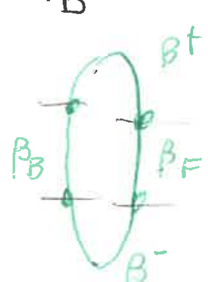
$\gamma \cap N = \gamma_F \sqcup \gamma_B$

Be the two arcs of intersection, in front and in back.

Thus $\beta_F \cup \gamma_F \sim \alpha$

$\beta_B \cup \gamma_B \sim \alpha$ } homotopic

Cartoon of β:



II

[Again there is the delicate issue of telling β_F apart from β_B? [FLP] deals with this by ruling out $X(\alpha_j)$ being a handle. But this is not needed – instead deal with the hyperelliptic directly.]

Define $\alpha^{\pm} = \partial N$ and $\alpha_R^+ \cup \alpha_L^+ = \alpha^+$ as separated by its intersection with β. [See above figure of N unwrapped].

Weights: Fix $\lambda \in \mathcal{ML}(S)$.

Define $a_i = i(\alpha_i, \lambda)$

$b_i = i(\beta_i, \lambda)$

$c_i = i(\gamma_i, \lambda)$.

Define $I : \mathcal{ML}(S) \longrightarrow \mathbb{R}_{\geq 0}^{3\xi}$

$\lambda \longmapsto ((a_i, b_i, c_i)_i)$

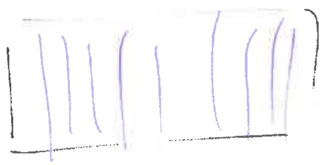
we will use these weights to define the twist $tw_\alpha(\lambda)$.

Normal forms: Isotope λ to be (i) Transverse to N

and (ii) minimize intersection with β^+ and β^-.

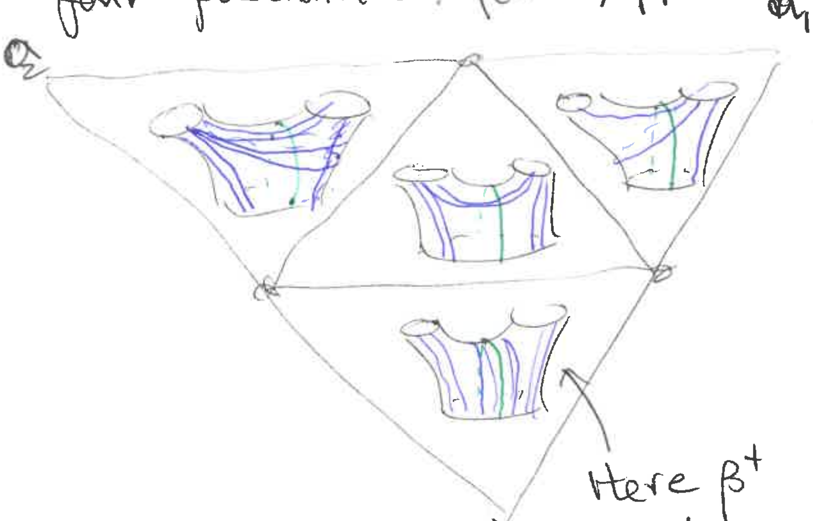
Thinking (as usual) of λ as a foliation we find

(i) $\lambda|N$ has no bigons, no singularities, so gives N the structure of a product



(ii) β^+ is in "efficient position" with respect to $\lambda|P^+$.

Here are cartoons of the four possibilities for $\lambda|P^+$

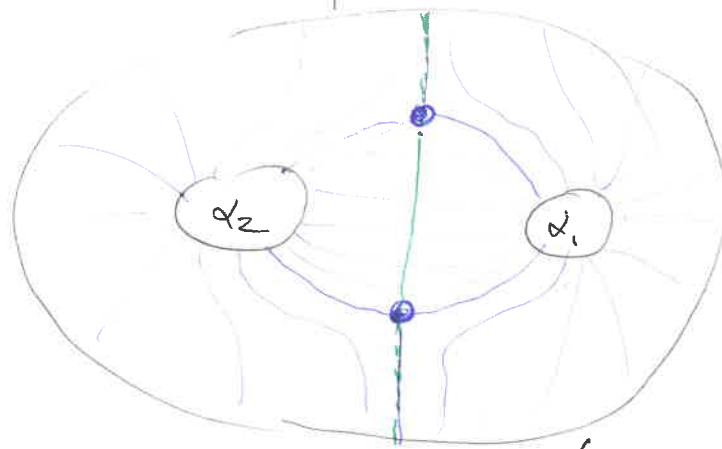


Here β^+ divides the measure of the band evenly.

