The complex AGM and periods of elliptic curves over $\mathbb{C}$

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Plan

1. Introduction and statement of the problem
2. AGM sequences, lattice chains and isogeny chains
3. Periods of elliptic curves over $\mathbb{C}$
4. Complex elliptic logarithms

Joint work with Thotsaphon Thongjunthug (Khon Kaen, Thailand)

See `http://arxiv.org/abs/1011.0914` and JNT 133 (2013), 2813–2841.

We will study three related classes of objects:

- Complex AGM sequences (first studied by Gauss!)
- Chains of lattices in $\mathbb{C}$
- Chains of 2-isogenies between elliptic curves over $\mathbb{C}$

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in order to give efficient computational solutions to these questions:

1. How can we compute a basis for the period lattice Λ of an elliptic curve E defined over $\mathbb{C}$, given by a Weierstrass equation?
2. Given a point $P = (x, y) \in E(\mathbb{C})$, how can we compute its elliptic logarithm $z \in \mathbb{C} \pmod{\Lambda}$?

The real AGM

Let a, b be *positive real* numbers. Set $a_0 = a$, $b_0 = b$, and for all $n \geq 0$,

$$a_{n+1} = \frac{1}{2}(a_n + b_n), \quad b_{n+1} = +\sqrt{a_n b_n}.$$

Then $\lim a_n$ and $\lim b_n$ both exist and are equal. Their common value is the *Arithmetic-Geometric Mean* $M(a, b)$.

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The AGM is well known and has been used for centuries in evaluating (real) elliptic integrals. For example:

$$\int_0^{\pi/2} \frac{dx}{\sqrt{a^2 \cos^2 x + b^2 \sin^2 x}} = \frac{\pi/2}{M(a, b)}.$$

Complex AGM sequences

We now consider pairs $a, b \in \mathbb{C}$ such that $ab(a^2 - b^2) \neq 0$.

A pair is *good* if $|a - b| \leq |a + b|$, or equivalently $\Re(a/b) \geq 0$.

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An *AGM sequence* is a sequence $((a_n, b_n))_{n=0}^{\infty}$, whose pairs $(a_n, b_n) \in \mathbb{C}^2$ satisfy

$$2a_{n+1} = a_n + b_n, \quad b_{n+1}^2 = a_n b_n$$

for all $n \geq 0$.

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There are uncountably many AGM sequences starting with (a_0, b_0).

$a_{n+1} = (a_n + b_n)/2$ and $b_{n+1} = \pm\sqrt{a_n b_n}$, with either choice of sign at each step.

Good sequences and optimality

An AGM sequence is

- *good* if (a_n, b_n) is good for all but finitely many n, else *bad*;
- *optimal* if (a_n, b_n) is good for all $n > 0$;
- *strongly optimal* if (a_n, b_n) is good for all $n \geq 0$;

For every starting pair (a_0, b_0) there is *exactly one* optimal AGM sequence, unless a_0/b_0 is real and negative, in which case there are two, with different signs of b_1.

These have the property that the ratios a_n/b_n in one of the sequences are the complex conjugates of those in the other.

Limits of AGM sequences

All AGM sequences have limits. More precisely:

For every AGM sequence $((a_n, b_n))_{n=0}^{\infty}$ starting at (a_0, b_0):

1. $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and are equal;
2. The common limit M is non-zero iff the sequence is good;
3. $|M|$ attains its maximum iff the sequence is optimal.

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The first two parts of this are elementary. The third (harder) implies

$$|M(a,b)| \geq |M(a,-b)| \iff |a-b| \leq |a+b|$$

where $M(a,b)$ denotes the optimal value.

Lattices and lattice chains

A *lattice* is a discrete free rank 2 $\mathbb{Z}$-module in $\mathbb{C}$.

