Projective morphisms according to Kawamata

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0 Introduction

X is a projective 3-fold with canonical singularities, $k = \mathbb{C}$; the terminology will be explained in 0.8 below.

Theorem 0.0 (on projective morphisms) *Let $D \in \operatorname{Pic} X$ be nef, and suppose that $aD - K_X$ is nef and big for some $a \in \mathbb{Z}$ with $a \geq 1$. Then $|mD|$ is free for every $m \gg 0$; equivalently, there exists a morphism to a projective variety $\varphi\colon X \to Z$ such that $\varphi_*\mathcal{O}_X = \mathcal{O}_Z$, and an ample $H \in \operatorname{Pic} Z$ such that $D = \varphi^* H$.*

0.1 Properties of φ

(a) *Vanishing:* $R^i\varphi_*\mathcal{O}_X = 0$ for $i > 0$, and in particular $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Z)$; furthermore, $H^i(Z, H^{\otimes m}) = 0$ for all $m \geq a$ and $i > 0$.

(b) *Relative anticanonical model:* φ factors as $X \xrightarrow{g} \overline{X} \xrightarrow{h} Z$ where g is birational, $\overline{X}$ has canonical singularities, $K_X = g^* K_{\overline{X}}$, and $-K_{\overline{X}}$ is relatively ample for h.

(c) *Cases according to* $\dim Z = \kappa_{\mathrm{num}}(D) = \kappa(D)$*:*

$\dim Z = 3$. Then $\varphi\colon X \to Z$ is birational, and Z has rational singularities.

$\dim Z = 2$. Then $\varphi\colon X \to Z$ is a *weak conic bundle*: Z is a normal surface with rational singularities, and the general fibre of φ is $\mathbb{P}^1$.

$\dim Z = 1$. Then $\varphi\colon X \to Z$ is a *weak del Pezzo fibre space*: Z is a nonsingular curve, and the general fibre A of φ is a surface with at worst Du Val singularities, such that $-K_A$ is nef and big.

$Z = \mathrm{pt}$. Then X is a *weak* $\mathbb{Q}$-*Fano* 3-*fold*, that is, $-K_X$ is nef and big; $H^i(\mathcal{O}_X) = 0$ for all $i > 0$, and $\operatorname{Pic} X$ is reduced[1] and torsion free; in this case $D = 0 \in \operatorname{Pic} X$.

Corollary 0.2 (finite generation) *If K_X is nef and big, that is, X is a* minimal model *of a 3-fold of general type) then $|mrK_X|$ is free for every $m \gg 0$, where r = index of X; in particular, the canonical ring is finitely generated.*

Proof Theorem 0.0 applies at once to $D = rK_X$. The final part comes from Zariski's projective normalisation: if m is such that $|mK_X|$ is free, then the canonical ring of X is a finite module over the subring generated by $H^0(mK_X)$.

0.3

The second corollary requires some setting up: write

$$N^1_{\mathbb{Q}}X = \{\text{Cartier divisors} \otimes \mathbb{Q}\}/\overset{\text{num}}{\sim}, \quad N^1X = N^1_{\mathbb{Q}}X \otimes \mathbb{R};$$
$$\text{and} \quad N_1X = \{\text{1-cycles} \otimes \mathbb{R}\}/\overset{\text{num}}{\sim};$$

by definition of numerical equivalence N^1X and N_1X are dual finite dimensional vector spaces. Let $\overline{\mathrm{NE}} = \overline{\mathrm{NE}}(X) \subset N_1X$ be the Kleiman–Mori closed cone of effective 1-cycles.