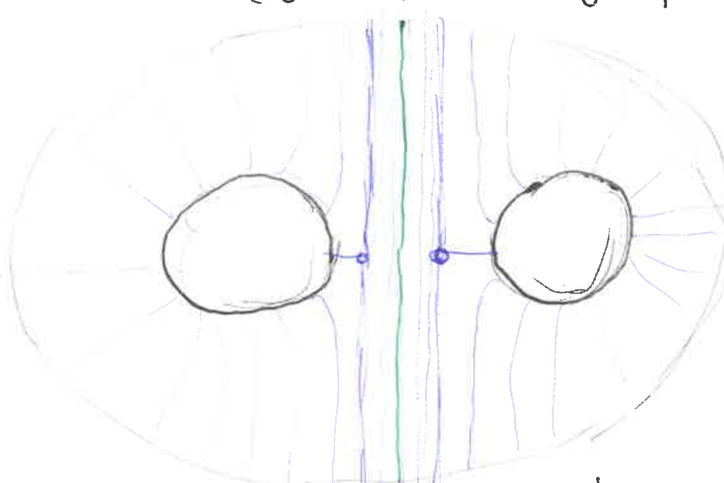
Here are the four cases again, as viewed "from above"

Case 1

$$\left.\begin{array}{l} a_0 < a_1 + a_2 \\ a_1 < a_2 + a_0 \\ a_2 < a_0 + a_1 \end{array}\right\} \text{all } \Delta \text{ inequalities.}$$



Case 2 ~~$a_0 > a_1 + a_2$~~ α_0 $a_0 > a_1 + a_2$

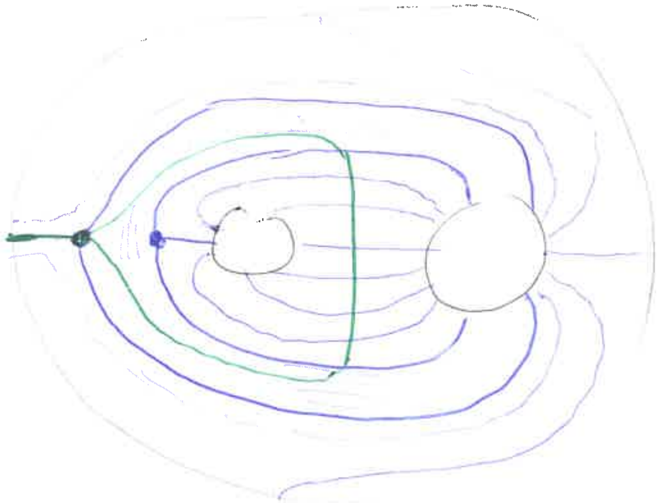


β^+ divides the measure of the band evenly.

~~Case 3~~ ~~$a_1 > a_2 + a_0$~~

$a_1 \not> a_2 + a_0$

Case 3 $a_1 > a_2 + a_2$

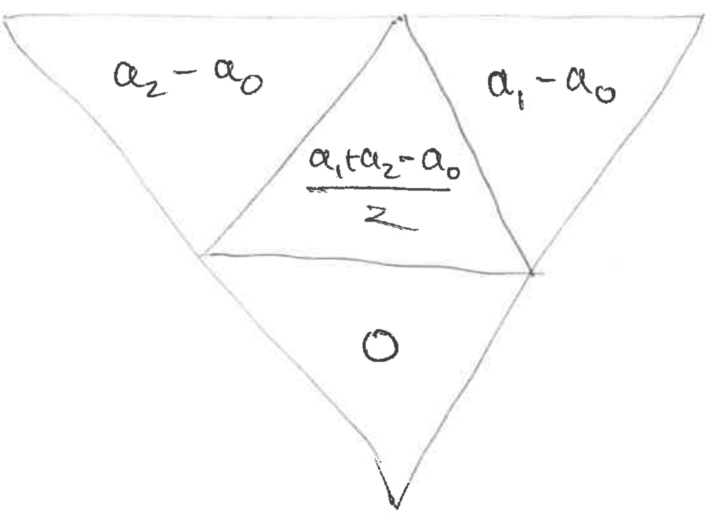


In this case β^+ is no longer embedded.

Upper and lower corrections

We can now define

$b^+ = i(\beta^+, \lambda|P^+)$, the upper correction



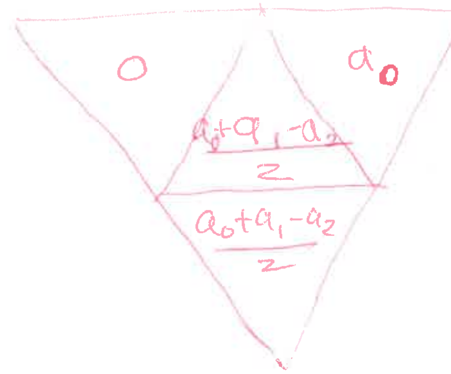
We define the lower correction b^- similarly with α_3, α_4 replacing α_1, α_2. Recall $\beta_N = \beta \cap N$

Define $i(\beta_N, \lambda|N) = b_N = b - (b^+ + b^-)$

Exercise: Write

$a_R^{\pm} = i(\alpha_R^{\pm}, \lambda)$ in terms of a_0, a_1, a_2, a_3, a_4

Solution:



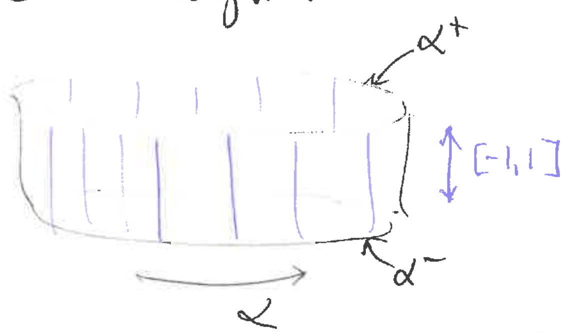
Geometry of N:

Take a homeomorphism

$$N \cong \alpha_0 \times [-1,1]$$

So that the leaves of λN go to vertical arcs of form $\{t\} \times [-1,1]$.
Also the measure of $\lambda|\alpha$ gives arclength.

N



Pull β_N, γ_N tight, rel boundary.

The arcs $\beta_F, \beta_B, \gamma_F, \gamma_B$ receive slopes.

Case analysis [again, following Appendix C of FLP]

Recall that

$$b_N = b_F + b_B = b - (b^+ + b^-)$$

$$c_N = c_F + c_B = c - (b^+ + b^-)$$

[and $c^\pm = b^\pm$]

Also

$$\cancel{a = a_R^+ + a_L^+ = a_R}$$

$$a = a_R^+ + a_L^+ = a_R^- + a_L^-$$

Case I

Lemma: the arcs $\beta_F, \beta_B, \gamma_F, \gamma_B$ all have positive slope in N iff

$$\boxed{c_N = b_N + 2a}$$

Furthermore in this case we have

$$c_F = b_F + a$$

$$c_B = b_B + a$$

and

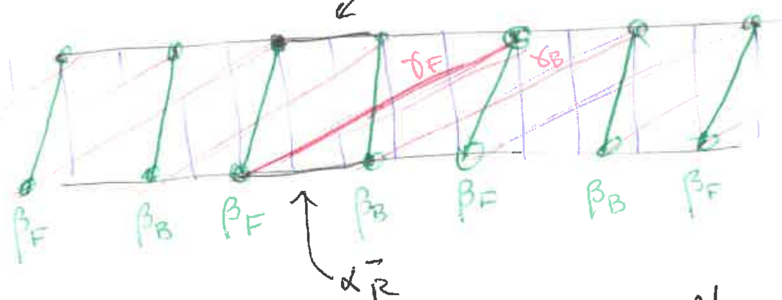
$$b_F + a_R^+ = a_R^- + b_B$$

Corollary: In this case

$$\begin{cases} b_F = \dfrac{c_N + a_R^- - a_R^+}{2} - r \\ c_F = \dfrac{c_N + a_R^- - a_R^+}{2} \end{cases}$$

Proof of Lemma

Picture of $\tilde{N}$



The equalities can be read off of the picture.

The proof of the converse follows from the observation that (as γ is a right twist of β) the slope of γ_F (of γ_B) is greater than that of β_F (of β_B).

Here are the characterising equalities in the other cases.

Case II: $b_N + a_R^+ = a_R^-$

Case III: $a_R^+ = b_N + a_R^-$

Case IV: $b_N + c_N = 2a$

Case V: $c_N + a_R^+ = a_R^-$

Case VI: $c_N + a_R^- = a_R^+$

Case VII: $b_N = c_N + 2a$

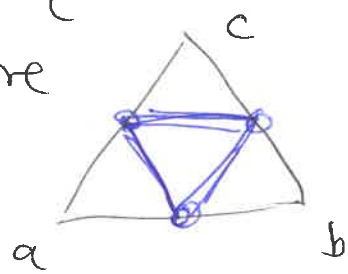
Conclusion: In all cases (for the normal form of $\lambda|P^{\pm}$) we find that the triple

$$(a, b_F, c_F)$$

lies in

$$\nabla = \left\{ (a,b,c) \in \mathbb{R}^3_{\geq 0} \;\middle|\; \begin{array}{l} a+b=c \text{ or} \\ b+c=a \text{ or} \\ c+a=b \end{array} \right\}$$

Picture



Projectively.

We now define $tw_\alpha(\lambda)$ as in the torus case.