A *lattice chain* is an infinite nested sequence of lattices

$$\Lambda_0 \supset \Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_n \supset \ldots$$

such that $[\Lambda_n : \Lambda_{n+1}] = 2$ and $\Lambda_{n+1} \neq 2\Lambda_{n-1}$ for all $n \geq 1$ (so Λ_0/Λ_n is cyclic of order 2^n).

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For each $n \geq 1$ we have $\Lambda_{n+1} = \langle w \rangle + 2\Lambda_n$ for some $w \in \Lambda_n \setminus 2\Lambda_{n-1}$.

Given Λ_0, there are *three* possibilities for Λ_1, and then *two* choices for Λ_n for $n \geq 2$.

The number of such chains starting with Λ_0 is uncountable.

Good lattice chains and limiting periods

Let

$$\Lambda_\infty = \bigcap_{n=0}^{\infty} \Lambda_n.$$

Then one of two possibilities occurs (since Λ_∞ has infinite index):

- if $\Lambda_\infty = \{0\}$, the chain is *bad*;
- if Λ_∞ is free of rank 1, the chain is *good*.

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In a good chain, $\Lambda_\infty = \langle w_\infty \rangle$ for some *primitive* period w_∞, called a *limiting period* of the chain. We then have $\Lambda_n = \langle w_\infty \rangle + 2^n \Lambda_0$.

Good and bad choices in lattice chains

$\Lambda_{n+1} \subset \Lambda_n$ is the *right choice* of sublattice of Λ_n if $\Lambda_{n+1} = \langle w \rangle + 2\Lambda_n$ where w is a *minimal* element in $\Lambda_n \setminus 2\Lambda_{n-1}$ (with respect to the usual complex absolute value).

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For a good chain $(\Lambda_n)_{n=0}^{\infty}$, the limiting period w_∞ is minimal in Λ_n for all but finitely many $n \geq 0$.

A chain is good if and only if $\Lambda_{n+1} \subset \Lambda_n$ is the right choice for all but finitely many $n \geq 1$.

Optimal chains

A lattice chain is *optimal* if $\Lambda_{n+1} \subset \Lambda_n$ is the right choice for *all* $n \geq 1$.

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There is usually *one* optimal chain for each of the three choices of Λ_1.

More precisely, a good chain is optimal if and only if w_∞ is a *minimal coset representative* of $2\Lambda_0$ in Λ_0; and the only situation in which minimal coset representatives are not unique (up to sign) is for *rectangular lattices* where the "diagonal" coset has a pair of minimal representatives (up to sign).

When Λ_0 is rectangular with orthogonal basis w_1, w_2 there are *four* optimal chains, including *two* with $\Lambda_1 = \langle w_1 + w_2 \rangle + 2\Lambda_0$.

Coset representatives and $\mathbb{Z}$-bases

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Every rectangular lattice Λ has precisely *four* optimal sublattice chains.

We will need these results to show that our AGM-based algorithm not only finds individual primitive periods, but actually gives a $\mathbb{Z}$-basis for the period lattice. This uses one more easy fact:

For $j = 1, 2, 3$, let w_j be minimal coset representatives for 2Λ in Λ. Then any two of the w_j form a $\mathbb{Z}$-basis for Λ.

Level 4 structure

We link AGM sequences with lattice chains via level 4 structure on elliptic curves and 2-isogenies.

There are bijections between the following sets:

1. "short" lattice chains $\Lambda_0 \supset \Lambda_1 \supset \Lambda_2$ with Λ_0/Λ_2 cyclic of order 4;
2. triples (E, ω, H) where E is an elliptic curve defined over $\mathbb{C}$, ω a differential on E, and $H \subset E(\mathbb{C})$ a cyclic subgroup of order 4;
3. unordered pairs of nonzero complex numbers a, b with $a^2 \neq b^2$, where the pairs a, b and $-a, -b$ are identified.