Corollary (contraction theorem) *Let F be a face of $\overline{\mathrm{NE}}(X)$ entirely contained in the half-space $\overline{\mathrm{NE}}_- = \{z \mid K_Xz < 0\}$, and suppose that there exists a nef class $d \in N^1_{\mathbb{Q}}X$ such that $d^\perp \cap \overline{\mathrm{NE}} = F$. Then there exists a morphism $\varphi = \mathrm{cont}_F\colon X \to Y$ with $\varphi_*\mathcal{O}_X = \mathcal{O}_Z$ and such that for every curve $C \subset X$,*

$$\varphi(C) = \mathrm{pt} \in Y \iff C \in F.$$

Proof Write

$$\overline{\mathrm{NE}}_+ = \{z \in \overline{\mathrm{NE}} \mid K_Xz \geq 0\},$$

and let Σ be the intersection of $\overline{\mathrm{NE}}_+$ with the unit sphere in N_1X. Then d is positive on Σ, and since Σ is compact, d is bounded away from zero; also K_X, considered as a linear form on N_1X, is bounded on Σ, so that for any sufficiently large $a \in \mathbb{R}$, $ad - K_X$ is positive on Σ, and then obviously positive on the whole of $\overline{\mathrm{NE}}$. If a is chosen so that in addition ad is represented by a divisior $D \in \operatorname{Pic} X$ then $D - K_X$ is ample on X by Kleiman's criterion, and Theorem 0.0 applies.

[1]Reduced and discrete is intended, because $H^1(\mathcal{O}_X) = 0$; see the proof in 1.7.

Remark In §5 I prove that under certain restrictions on the singularities of X, if K_X is not nef, then there always exists a face F satisfying the hypotheses of Corollary 0.3, and in fact F can be taken to be a ray R. This is a weak form of the conjectured "Theorem on the Cone" for singular 3-folds.

In [9], 4.18, I outlined a program in five steps for constructing minimal models of 3-folds. The results of this paper cover Steps 2 and 3 of this program in a fairly satisfactory way.

0.4

The following is an effective statement that can be obtained by the method of proof of Theorem 0.0:

Corollary *Let X, D, a be as in Theorem 0.0.*

(i) If $m \geq 2a+2$ then the general element of $M = |mD|$ is reduced and has only ordinary double curves along 1-dimensional components of $\operatorname{Sing} X$.

(ii) If $m \geq 3a+3$ the general element of M has only double curves, and only ordinary double curves if $m \geq 6a+6$.

0.5

The following result is proved in §4, using the notation, and in one place the method, of the proof of Theorem 0.0.

Theorem (Shokurov [12]) *Suppose that $-K_X \in \operatorname{Pic} X$ is big and nef (that is, X is a weak Fano 3-fold). Then the general element $S \in |-K_X|$ is a K3 surface with at worst Du Val singularities.*

It follows from the theory of linear systems on K3s, applied to the minimal resolution of S, that if $|-K_X|$ is not free then its scheme theoretic base locus is isomorphic to $\mathbb{P}^1$ or to a (reduced) point.

0.6 Discussion

Kawamata's method is a higher dimensional analog of the Kodaira–Ramanujam–Bombieri connectedness method for surfaces. The big drawback is that the method as it stands is not effective: whereas the method for surfaces allows us to choose a point $P \in X$, construct a divisor D with $P \in \operatorname{Sing} D$, and conclude that P is not a base point of $|D + K_X|$, the method proves only that there is *some* base component B of $|mD|$ of "maximal multiplicity" (see 1.4), and that then there is a b_0 such that for $b \geq b_0$, B is not a base component of $|bD|$.

Problems 0.7 (a) Make Theorem 0.0 effective; in particular, if the canonical class $K_X \in \operatorname{Pic} X$ is nef and big, prove that $|mK_X|$ is free for $m \geq$ some reasonable bound (say 10).

(b) Does Theorem 0.0 hold for $\dim X \geq 4$ (assuming if necessary that $\kappa(D) \geq 0$)? The present proof fails to go through at one point, namely Proposition 1.5, at which higher Chern classes turn up in the formula for $h^0((bf^*D + A)_{|B})$.

