CHAPTER IV

THE TABLES

Introduction to the tables

Table 1. Elliptic curves.

In Table 1 we give details of each computed elliptic curve E of conductor N for $N \leq 1000$, arranged by conductor and isogeny class. The first curve in each class[1] is the 'strong Weil curve' E_f computed from the periods of the newforms f, and it is followed by the isogenous curves, if any. There are 2463 isogeny classes, and 5113 curves in all.

The table contains the coefficients of minimal models of all the curves. Each curve has a code of the form $N\mathrm{X}i$, where N is the conductor, X is the letter code for the corresponding newform or isogeny class, and i is the number of the curve in its class. The order of the isogeny classes for each N is the order in which the corresponding newforms were found.[2]

After the first curve in each class, the other curves in the class (if any) are listed in the order in which they were found by the isogeny program, as described in Section 3.8. The isogeny information given in the last column was also recorded by that program. For each curve for $N \leq 200$ we also give in parentheses the Antwerp code of each, as in [2]. Thus curves 11A1, 11A2 and 11A3 are the Antwerp curves 11B, 11C and 11A in that order. We hope that this new system of identifying codes will not cause confusion; it was the most natural, given the way the curves were found.

The other data in this table is, for each curve: the rank and number of torsion points; the factorization of the discriminant and j-invariant; the local index c_p and the Kodaira symbol at each bad prime p; and the isogenies of prime degree.

When the number of torsion points is of the form $4k$, one can tell whether the torsion subgroup is cyclic (C_{4k}) or not ($C_{2k} \times C_2$) by seeing whether the number of 2-isogenies is 1 or 3 (respectively), since this number is the same as the number of points of order 2.

We have not indicated on this table either the presence of complex multiplication, or when a curve is a twist of one elsewhere in the table. These omissions were made to save space. Complex multiplication may be determined most easily by referring to the table in Section 3.9, where a complete list of the rational j-invariants of curves with complex multiplication was given. For example, the isogeny class 49A1-2-3-4 consists of four complex multiplication curves; 49A1,3 have $j = -3375 = -15^3$ and CM by -7, while 49A2,4 have $j = 16581375 = 255^3 = 3^3 5^3 17^3$ and CM by -28. All other curves with these complex multiplications are twists of these. Within the range of the tables here we find their -3-twists at 441D1-2-3-4 and their

[1] with the single exception of class 990H, where curve H3 is the strong Weil curve E_f, owing to a slip; we have not changed the numbering, in order to maintain consistency with the first edition of these tables.

[2] Roughly speaking, this is in lexicographic order of the vector of Hecke eigenvalues, but we claim no uniformity here; as the program evolved its strategy changed at least twice: once when we first started using the W operators, and again when we changed the order of searching for eigenvalues of T_p from $\ldots, -2, -1, 0, 1, 2, \ldots$ to $0, 1, -1, 2, -2, \ldots$, after realizing that small values of a_p were more likely to occur. Unfortunately, this means that the current version of the program recomputes the newforms in a different order; for consistency, we now have to build into the program the permutations necessary to produce output in the order fixed by the first edition of the tables.

-4-twists at 784H1-2-3-4. As remarked earlier, the presence of complex multiplication can also be spotted in Table 3, where half the Hecke eigenvalues will be zero.

In the non-complex multiplication case, unless N is divisible by D^2 the twist with discriminant D of a (non-complex multiplication) newform at level N will have level $\operatorname{lcm}(N, D^2) > N$; thus a necessary condition for a newform to have a twist earlier in the table is that its level should be divisible by a square. It is then usually easy to see which newform at a lower level is the twist. Table 3 can help here, since twisted newforms have the same Hecke eigenvalues a_p up to sign. Thus we can determine when two isogeny classes of curves are twists of each other, since each class corresponds to a newform. Once two isogeny classes have been identified as twists of each other, one can determine which curves in the first class are twists of which in the second by comparing j-invariants.