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$(1) \leftrightarrow (2)$ is clear. For $(2) \leftrightarrow (3)$ we use the elliptic curve

$$E_{\{a,b\}}: \quad Y^2 = 4X(X + a^2)(X + b^2)$$

on which $P_{\{a,b\}} = (ab, 2ab(a + b))$ has order 4 and $2P = T = (0, 0)$.

Modular functions

Up to homothety our level 4 structures are parametrized by points τ in the affine modular curve $Y_0(4) = \Gamma_0(4)\backslash\mathcal{H}$ where $\mathcal{H}$ is the upper half-plane.

The ratio a/b is a modular function (of τ).

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The ratio a/b is a modular function (of τ). In fact,

- $\tau \mapsto \kappa(\tau) = a/b$ is a hauptmodul for $\Gamma_0(4) \cap \Gamma(2)$;
- $\tau \mapsto \lambda(\tau) = a^2/b^2$ is a hauptmodul for $\Gamma(2)$;
- $\tau \mapsto f(\tau) = a/b + b/a$ is a hauptmodul for $\Gamma_0(4)$;
- $$f(\tau) = 2(1 + \lambda(2\tau))/(1 - \lambda(2\tau))$$

  where $\lambda(\tau)$ is the classical Legendre elliptic function on $\Gamma(2)$.

Application of modular functions

By studying the values taken by the modular function f on the upper half-plane, one can prove:

Theorem

Let $\Lambda_0 \supset \Lambda_1 \supset \Lambda_2$ be a short lattice chain corresponding to the unordered pair $\{a, b\}$ and modular parameter $f = a/b + b/a$.
The following are equivalent:

1. *Λ_2 is the right choice of sublattice of Λ_1;*
2. *the pair (a, b) is good;*
3. $\Re(f) \geq 0$.

The link: isogeny chains

Let Λ_0 be the period lattice of an elliptic curve $E_0/\mathbb{C}$ with Weierstrass equation

$$E_0: \quad Y_0^2 = 4(X_0 - e_1^{(0)})(X_0 - e_2^{(0)})(X_0 - e_3^{(0)}).$$

Let

$$a_0 = \pm\sqrt{e_1^{(0)} - e_3^{(0)}}, \quad b_0 = \pm\sqrt{e_1^{(0)} - e_2^{(0)}}.$$

Fixing the order and signs of a_0, b_0 corresponds to fixing a point P of order 4 on E_0 with $2P = T = (e_1^{(0)}, 0)$, and to fixing a short lattice chain $\Lambda_0 \supset \Lambda_1 \supset \Lambda_2$.

AGM sequences $((a_n, b_n))_{n=0}^{\infty}$ starting from (a_0, b_0) now correspond to lattice chains (Λ_n) with the same three starting terms.

The link: isogeny chains

For each $n \geq 0$ let

$$e_1^{(n)} = \frac{a_n^2 + b_n^2}{3}, \quad e_2^{(n)} = \frac{a_n^2 - 2b_n^2}{3}, \quad e_3^{(n)} = \frac{b_n^2 - 2a_n^2}{3}.$$

and E_n the curve with equation $Y_n^2 = 4(X_n - e_1^{(n)})(X_n - e_2^{(n)})(X_n - e_3^{(n)})$.

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Λ_n is the period lattice of E_n and there are 2-isogenies $\varphi_n : E_n \to E_{n-1}$ induced by $\mathbb{C}/\Lambda_n \to \mathbb{C}/\Lambda_{n-1}$, which fit together to form an isogeny chain:

$$\cdots \longrightarrow E_n \xrightarrow{\varphi_n} E_{n-1} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0$$

where $\varphi_n((e_1^{(n)}, 0)) = (e_1^{(n-1)}, 0)$ for all $n \geq 1$.