(c) The following statement would be very useful in many different contexts, in particular in (b) above:

Conjecture *If V is a nonsingular projective 3-fold and $c_2(V) \cdot H < 0$ for some ample H then the subsheaf $E \subset \Omega^1_V$ breaking the stability of Ω^1_V is orthogonal to a foliation of V by rational subvarieties.*

(d) If $K_X \overset{\text{num}}{\sim} 0$ it follows from Theorem 0.0 that D is nef and big if and only if $|mD|$ is free for $m \gg 0$, and defines a birational morphism $\varphi\colon X \to Z$; then Z also has canonical singularities and $K_X = \varphi^* K_Z$. What happens when D is nef but $\kappa_{\text{num}}(D) = 1$ or 2? In this case it is certainly possible that $h^0(mD) = 0$ for all $m > 0$ (because D may be numerically but not linearly equivalent to 0 on an Abelian factor of X).

Conjecture *There exists an $m > 0$ and a free linear system $|L|$ with $L \overset{\text{num}}{\sim} mD$. Hence there is a morphism $\varphi\colon X \to Z$ such that φ contracts precisely the curves $C \subset X$ such that $DC = 0$.*

(e) It would be interesting to know what kind of singularities the map $\varphi\colon X \to Z$ can have in the cases $\dim Z = 3$ or 2 of Proposition 0.1, (c). In the birational case, Z has singularities that are more general than canonical, but presumably much more restricted than general rational singularities.

0.8 Preliminaries and terminology

a. $\mathbb{Q}$-divisors Let X be a projective normal variety; a $\mathbb{Q}$-divisor $D \in \operatorname{Div} X \otimes \mathbb{Q}$ is $\mathbb{Q}$-*Cartier* if $rD \in \operatorname{Pic} X$ for some $r \in \mathbb{Z}$, $r > 0$. Intersection numbers and cycles are defined for $\mathbb{Q}$-Cartier divisors in the obvious way:

$$D_1 \cdots D_k =_{\text{def}} \frac{1}{r_1 \cdots r_k}(r_1 D_1) \cdots (r_k D_k),$$

where the right-hand side is the intersection cycle of Cartier divisors defined by any of the usual procedures.

b. Nef $D \in \operatorname{Div} X \otimes \mathbb{Q}$ is *nef* if it is $\mathbb{Q}$-Cartier and for every curve $C \subset X$,

$$DC =_{\text{def}} \frac{1}{r}(rD)C \geq 0.$$

By Kleiman's ampleness criterion, D is nef if and only if D is numerically equivalent to a limit of ample $\mathbb{Q}$-Cartier divisors; in particular, if $D_1, \dots, D_k$ are nef and Z is an effective cycle of codimension l then $D_1 \cdots D_k Z$ is a limit of effective cycles of codimension $k + l$.

c. $\kappa_{\text{num}}(D)$ **and big** If D is nef then the *characteristic dimension* or the *numerical Kodaira dimension* of D is defined to be

$$\kappa_{\text{num}}(D) = \max\bigl\{k \bigm| D^k \overset{\text{num}}{\not\sim} 0\bigr\}.$$

Then $\max\{0, \kappa(D)\} \leq \kappa_{\text{num}}(D) \leq n$ where $n = \dim X$ and $\kappa(D) = \kappa(X, D)$ is the Iitaka D-dimension of X, and it is easy to see (using vanishing, so only in characteristic 0) that the following are equivalent:

(i) $\kappa_{\text{num}}(D) = n$;

(ii) $D^n > 0$;

(iii) $h^0(X, mrD) \sim m^n$ as $m \to \infty$;

(iv) for every ample $H \in \operatorname{Pic} X$ there is an $m > 0$ such that $mrD \overset{\text{lin}}{\sim} H + M$ where $M \in \operatorname{Pic} X$ is effective;

(v) $\kappa(D) = n$.

If this happens, I say that D is *big*.