For example, consider level $N = 704 = 2^6 \cdot 11$. At this level three twists operate, with $D = -4$, -8 and $+8$. The 12 newforms form 6 pairs which are -4-twists of each other: A–K, B–C, D–F, E–I, G–J and H–L. Their ± 8-twists, however, are all at lower levels. 704A and 704K are ± 8 twists of 11A and 176B; 704D and 704F of 44A and 176C; 704E and 704I of 88A and 176A; 704G and 704J of 352A and 352C; 704B and 704C of 352B and 352D; and 704L and 704H of 352E and 352F (respectively). Thus of the 24 newforms, we only have 6 up to twists, whose first representatives are 11A, 44A, 88A, 352A, 352B and 352E. The first of the corresponding sets of isogeny classes consists of three curves, linked by 5-isogenies: 11A3-1-2; 176B1-2-3; 704A1-2-3; 704K1-2-3 (respectively). (Note that in order to keep to our convention that the first curve in each class is the 'strong Weil curve', we were not able to number the curves in the classes in such a way that the numbers in twisted classes correspond: being the 'strong' curve is not preserved under twisting, as this example shows.) Also the classes 44A, 176C, 704D, 704F each consist of a pair of curves linked by 3-isogeny: in this case the first curves do all correspond under twisting. Each of the other isogeny classes consists of a single curve.

Table 2. Mordell–Weil generators.

The second table contains generators for the Mordell–Weil group (modulo torsion) of the first curve[3] in each isogeny class of curves of positive rank. In the case of rank 1 curves, this generator P is unique up to replacing P by $\pm P + Q$ where Q is a torsion point; in rank 2 cases, the given generators P_1, P_2 could be replaced by $aP_1 + bP_2 + Q_1$ and $cP_1 + dP_2 + Q_2$, where $ad - bc = \pm 1$ and Q_1, Q_2 are torsion points.

To save space, we only list generators for one curve in each isogeny class. (Generators for the other curves may be obtained from the author's ftp site given below.) As in Table 1, for $N \leq 200$ we give in brackets the Antwerp code of the curve.

For $N \leq 200$ there are some discrepancies with Table 2 of [2]: the generators for 143A and 154C are omitted there; the point $(0, 2)$ on 155D has order 5, with $(2, 5)$ being a generator of infinite order; and on 170A, a generator is $P = (0, 2)$, and the point $(2, 1)$ given in [2] is $-2P$.

Table 3. Hecke eigenvalues.

In Table 3 we give the Hecke eigenvalues for all the rational newforms at all levels up to $N = 1000$. As in [2], we give the eigenvalue a_p for all $p < 100$ not dividing N, and the W_q eigenvalue ε_q for all primes q dividing N.

Almost all of these numbers could be computed from the modular curves themselves, as listed in Table 1, by the formulae of Section 2.6. For $p \nmid N$, the eigenvalue a_p is equal to the trace of Frobenius of the curve, which is easily computed as in Section 3.9. When $q \mid\mid N$, we have $-\varepsilon_q = a_q = 1 + q - |E(\mathbb{F}_q)|$. However when $q^2 \mid N$ we cannot recover ε_q this way, since then $a_q = 1 + q - |E(\mathbb{F}_q)| = 0$. We did in fact check in each case that the Hecke eigenvalues and traces of Frobenius agreed for all $p < 1000$.

[3]except for class 990H, where we give the generator for curve 990H3 as this is the strong Weil curve

Each newform is identified by its level N and a letter, as in Table 1. Thus 50A is the newform corresponding to the curve 50A1 (and by isogeny to 50A2,3,4), while 50B corresponds to curves 50B1,2,3,4. As in Table 1, for $N \leq 200$ we give in brackets the Antwerp codes, for ease of cross-reference.

Table 4. Birch–Swinnerton-Dyer data.

In this table we present the data pertaining to the Birch–Swinnerton-Dyer conjectures for the modular curves $E = E_f$ of conductor N up to 1000 attached to rational newforms f in $S_2(N)$. In each case we list the following quantities:

- the rank r
- the period $\Omega(f) = \Omega(E)$;
- the value of $L^{(r)}(E,1)/r! = L^{(r)}(f,1)/r!$;
- the regulator R of $E(\mathbb{Q})$;
- the ratio $L^{(r)}(E,1)/r!\Omega R$;
- and finally the quantity S defined as

$$S = \frac{L^{(r)}(f,1)}{r!\ \Omega(f)} \bigg/ \frac{(\prod c_p)\ R}{|E(\mathbb{Q})_{\text{tors}}|^2}\ .$$