The limit

Assume that the AGM sequence (a_n, b_n) is good, with nonzero limit M. Then

$$\lim_{n\to\infty} e_1^{(n)} = \frac{2}{3}M^2; \qquad \lim_{n\to\infty} e_2^{(n)} = \lim_{n\to\infty} e_3^{(n)} = \frac{-1}{3}M^2.$$

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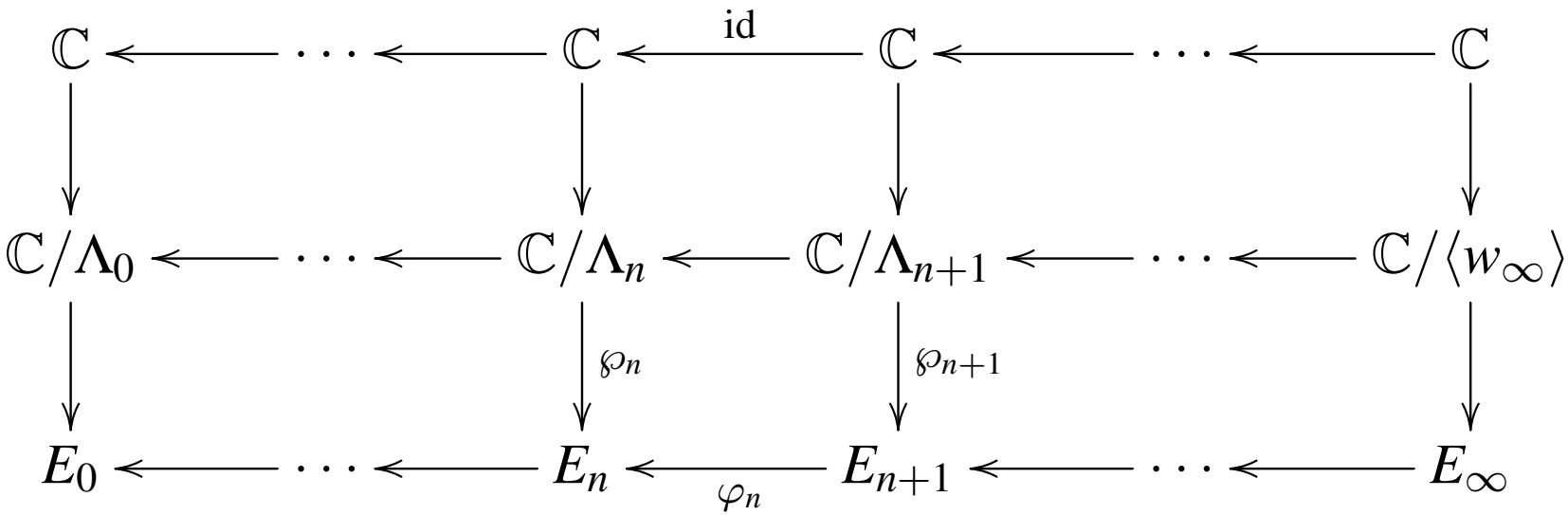
Equivalently, the lattice chain (Λ_n) is good, with limiting period w_∞:



where E_∞ is the singular curve $Y^2 = 4(X - 2M^2/3)(X + M^2/3)^2$.

Summary so far

We have established bijections between three sets:

1. all AGM sequences starting at (a_0, b_0);
2. all isogeny chains starting with the short chain $E_2 \to E_1 \to E_0$;
3. all lattice chains starting with the short chain $\Lambda_0 \supset \Lambda_1 \supset \Lambda_2$.

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such that for all n,

1. $E_n \cong E_{\{a_n,b_n\}} \cong \mathbb{C}/\Lambda_n$;
2. $\Lambda_n \supset \Lambda_{n+1} \supset \Lambda_{n+2}$ is a short chain;
3. Λ_{n+2} is the right choice of sublattice of Λ_{n+1} if and only if (a_n, b_n) is a good pair;
4. the lattice chain (Λ_n) is good (respectively, optimal) if and only if the sequence $((a_n, b_n))$ is good (respectively, optimal).