Final remark [after chatting with Aaron]

We have defined DT in terms of I

$$\begin{tikzcd}
 & \mathcal{ML}(S) \arrow[dl, "DT"', "\cong"] \arrow[dr, "I"] \arrow[d, phantom, "\circ"] & \\
(\mathbb{R}^{\xi}_{\geq 0} \times \mathbb{R}^{\xi})/\sim \arrow[rr, hook] & & \mathbb{R}^{3\xi}
\end{tikzcd}$$

Most of the work was understanding the inverse of the bottom arrow, i.e. how to define $tw_\alpha(\lambda)$ in terms of $(a_i, b_i, c_i)_i$

Coordinates on $\mathcal{ML}(S)$ III

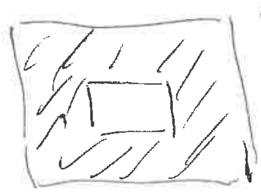
[2018-09-18, Fields]

V Train tracks

Index: Suppose R is a compact, connected, oriented surface with polygonal boundary. At the vertices of ∂R we have a bit of information: inward versus outward

Pictures

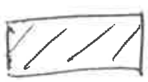
bigon, 2 outward



4 of each type.

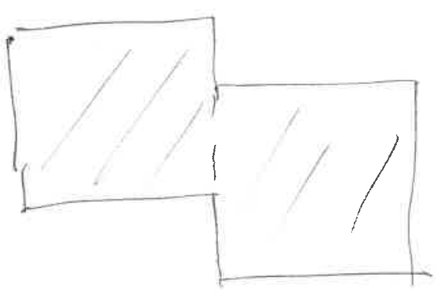
Define: index(R) =

$$= \chi(R) + \frac{1}{4}\left(\#\text{inward} - \#\text{outward}\right)$$

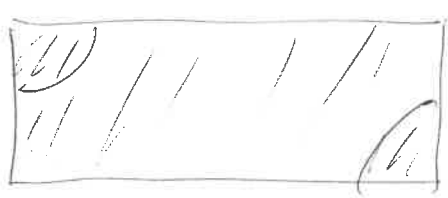
Picture	Name	Index
	disk nullgon	1
	bigon	1/2
	trigon	1/4
	annulus	0
	rectangle	0.

If we glue carefully we find

$$\mathrm{index}(R \cup R') = \mathrm{index}(R) + \mathrm{index}(R')$$



index 0.



$\frac{1}{4}+\frac{1}{4}-\frac{1}{2}=0$

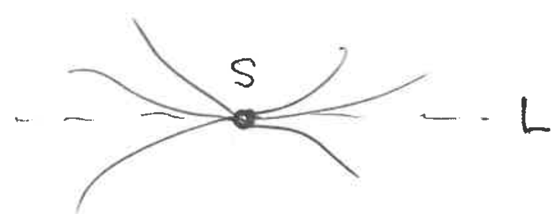
Corollary: If S is a union of rectangles then $\mathrm{index}(S) = 0$.

Pre-tracks:

Suppose $\tau \subset S$ is a smooth, locally finite graph. We call the vertices switches and the edges branches. To be a pre-track every switch $s \in \tau^{(0)}$ must have a tangent line L with

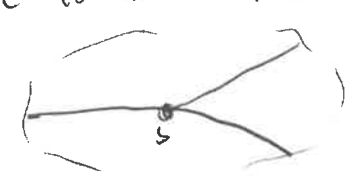
(i) all adj. branches asymptotic to L and

(ii) branches enter s along both components of $L - s$.



[So: No stops allowed]

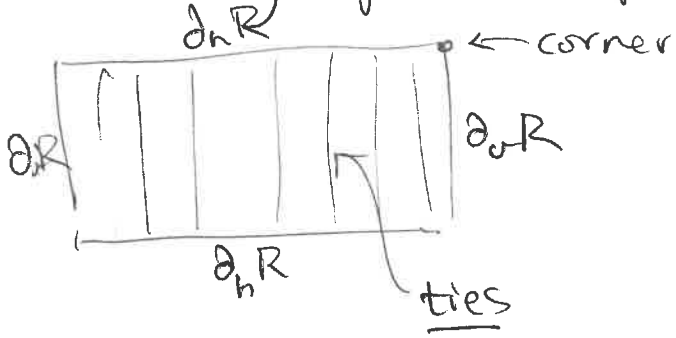
A switch s is generic if it is modelled on



ie has degree 3.

Tie neighborhoods:

the anatomy of a rectangle R



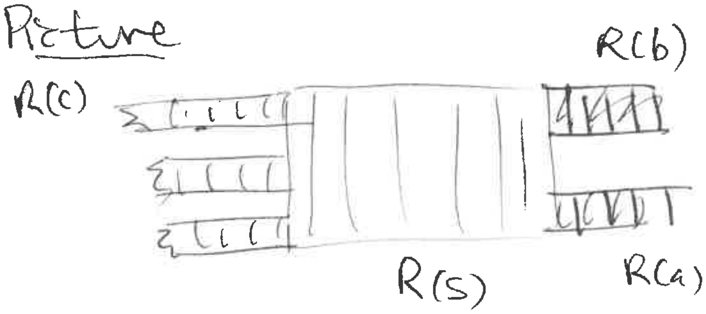
(i) $\partial R = \partial_v R \cup \partial_h R$
(ii) R is foliated by vertical ties.