(d) Round-up $\lceil\ \rceil$ For $r \in \mathbb{R}$, write $\lceil r \rceil$ for the smallest integer $\geq r$, the *round-up* of r; (the Gauss symbol $[\ ]$ is "round-down", and is related by $\lceil r \rceil = -[-r]$). If $D = \sum q_i F_i$ with F_i distinct prime divisors, and $q_i \in \mathbb{Q}$, write $\lceil D \rceil = \sum \lceil q_i \rceil F_i$. Note that $\lceil\ \rceil$ is a function on divisors, not on divisor classes, although if $D = D_1 + D_2$, with $D_2 \in \operatorname{Div} X \otimes \mathbb{Q}$, and $D_1 \in \operatorname{Pic} X$ (that is, D_1 defined only up to linear equivalence), then $\lceil D \rceil = D_1 + \lceil D_2 \rceil \in \operatorname{Pic} X$ is well defined. Thus I will usually write "=" of $\mathbb{Q}$-divisors to indicate that the fractional parts are equal and the integer parts are linearly equivalent.

Note also that if $f\colon Y \to X$ is a birational morphism, and $rK_X \in \operatorname{Pic} X$, then the isomorphism of $\omega_X^{[r]}$ and $\omega_Y^{[r]}$ on the locus where f is an isomorphism extends to a canonical isomorphism

$$f^*\omega_X^{[r]} \otimes \mathcal{O}_Y(D) \xrightarrow{\simeq} \omega_Y^{[r]},$$

where D is a Weil divisor made up of exceptional divisors of f (effective if X has canonical singularities). I write equality of $\mathbb{Q}$-divisors $K_Y = f^*K_X + \Delta$ where $\Delta = \frac{1}{r}D$ to describe this.

Lemma 0.9 *(i) If D is nef then $\kappa_{\mathrm{num}}(D) \geq \kappa(D)$;*

(ii) if D is nef with $\kappa_{\mathrm{num}}(D) \geq k$ and H is nef and big then $D^kH^{n-k} > 0$;

(iii) if D is an effective Weil divisor which is nef and has $\kappa_{\mathrm{num}}(D) \geq 2$ then $\operatorname{Supp} D$ is connected in codimension 1, in the sense that if $D = D_1 + D_2$ with D_1, D_2 effective and with no common divisors, then the intersection $\operatorname{Supp} D_1 \cap \operatorname{Supp} D_2$ has at least one component of dimension $n-2$.

Proof (i) If $\kappa(D) = k$ then for a suitable $m > 0$ such that $mD \in \operatorname{Pic} X$, $|mD|$ defines a dominant rational map $X \dashrightarrow Z$ to a k-dimensional projective variety. Resolving indeterminacy gives

$$\begin{array}{ccc} & Y & \\ {}^{f}\swarrow & & \searrow^{\varphi} \\ X & \dashrightarrow & Z, \end{array}$$

where f, φ are morphisms, and $|f^*mD| = |L| + F$, where $|L|$ is free with $L^k > 0$ and F is effective. Then

$$(mD)^k = (f^*mD)^k = (L+F)^k \geq L^k > 0,$$

which holds because for each i with $0 \leq i < k$,

$$(f^*mD)^{i+1}L^{k-i-1} = (f^*mD)^i(L+F)L^{k-i-1} \geq (f^*mD)^iL^{k-i},$$

using the fact that both L and f^*mD are nef.

(ii) follows by a similar argument using the fact that some multiple of H is of the form an ample divisor plus an effective divisor.

(iii) Assuming that $\operatorname{Supp} D_1 \cap \operatorname{Supp} D_2$ has codimension ≥ 3 in X, it will not meet a general surface sections S of X, so that both D_1 and D_2 are $\mathbb{Q}$-Cartier divisors in a neighbourhood of S. Writing $\widetilde{S} \to S$ for a resolution of S, and $'$ for the pullback of a divisor of X to $\widetilde{S}$, I have $D_1'D_2' = 0$, but $(D_1')^2, (D_2')^2 \geq 0$ (because D is nef), and $(D_1' + D_2')^2 > 0$ (because $\kappa_{\mathrm{num}}(D) > 2$), and this contradicts the index theorem.