Computing periods via AGM

We now show how every primitive period w_1 of E_0 may be expressed in terms of the limit of a suitable AGM sequence.

w_1 determines a good lattice chain with $\Lambda_n = \langle w_1 \rangle + 2^n \Lambda_0$ (and conversely, since $\cap_n \Lambda_n = \langle w_1 \rangle$).

The lattice chain in turn determines a good AGM sequence $((a_n, b_n))$ starting at a pair (a_0, b_0) such that $E_0 \cong E_{\{a_0, b_0\}}$.

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Theorem

Let (Λ_n) be a good lattice sequence with limiting period w_1 (generating $\cap \Lambda_n$, and defined up to sign). Then for all $z \in \mathbb{C} \setminus \Lambda_0$ we have

$$\lim_{n\to\infty} \wp_{\Lambda_n}(z) = \left(\frac{\pi}{w_1}\right)^2 \left(\frac{1}{\sin^2(z\pi/w_1)} - \frac{1}{3}\right)$$

$$\lim_{n\to\infty} \wp'_{\Lambda_n}(z) = -2\left(\frac{\pi}{w_1}\right)^3 \left(\frac{\cos(z\pi/w_1)}{\sin^3(z\pi/w_1)}\right).$$

The period formula

Taking $z = w_1/2$ we find:

Corollary

In the above notation, let (Λ_n) be a (good) lattice chain, with limiting period w_1, associated to the elliptic curve E_0 and the (good) AGM sequence $((a_n, b_n))$ with non-zero limit M. Then $M = \pm\pi/w_1$, so that the period w_1 may be determined up to sign by

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Choosing different good AGM sequences we obtain all primitive periods in the coset $w_1 + 4\Lambda_0$; the optimal sequence's limit gives the minimal such period.
Choosing the sign of b_0 so that (a_0, b_0) is also good, the strongly optimal AGM limit gives the minimal period in the coset $w_1 + 2\Lambda_0$.

Another corollary

Corollary

$|\mathrm{AGM}(a_0, b_0)|$ *attains its maximum among all limits of AGM-sequences starting at* (a_0, b_0) *if and only if the sequence is optimal.*

Conclusion: computing periods I

Let E be an elliptic curve over $\mathbb{C}$ given by the Weierstrass equation

$$Y^2 = 4(X - e_1)(X - e_2)(X - e_3),$$

with period lattice Λ. Set

$$a_0 = \sqrt{e_1 - e_3}, \quad b_0 = \sqrt{e_1 - e_2},$$

where the signs are chosen so that (a_0, b_0) is good, i.e.,

$$|a_0 - b_0| \leq |a_0 + b_0|,$$

and let

$$w_1 = \frac{\pi}{\mathrm{AGM}(a_0, b_0)},$$

using the optimal value of the AGM. Then w_1 is a primitive period of E, and is a minimal period in its coset modulo 2Λ.

Define w_2, w_3 similarly by permuting the e_j; then any two of w_1, w_2, w_3 form a $\mathbb{Z}$-basis for Λ.

Conclusion: computing periods II

Let E be an elliptic curve over $\mathbb{C}$ given by the Weierstrass equation

$$Y^2 = 4(X - e_1)(X - e_2)(X - e_3),$$

with period lattice Λ. Order the roots (e_1, e_2, e_3) of E, so that the signs of $a = \sqrt{e_1 - e_3}$, $b = \sqrt{e_1 - e_2}$, $c = \sqrt{e_2 - e_3}$ may be chosen to satisfy

$$|a - b| \leq |a + b|, \quad |c - ib| \leq |c + ib|, \quad |a - c| \leq |a + c|.$$

Define

$$w_1 = \frac{\pi}{M(a, b)}, \quad w_2 = \frac{\pi}{M(c, ib)}, \quad w_3 = \frac{i\pi}{M(a, c)}.$$

Then each w_j is a primitive period, minimal in its coset modulo 2Λ, and any two of the w_j form a $\mathbb{Z}$-basis for Λ.