Define $N(\tau)$ to be the tie neighborhood of τ: we have

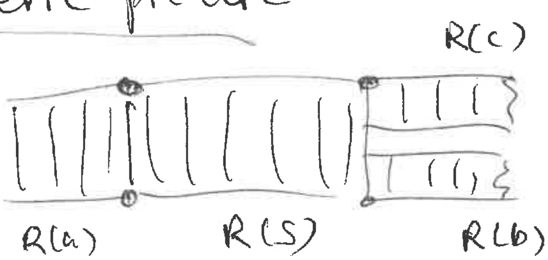
$$N(\tau) = \Big(\bigcup_{s \in \tau^{(0)}} R(s)\Big) \cup \Big(\bigcup_{b \in \tau^{(1)}} R(b)\Big)$$

where we glue $R(b)$ to $R(s)$ if it is incident, and we ensure that $R(b) \cap R(b') = \emptyset$

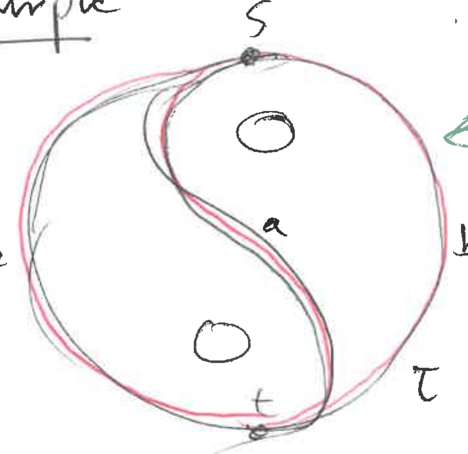
Picture



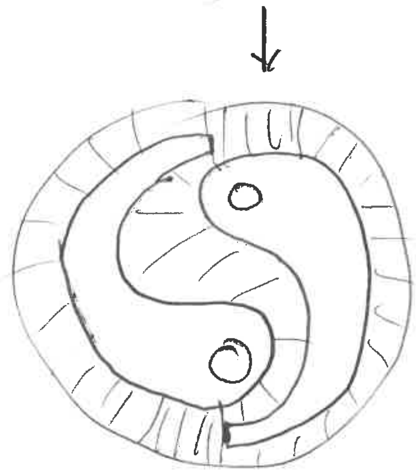
Generic picture



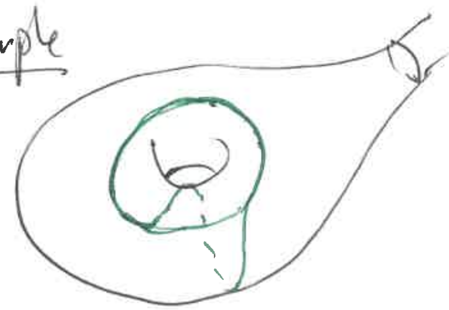
Example



← green is better :)



Example



Tracks: Suppose $\tau \subseteq S$. Set $\mathring{N}(\tau)$ to be the interior of $N(\tau)$. Then τ is a train track if τ is finite (aka compact) and satisfies

P

Index condition

Every component of

$$S - \mathring{N}(\tau)$$

has negative index.

This rules out nullgons, bigons, rectangles, annuli.

Rmk: Sometimes we want to also rule out "smooth horiz boundaries" ie we require a components of $\partial_h N(\tau)$ to meet some corner.

Easy Corollary: If $\tau \subset S$ is a train track then $\chi(S) < 0$ and $S \ncong S_{0,3}$

Train lines, rays, and routes

Suppose I is one of $[0,1]$, $[0,\infty)$, or $\mathbb{R}$.

A map

$$p : I \longrightarrow N(\tau)$$

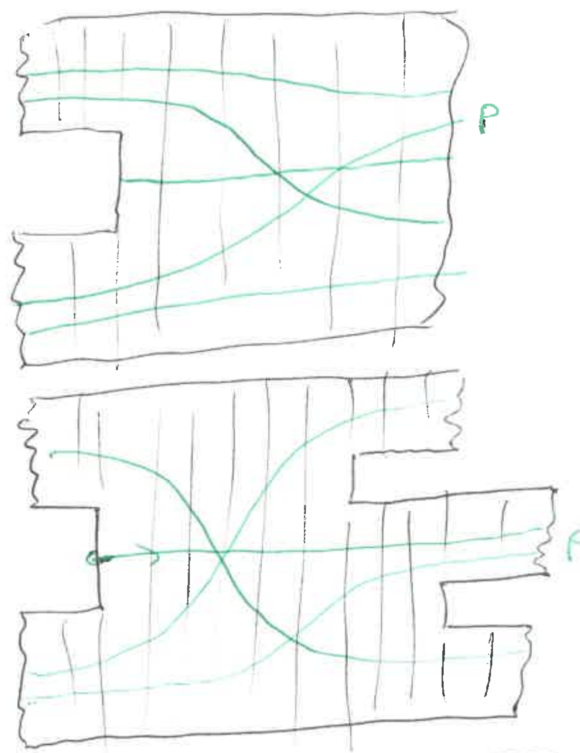
is a train route if (i) $p(\partial I) \subseteq \partial_v N$

(ii) $p(I) \cap \partial_h N = \emptyset$

(iii) p is transverse to the ties of N.