Note that some of these quantities are computed from the newform f, while others are computed from the curve E. Some can be obtained from either: for example, we know that the analytic rank is in each case equal to the Mordell–Weil rank. When the rank is 0, we use the exact value of the ratio ratio $L(f,1)/\Omega(f)$ obtained via modular symbols; dividing by the rational number $\prod c_p / |E(\mathbb{Q})_{\text{tors}}|^2$, we thus obtain an exact rational value for S. This was equal to 1 in all but four cases: $S = 4$ for 571A1, 960D1 and 960N1, and $S = 9$ for 681B1. These are consistent with the data concerning the order of $Ш(E/\mathbb{Q})$ coming from the two-descent which we carried out: in the three cases 571A1, 960D1 and 960N1 and in no other cases there are homogeneous spaces (2-coverings of E) which have points everywhere locally, but on which we could find no rational point. Since these curves are modular, and we know that $L(E,1) \neq 0$ in each case, we know that the rank is in fact 0 (by Kolyvagin's result), so that $|Ш[2]| = 4$ in each case. For the Birch–Swinnerton-Dyer conjecture to hold we would need to establish $|Ш| = 4$. This should be possible using the methods of Rubin and Kolyvagin.

When the rank is positive the ratio is computed from three approximations to the values of $L^{(r)}(f,1)$, R and $\Omega(E)$. Thus in these cases we only compute an approximation to the value of S; but in all cases in the tables this value was equal to 1 to within the accuracy of the computation. These values are listed as 1.0 in the table to emphasize the fact that they were obtained as approximations.

Table 5. Parametrization degrees.

This table shows, for each newform f, the degree of the modular parametrization $X_0(N) \to E_f$, together with the prime factorization of the degree.

Some remarks on the computations

We have implemented the algorithms described in Chapter 2 and run them for all N up to 5077. For a summary of the results obtained over this range, see the end of this section.

In the first phase of the computation for each N, we worked in the smaller space $H^+(N)$ and found rational newforms, storing in a data file the number of forms, and for each, the rational number $L(f,1)/\Omega(f)$ and a certain number of Hecke eigenvalues including all the W_q eigenvalues. We also stored enough modular symbol information that if we needed more Hecke eigenvalues later we could resume the calculation with the minimum of repetition. At this stage we already had all the data given in Table 3 below, and knew the sign of the functional equation (and hence the parity of the analytic rank) and whether or not $L(f,1)=0$. Thus we had enough to guess the analytic rank as 0, 1 or 2; and in each case this preliminary estimate turned out to be correct, since no curves of rank greater than 2 were found for $N<5077$.

The second phase was to work in $H(N)$, using the eigenvalues already known, to find the period lattices, the approximate c_4 and c_6 invariants of the modular curves, and hence their (rounded) coefficients. These coefficients were stored. We also stored information about the twisting primes l and matrices used to evaluate the periods. In some cases the first pass through this phase did not produce c_4 and c_6 to sufficient precision; this tended to happen when their values were large and when the auxiliary primes l were also large. In such cases we went back to phase 1 to compute more eigenvalues, so that we could evaluate more terms of the relevant series. The re-evaluation of the periods given more a_p was then very fast, as we had all the relevant information stored and merely had to sum the series. In very few cases did we need to use a_p for $p>5000$.

We then implemented and ran the algorithms of Chapter 3 on the resulting curves, including finding all curves isogenous to them. Checking that the c_4 and c_6 invariants were indeed those of a minimal model of a curve of conductor N was in fact done by the program which computed them in the first place, so that we could tell when sufficient accuracy had been obtained. Starting from a file containing the coefficients of the original 'strong' curves E_f, we ran the isogeny program to produce a larger file with the complete list of curves, together with information on the degrees of the isogenies linking them. This file was then used by a further program which produced the TEX source code for Table 1, with all the other data there being recomputed or read from files.

The number of torsion points was computed by the method of Section 3.3.