Special Case I: Real Curves with $\Delta > 0$

Here the e_j are all real and we may order them so that $e_1 > e_2 > e_3$.

We obtain a rectangular basis for the period lattice by setting

$$w_1 = \pi / \operatorname{AGM}(\sqrt{e_1 - e_2}, \sqrt{e_1 - e_3}),$$
$$w_2 = \pi i / \operatorname{AGM}(\sqrt{e_2 - e_3}, \sqrt{e_1 - e_3})$$

with all square roots positive; then w_1 and w_2/i are both real and positive.

Special Case II: Real Curves with $\Delta < 0$

Order the roots so that $e_1 \in \mathbb{R}$ and $e_2 = \overline{e_3}$.

Set $a_0 = \sqrt{e_1 - e_3} = x + yi$; we may assume that $x, y > 0$ by swapping e_2, e_3 or changing the sign of a_0 if necessary. Set $r = \sqrt{x^2 + y^2} > 0$ and $b_0 = \sqrt{e_1 - e_2} = x - yi$. Now

$$w_+ = \pi / \operatorname{AGM}(a_0, b_0) = \pi / \operatorname{AGM}(x + yi, x - yi) = \pi / \operatorname{AGM}(x, r)$$

is a real period, and

$$w_- = \pi / \operatorname{AGM}(-a_0, b_0) = \pi i / \operatorname{AGM}(y - xi, y + xi) = \pi i / \operatorname{AGM}(y, r).$$

is an imaginary period.

These periods span a sublattice of index 2 in the period lattice, for which a $\mathbb{Z}$-basis may be taken to be

$$w_1 = w_+ \qquad \text{and} \qquad w_2 = (w_+ + w_-)/2,$$

with $\Re(w_2/w_1) = 1/2$.