We do not require p to be an embedding.

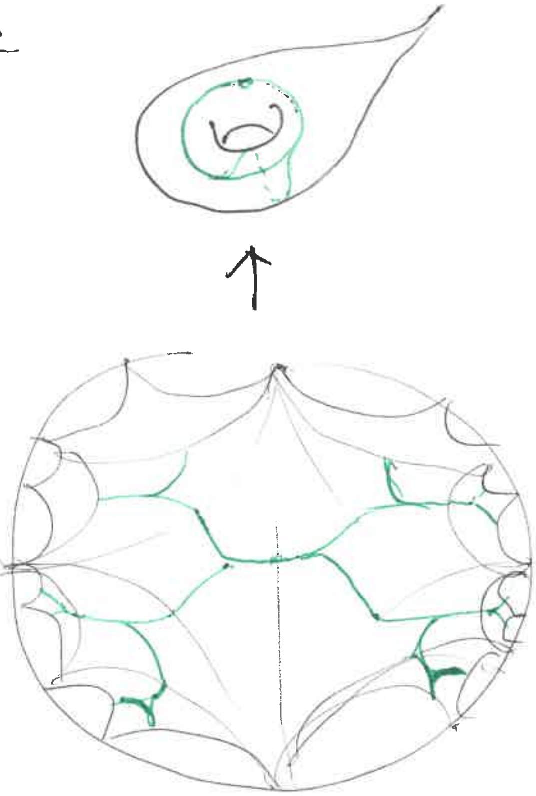
Eg



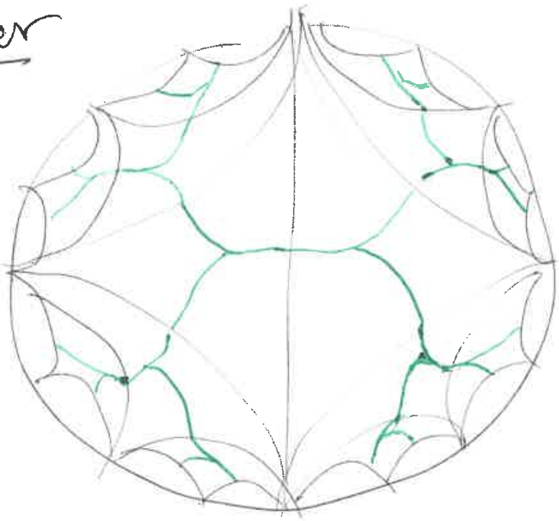
We equip (S, τ) with a combinatorial metric [where it costs 1 to cross a rectangle of N or a region of $S - \mathring{N}$]

Definition: Let $\tau^S \subseteq \tilde{S}$ be the lift of τ to the universal cover of S.

Picture



Better



$\tau^s \subseteq \hat{S}$.

Theorem: Train routes (rays or lines) $\rho : I \to N(\tau^s)$ are uniform quasi-geodesics.

Corollary: Train rays ρ in τ^s have endpoints $\rho(\infty) \in \partial_\infty \tilde{S} = \partial_\infty \pi_1(S)$

$\uparrow$ Bowditch ∂.

Thus train lines ρ in τ^s give geodesics:

$$[\![\rho(-\infty), \rho(\infty)]\!] \in \mathcal{G}(S) = \frac{\partial_\infty(\hat{S}) \times \partial_\infty(\hat{S}) - \Delta}{\text{flip}}.$$

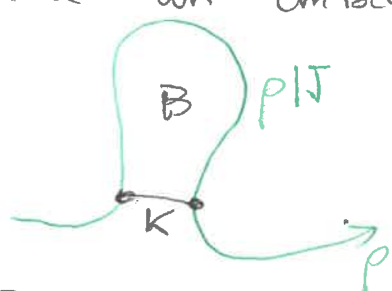
Proof Sketch of Theorem.

[due to Casson?]

Step 1: $\rho : I \to N(\tau^s)$ crosses every rectangle at most once.