The rank was first guessed as the smallest possible value consistent with the Birch–Swinnerton-Dyer conjecture, given that we knew whether $L(E,1)$ was zero and the sign of the functional equation; this value $r=0$, 1 or 2 was then confirmed as the analytic rank by the computation of $L^{(r)}(f,1)$. When $r=0$ or 1 it then follows that the Mordell–Weil rank of $E(\mathbb{Q})$ is also r by Kolyvagin's results. In almost all cases, including all of those where $r=2$, we verified this using our two-descent rank program. The exceptional cases were for curves with no rational two-torsion and very large coefficients, where the two-descent would have taken too long. In these cases we did know (independently of Kolyvagin) that the given value of r was a lower bound, since we always found that number of independent generators of infinite order.

The case $r=2$ occurs only 18 times in these tables, with the following curves: 389A, 433A, 446D, 563A, 571B, 643A, 655A, 664A, 681C, 707A, 709A, 718B, 794A, 817A, 916C, 944E, 997B and 997C. In each case, there are no isogenous curves.

Apart from the rank, all the other data in Table 1 was computed by our implementation of Tate's algorithm.

The generators given in Table 2 were obtained by the methods of Section 3.5, where we first searched for the expected number of independent points of infinite order, and then refined these where necessary to be sure we had generators of the curves (modulo torsion), and not of a subgroup of finite index. In most cases the generators were found immediately; the hard ones to find were those with most digits, and particularly those which are not integral. For the last few to be found, the modular refinements mentioned in Chapter 3 were essential. (For the record, the generator of 873C1 was first found not by search, but via Heegner points, during the July 1989 Durham meeting on L-series; by day we learned about the latest results of Rubin and Kolyvagin, and about Heegner points, while by night we applied some of these ideas in a new program, which eventually came up with the elusive point.) We also found generators on the isogenous curves; see [16] for details of this.

Table 4 was produced by a program which took as data the Hecke eigenvalues, sign of functional equation (and hence the analytic rank) of the newform f; the coefficients of the curve E_f attached to f; and the generators. Then the value $L^{(r)}(f,1)$ and the period Ω were computed from the eigenvalues as in Section 2.13; and the regulator from the heights of the generators as in Section 3.4. We also recomputed the local factors c_p and number of torsion points, and from all these could obtain the conjectured value S of the order of the Tate–Shafarevich group Ш.

Finally, the data in Table 5 was obtained by implementing the formula for $\deg(\varphi)$ given in Section 2.15. The degree was computed during the second phase of each computation (working in $H(N)$) and stored on file with the other data.

All the tables were produced as follows, to minimize the risk of transcription error. The programs which computed the numbers themselves wrote the results to data files; separate programs read in this data and added the TeX formatting characters, producing the TeX source files, which were then processed in the usual way. Thus none of the numbers in the tables was typed by human hand at any stage. Many consistency checks were applied along the way. Almost all the errors in the earlier versions of the tables arose as a result of using old versions of data files by mistake, rather than from errors in the programs. We sincerely hope that all such slips have been avoided here.

More details of the specific layout of each of the five sets of tables is given immediately before each set below.

Summary of results obtained up to 5077

Although the tables included here only contain results for levels up to 1000, we have to date carried out the computations described above for all levels N up to 5077. For each N we found the rational newforms, and computed many Hecke eigenvalues for each; for each form we computed a period lattice, and hence found a corresponding curve of conductor N. For each curve, we verified by 2-descent (in most cases, including all cases of rank 2) that the rank was equal to the analytic rank, and by finding the Mordell-Weil group of each curve (again, in most cases) we were able to compute the value predicted by the Birch Swinnerton-Dyer conjecture for the order of the Tate–Shafarevich group.

We summarize the results obtained in the following table, where for brevity we only give the numbers of newforms found, subdivided by rank. The first examples of each rank are: $N = 11$ for rank 0; $N = 37$ for rank 1; $N = 389$ for rank 2; and $N = 5077$ for rank 3.

Rational newforms for $\Gamma_0(N)$, $N \leq 5077$

Range of N	Total	$r=0$	$r=1$	$r=2$	$r=3$
1–1000	2463	1321	1124	18	0
1001–2000	3391	1575	1737	79	0
2001–3000	3661	1663	1852	146	0
3001–4000	3837	1665	2006	166	0
4001–5000	3962	1690	2092	180	0
5001–5077	284	121	148	14	1
1–5077	17598	8035	8959	603	1

For conductors $N > 1000$, tables of the curves and related data may be obtained online from `http://www.maths.nott.ac.uk/personal/jec/ftp/data/INDEX.html`.