The elliptic logarithm problem

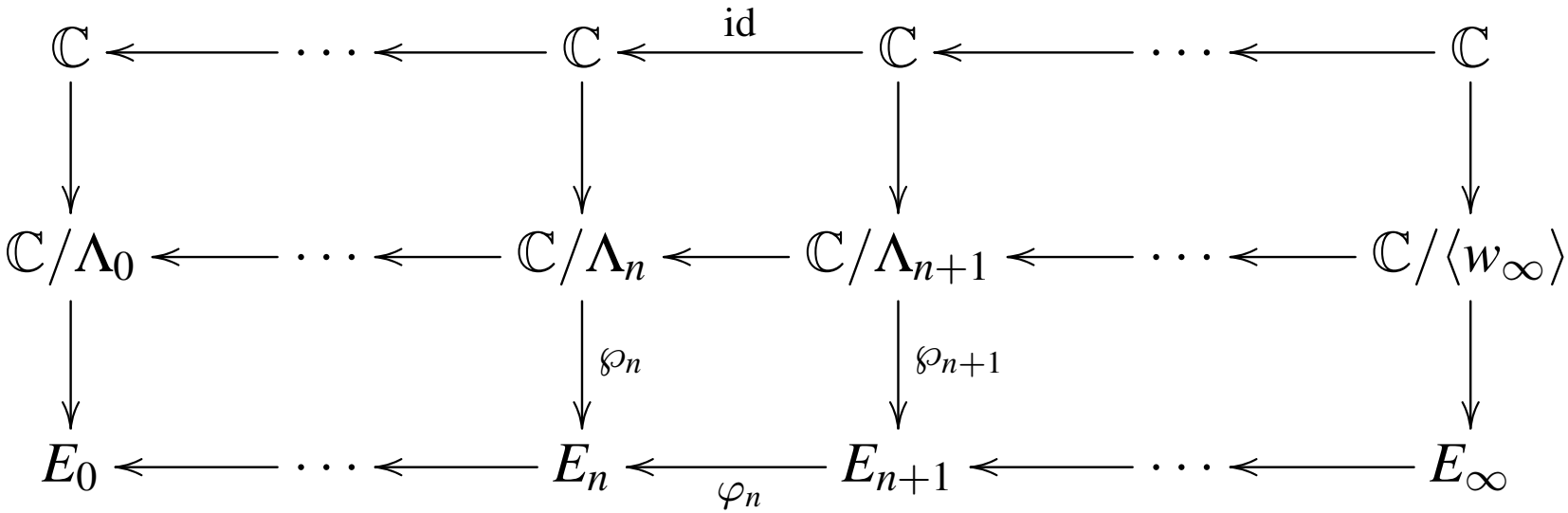
We wish to invert the map

$$\mathbb{C} \longrightarrow \mathbb{C}/\Lambda \xrightarrow{\wp\Lambda} E(\mathbb{C}).$$

The *elliptic logarithm* of $P = (x, y) \in E(\mathbb{C})$ is any $z \in \mathbb{C}$ such that

$$\wp_\Lambda(z) := (\wp(z;\Lambda), \wp'(z;\Lambda)) = (x, y).$$

Recall the diagram:



Given $z \in \mathbb{C}$ define $P_n = \wp_n(z)$ for $n \geq 0$.

Then $\varphi_n(P_{n+1}) = P_n = (x_n, y_n)$ for $n \geq 0$.

If $\lim P_n = (x_\infty, y_\infty)$ then one can recover z from x_∞, y_∞ using formulae given earlier for $\lim \wp_n(z), \lim \wp_n'(z)$.

Coherent sequences

Conversely, to each $P = P_0 \in E_0(\mathbb{C})$ there are uncountably many such "coherent sequences" (P_n), of which only a countable number arise in this way, one for each choice of z with $\wp_0(z) = P_0$. (These z values form a whole coset of Λ_0 in $\mathbb{C}$.) We call the latter "good sequences".

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We will recursively compute the point sequence $(P_n) = ((x_n, y_n))$ from $P = (x, y)$. At each stage there will be two choice of preimage $P_{n+1} \in \varphi_n^{-1}(P_n)$. Of these, we can specify one as the "right choice" in such a way that the countable number of good sequences are exactly those in which all but a finite number of choices are right.

Choices

To any coherent sequence (P_n) we associate the nested sequence of cosets $C_n = z_n + \Lambda_n$, where z_n is any elog of P_n.

$$\cdots \supset C_n \supset C_{n+1} \supset \ldots.$$

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Each Λ_n-coset C_n splits into two Λ_{n+1}-cosets, one of which is C_{n+1}.

The point sequence is good if and only if $\cap C_n \neq \emptyset$. In this case the intersection is a coset of Λ_∞.

The *right choice* of C_{n+1} (or equivalently of P_{n+1}) is the one containing the *minimal* element of C_n.

Discreteness implies that $\cap C_n \neq \emptyset$ iff C_{n+1} is the right choice for all but finitely many n.

New coordinates

It turns out to yield simpler formulae if instead of coordinates (x_n, y_n) on E_n we use coordinates (t_n, r_n) where $r_n^2 = (t_n^2 + a_{n-1}^2)/(t_n^2 + b_{n-1}^2)$, initialised by

$$r_1 = \sqrt{\frac{x_0 - e_3}{x_0 - e_2}}, \qquad t_1 = \frac{y_0}{2r_1(x_0 - e_2)} = \sqrt{x_0 - e_1}.$$

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The formula for the 2-isogeny now gives

$$r_n = \pm\sqrt{\frac{a_{n-1}(r_{n-1} + 1)}{b_{n-2}r_{n-1} + a_{n-2}}}, \qquad t_n = r_n t_{n-1}.$$

and the "right choice" is to take $\Re(r_n) \geq 0$.