Index Theorem 0.10 *Let D, A be $\mathbb{Q}$-Cartier divisor on a normal projective n-fold X with $n \geq 2$, such that D is nef, $D \overset{\mathrm{num}}{\not\sim} 0$. Then* *(i) for ample $\mathbb{Q}$-divisors $H_1, \dots, H_{n-2}$,*

$$DAH_1 \cdots H_{n-2} = 0 \implies -A^2 H_1 \cdots H_{n-2} \geq 0;$$

in particular, if $n \geq 3$ and $DAH_1 \cdots H_{n-3} \overset{\text{num}}{\sim} 0$ (as a 1-cycle) then $-A^2 H_1 \cdots H_{n-3} \in \overline{\mathrm{NE}}(X)$.

(ii) If for some ample $H_1, \dots, H_{n-2}$,

$$DAH_1 \cdots H_{n-2} = A^2 H_1 \cdots H_{n-2} = 0$$

then $A \overset{\text{num}}{\sim} qD$ for some $q \in \mathbb{Q}$, and if $q \neq 0$ then $D^2 \overset{\text{num}}{\sim} 0$, that is, $\kappa_{\text{num}}(D) = 1$.

Proof Let $S = L_1 \cap \cdots \cap L_{n-2}$ be a reduced irreducible surface complete intersection, with $L_i \in |m_i H_i|$ (where $m_i H_i \in \operatorname{Pic} X$); let $f\colon \widetilde{S} \to S$ be a resolution, and let $'$ denote the pullback of $\mathbb{Q}$-Cartier divisors of X to $\widetilde{S}$.

Now D' is nef on $\widetilde{S}$ and $D' \overset{\text{num}}{\not\sim} 0$; also $D'A' = mDAH_1 \cdots H_{n-2}$ and $(A')^2 = mA^2 H_1 \cdots H_{n-2}$ (where $m = \prod m_i$), so that (i) is just a restatement of the usual index theorem. If $(A')^2 = 0$ then $A' \overset{\text{num}}{\sim} qD'$ on $\widetilde{S}$; the value of q can be determined by

$$A'H_1' = mAH_1^2 H_2 \cdots H_{n-2} = qmDH_1^2 H_2 \cdots H_{n-2} = qD'H_1',$$

since $D'H_1' \neq 0$, and so q does not depend on the choice of m_i and $L_i \in |m_i H_i|$.

I now claim that for every curve $C \subset X$, $(A - qD)C = 0$. To see this, note that for $m_i \gg 0$ such that $m_i H_i \in \operatorname{Pic} X$, $\mathcal{I}_C \cdot \mathcal{O}_X(m_i H_i)$ is generated by its H^0, where $\mathcal{I}_C$ is the ideal defining C, so that choosing $L_i \in |m_i H_i|$ to contain C, but otherwise general, the intersection $S = L_1 \cap \cdots \cap L_{n-2}$ is reduced and irreducible. Now let $f\colon \widetilde{S} \to S$ be its resolution, and $\widetilde{C} \subset \widetilde{S}$ any irreducible curve such that $f_{|\widetilde{C}}\colon \widetilde{C} \to C$ is generically finite, of degree d say. Then

$$0 = (A' - qD')\widetilde{C} = d(A - qD)C. \quad \text{Q.E.D.}$$

0.11 Vanishing

The following result is the main technical tool of this paper.

Vanishing *If Y is a nonsingular variety and $N \in \operatorname{Div} Y \otimes \mathbb{Q}$ is nef and big, and the fractional part of N is supported on a divisor with normal crossings, then*

$$H^i(Y, \lceil N \rceil + K_Y) = 0 \quad \text{for } i > 0.$$

In Kawamata's treatment [5] this is an easy formal consequence of Kodaira vanishing.