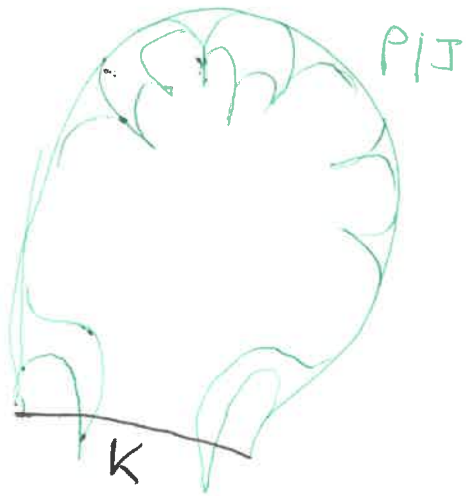
[Exercise]

Step 2: Suppose $J \subset I$ gives a route $\rho|J$ violating the inequality via an embedded bigon



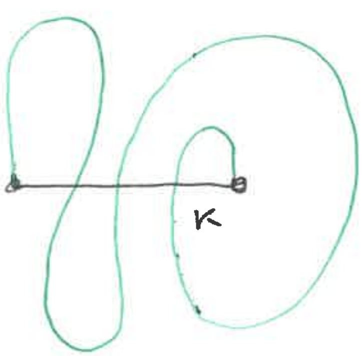
So $\rho|J$ is "long" the arc K is "short" and $\rho|J \cap K = \emptyset$. So $\rho|J$ and K co-bound a bigon (of index ½).

Picture

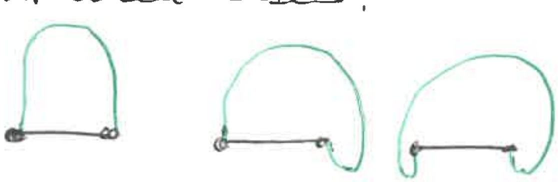


So K crosses a small number of regions - each of these contributes a definite positive at most amount of index. Since $\rho | J$ is long the area of B is large so B has neg. index $\divideontimes$.

Step 3 General case



There are actually three embedded cases.



case			
index	$\frac{1}{2}$	1	$\frac{3}{2}$

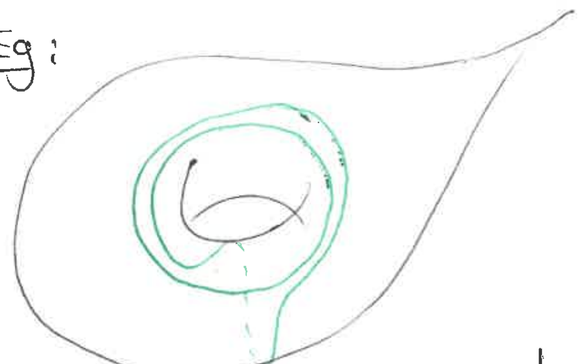
Carried curves:

(R)

If $\alpha \subset S$ a curve, is
(i) embedded ~~contained~~ in $N(\tau)$
(ii) $\alpha \cap \partial N = \emptyset$
(iii) α is transverse to the ties of N

we say α is carried by τ (and we write $\alpha < \tau$)

Eg:



α is carried by the above track.

Corollary [of Quasi-geod Thm]

If $\alpha < \tau$ then α is essential and non-peripheral.

Definition: $\tau \subset S$ is recurrent if for every branch $b \in \tau^{(1)}$ there is a curve $\alpha < \tau$ so that α crosses $R(b)$.

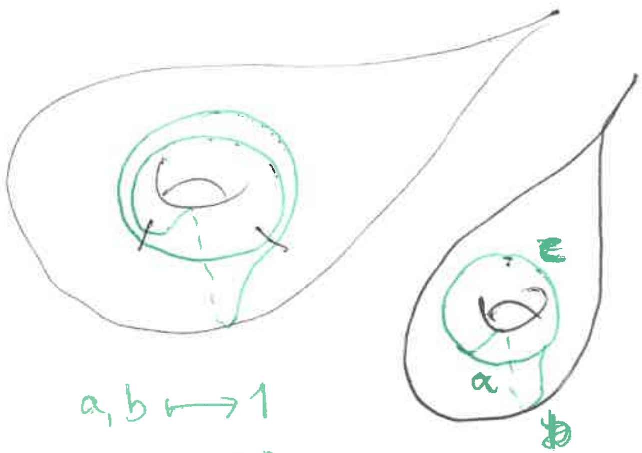
Ex: 

The branch b is not recurrent.

Weights: Suppose $\alpha < \tau$ is a carried curve. We may define a function

$$\mu_\alpha : \tau^{(1)} \longrightarrow \mathbb{N}$$

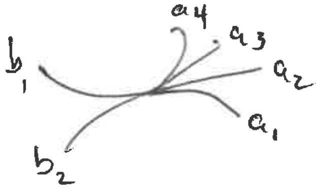
$$b \longmapsto |\alpha \cap R(b)|$$



$a, b \longmapsto 1$

$c \longmapsto 2$

Note that μ_α satisfies the switch equalities:



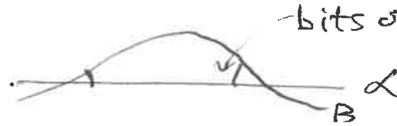
$$\sum_i \mu_\alpha(a_i) = \sum_j \mu_\alpha(b_j)$$

and positivity: $\mu_\alpha(b) \geq 0$ for all b.

Theorem: Suppose $\alpha, \beta < \tau$ are carried curves. Then $\alpha \simeq \beta$ (isotopic) if and only if $\mu_\alpha = \mu_\beta$.