The elliptic logarithm algorithm

Input: An elliptic curve E defined over $\mathbb{C}$, with roots e_1, e_2, e_3
A point $P = (x, y) \in E(\mathbb{C})$ with $y \neq 0$

Initialization:

1. Set $a_0 = \sqrt{e_1 - e_3}$ and $b_0 = \sqrt{e_1 - e_2}$, with $|a_0 - b_0| \leq |a_0 + b_0|$.
2. Set $r = \sqrt{(x - e_3)/(x - e_2)}$ with $\Re(r) \geq 0$.
3. Set $t = -y/(2r(x - e_2))$ (so $t^2 = x - e_1$).

Iteration:

4. Repeat the following for $n = 1, 2, \ldots$:
   1. Let
   $$a_n = (a_{n-1} + b_{n-1})/2, \quad b_n = \sqrt{a_{n-1}b_{n-1}}.$$
   with $|a_n - b_n| \leq |a_n + b_n|$.
   2. Let $r \leftarrow \sqrt{(a_n(r+1))/(b_{n-1}r + a_{n-1})}$ with $\Re(r) \geq 0$.
   3. Let $t \leftarrow rt$.

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Output:

$$z_P = \frac{1}{M} \arctan\left(\frac{M}{T}\right).$$

Implementations

We have implemented the algorithm in both SAGE and MAGMA.

The Sage version (by JEC) was merged in version 4.4 of Sage after ticket #6390 was positively reviewed by Chris Wuthrich (Nottingham). We also needed to re-implement the complex AGM since previously SAGE used PARI/GP's `agm` function, which does not give "optimal" values. That was done jointly by JEC and Robert Bradshaw (Google) (in Cython for efficiency).

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The Magma version was implemented by TT, who also provided the examples.

We will only give one example here. For more examples, see `http://www.sagemath.org/doc/reference/sage/schemes/elliptic_curves/period_lattice.html`!

Example

Let E be the elliptic curve over $\mathbb{C}$ given by the Weierstrass equation

$$E: \quad Y^2 = 4(X - e_1)(X - e_2)(X - e_3)$$

with

$$e_1 = 3 - 2i, \quad e_2 = 1 + i, \quad e_3 = -4 + i.$$

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Now

$$a_0 = 2.703310295347530788 67\ldots - i0.554875258893342750 23\ldots$$
$$b_0 = 1.674149228035540040 44\ldots - i0.895977476129838124 71\ldots$$
$$c_0 = 2.236067977499789696 40\ldots$$

satisfy

$$a_0^2 = e_1 - e_3, \quad b_0^2 = e_1 - e_2, \quad c_0^2 = a_0^2 - b_0^2,$$

and

$$|a_1 - b_1| < |a_1 + b_1|, \quad |c_1 - ib_0| < |c_0 + ib_0|, \quad |a_0 - c_0| < |a_0 + c_0|.$$

Example (continued)

The minimal periods are

$$w_1 = 1.2921515174871305190 4\ldots + i0.4475921810781889660 8\ldots,$$
$$w_2 = 1.4266137345178450758 7\ldots - i0.8096384805630188210 7\ldots,$$
$$w_3 = -0.1344622170307145568 2\ldots + i1.2572306616412077871 5\ldots;$$

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Next, we wish to compute an elliptic logarithm of the point

$$P = (2 - i, 8 + 4i) \in E(\mathbb{C})$$

(which has infinite order). We find

$$z_P = -0.7221299791400229912 6\ldots + i0.0171712241265090224 9\ldots.$$

Computing canonical heights

If E is an elliptic curve defined over a number field K, then the *canonical* or *Néron-Tate height* of over all places of K of *local heights*.

At finite (non-archimedean) places this is easy to compute.

At real and complex places there are several methods available.

Mestre showed how to use real AGM sequences (and the same chain of 2-isogenies as for computing periods) to compute the local height of a point at a real place.

Work is in progress to extend this to complex places.