0.12 Acknowledgement

I am extremely grateful to Y. Kawamata for sending me his brilliant series of preprints [2]–[3] from which the ideas in this article are mostly plagiarised. Our immense debt to S. Mori's work will be clear to the reader.[2]

1 Proof of Theorem 0.0 assuming $\kappa(D) \geq 0$

Preliminary Lemma 1.1 *$H^0(mD) = 0$ for at most 3 values of $m \geq a$. (See also Lemma 1.8 below.)*

Proof It follows easily from Riemann–Roch and vanishing (see Corollary 3.2 for the details) that $h^0(mD)$ is a polynomial in m of degree ≤ 3 for $m \geq a$. In §2 below it is shown that this polynomial is not identically zero, and hence has at most 3 zeros. Q.E.D.

1.2 Construction

Let $M \subset |mD|$ be any linear system with $\dim M \geq 0$, $\operatorname{Bs} M \neq 0$. Then there exists a resolution $f\colon Y \to X$, a divisor with normal crossings $\sum F_j$ (for $j \in J$) on Y, and constants a_j, r_j, p_j such that

(1) $K_Y = f^*K_X + \sum a_j F_j$ with $a_j \in \mathbb{Q}$, $a_j \geq 0$ and $a_j > 0$ only if F_j is exceptional for f;

(2) $f^*M = L + \sum r_j F_j$ where L is a free linear system, $r_j \in \mathbb{Z}$, $r_j \geq 0$, and $r_j > 0$ for at least one $j \in J$ (if $\dim M = 0$ then $L = 0$);

(3) $f^*(aD - K_X) - \sum p_j F_j$ is an ample $\mathbb{Q}$-divisor on Y, where $p_j \in \mathbb{Q}$, $0 \leq p_j \ll 1$.

Note for further use that a very slight increase in one of the p_j does not affect the truth of (3).

Remark (Shokurov [13], p. 436, see also 4.3 below) There is no loss of generality in assuming that $r_j \geq a_j$ if $f(F_j)$ is a curve.

[2] Essentially all the results of this paper have been generalised to all dimensions in 2 preprints by Shokurov [13] and Kawamata [4]. Shokurov's paper also sidesteps the difficult proof of §2. I believe that some form of the other main result (Theorem 5.3) is proved in Shokurov [14]. (Note added in 1983–84.)

Proof Let $H \in \operatorname{Pic} X$ be ample. Since $aD - K_X$ is big, for m large enough $h^0(m(aD - K_X) - H) \neq 0$. Choosing $D_1 \in |m(aD - K_X) - H|$ it follows that for every $\varepsilon_1 \in \mathbb{Q}$, $0 < \varepsilon_1 \ll 1$, the $\mathbb{Q}$-divisor $aD - K_X - \varepsilon_1 D_1$ is ample on X.

Now choose a composite of blowups $f\colon Y \to X$ which resolves the singularities of X and the base locus of M, and such that the exceptional locus of f and the inverse image of D_1 form a divisor with normal crossing $\sum F_j$. By construction of f it is clear that there exists an effective divisor $D_2 = \sum c_j F_j$ such that $-D_2$ is relatively ample for f; hence choosing ε_2 with $0 < \varepsilon_2 \ll \varepsilon_1$, and setting $f^*\varepsilon_1 D_1 + \varepsilon_2 D_2 = \sum p_j F_j$ gives (3). Q.E.D.

1.3 The method

Fix the set-up of 1.2. For $b \in \mathbb{Z}$, $c \in \mathbb{Q}$ with $c \geq 0$ and $b \geq cm + a$, the $\mathbb{Q}$-divisor

$$\begin{aligned} N = N(b,c) &= bf^*D + \sum(-cr_j + a_j - p_j)F_j - K_Y \\ &\overset{\text{num}}{\sim} cL + f^*((b - cm)D - K_X) - \sum p_j F_j \end{aligned}$$

is ample on Y, and has fractional part supported in $\sum F_j$. Vanishing gives $H^i(\lceil N \rceil + K_Y) = 0$ for $i > 0$, and I have