Proof: If $\mu_\alpha = \mu_\beta$ then there is a tie preserving isotopy taking α to β. Conversely suppose $\alpha \simeq \beta$. ~~lift to $\tilde{S}$ and build a fundamental rectangle~~ ~~Picture [assuming $\alpha \cap \beta \neq \phi$]~~

Either $\alpha \cap \beta = \emptyset$ and they cobound an annulus A or $\alpha \cap \beta \neq \emptyset$ and they cobound an innermost bigon B. ~~[index(A) = 0 and index(B) = 2]~~ Since index$(A) = 0$ we deduce $A \subseteq N$ via the usual index argument. The bigon case is left as an exercise.



bits of tie to cut off corner.

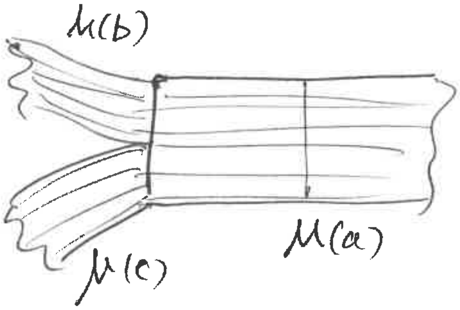
Transverse Measures

We call $\mu : \tau^{(1)} \longrightarrow \mathbb{R}$ a transverse measure on τ if it satisfies the switch equalities and positivity.

Define $M(\tau) \subseteq \mathbb{R}^{|\tau^{1}|}$ to be the core of transverse measures.

Remark: The integral points of $M(\tau)$ are weights of (multi)-curves.

Laminations: Suppose $\mu \in M(\tau)$. We build a partial foliation in S by gluing rectangles (the cusp model)



There are two kinds of leaves: nonsingular and singular leaves.

Nonsingular leaves give train lines. So By Thm. Quasi-geod the nonsingular leaves give a measured lamination.

Theorem: The function

$$\begin{array}{ccc} \Lambda : M(\tau) & \longrightarrow & \mathcal{ML}(S) \\ \mu & \longmapsto & \lambda_\mu \end{array}$$

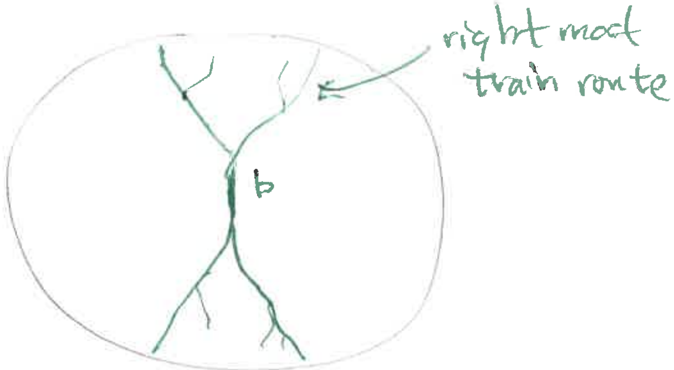
is continuous and injective.

Pf: Continuity is too long to prove now.

~~Splits: We have several local moves on tracks~~

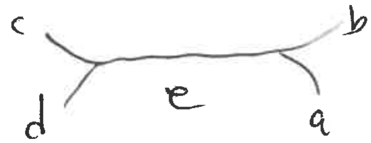
Injectivity is "easy": if $\mu(b) \neq \nu(b)$ then

~~the~~ any tie of b gives a family of geodesics

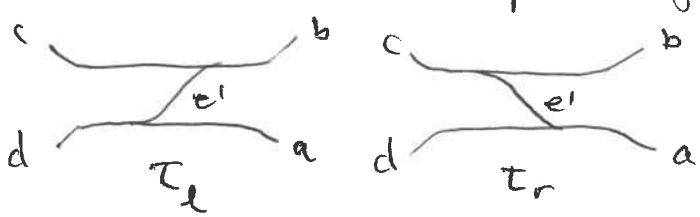


$\Lambda(\mu)$ and $\Lambda(\nu)$ give this family different measure. //

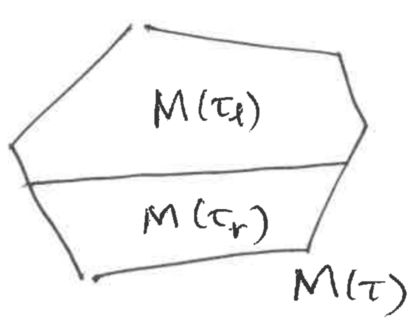
<u>Splits</u>: Suppose ~~e~~ e is a large branch



There are two splits of e



and between them they ~~ext~~ carry all laminations of $M(\tau)$



If e' is the new branch then

left $e = e' + a + c$

right $e = e' + b + d$

So the change of coordinates is <u>linear</u>

[and this is so nice it can actually be used to compute!]