$$\lceil N \rceil + K_Y = bf^*D + \Sigma,$$

where I can write

$$\Sigma = \sum \lceil -cr_j + a_j - p_j \rceil F_j = A - B,$$

with A, B effective divisors not having any common components. Since all of $c, r_j, a_j, p_j \geq 0$, A consists of components F_j with $a_j > 0$, and by 1.2, (1) these must be exceptional for f. Hence

$$H^0(X, bD) = H^0(Y, bf^*D) = H^0(Y, bf^*D + A).$$

Now $H^1(bf^*D + A - B) = 0$ implies that

$$H^0(Y, bf^*D + A) \twoheadrightarrow H^0(B, (bf^*D + A)_B).$$

In 1.4 below, it is shown how to adjust the parameter c and the p_j so that B is one of the irreducible components $B = F_0$ of $\sum F_j$, and $-cr_0 + a_0 - p_0 = -1 \in \mathbb{Z}$. From now on, I write $'$ to denote the pullback to B of a divisor on X or Y. Then

$$bf^*D + A = \lceil N \rceil + K_Y + B,$$

so that

$$bD' + A' = (\lceil N \rceil)' + K_B.$$

Now $B = F_0$ appears in N with integral coefficient, so that (see 0.8, (d) for the abuse of notation)

$$(\lceil N \rceil)' = \lceil N' \rceil\,,$$

and N is an ample $\mathbb{Q}$-divisor on B with fractional part supported on the divisor with normal crossing $\sum_{j\neq 0} F'_j$. Hence vanishing applies again to give $H^i(bD' + A') = 0$ for $i > 0$, so that $h^0(bD' + A') = 0$ is a polynomial in b. The subtle part of the argument, Proposition 1.5, is to show that the polynomial cannot be identically zero; this is the only point at which the condition $\dim X = 3$ is used. The method here is due to Xavier Benveniste [1], and improves Kawamata's original proof.

1.4 Selecting a base component of maximal multiplicity

Set $c = \min(a_j + 1 - p_j)/r_j$, taken over $j \in J$ with $r_j > 0$; since $p_j \ll 1$ and $a_j \geq 0$, it follows that $c > 0$. Suppose that $0 \in J$ is one of the indices for which the minimum value occurs; on increasing the corresponding p_0 slightly, c decreases, so that the minimum occurs only for this one component F_0. Then by definition of c,

$$-cr_0 + a_0 - p_0 = -1 \quad \text{and} \quad -cr_j + a_j - p_j > -1 \quad \text{for } j \in J,\ j \neq 0;$$

hence $B = F_0$.

Proposition 1.5 *(i) If $D' \overset{\text{num}}{\sim} 0$ then $h^0(bD' + A') = 1$ for every $b \in \mathbb{Z}$;*

(ii) if $D' \overset{\text{num}}{\not\sim} 0$ then $h^0(bD' + A') > 0$ for every $b \geq cm + a + 1$.

Proof (i) Assume $D' \overset{\text{num}}{\sim} 0$; then for every $b \in \mathbb{Z}$, the $\mathbb{Q}$-divisor

$$N' = bD' + \sum\nolimits_{j\neq 0}(-cr_j + a_j - p_j)F'_j - K_B$$

is ample on B, so that $H^i(\lceil N' \rceil + K_B) = 0$ for $i > 0$, and

$$h^0(bD' + A') = \chi(bD' + A') = \text{const.};$$

for $b = 0$, $h^0(A') \geq 1$ since A' is effective. However, $h^0(bD' + A') \leq 1$ for $b \geq cm + a$, in view of the fact that

$$H^0(Y, bf^*D) = H^0(Y, bf^*D + A) \twoheadrightarrow h^0(bD' + A').$$

(ii) Set

$$p(b) = \frac{1}{2}(D')^2 b^2 + \frac{1}{2}D'(2A' - K_B)b + \frac{1}{2}((A')^2 - A'K_B) + \chi(\mathcal{O}_B),$$

so that

$$0 \leq h^0(bD' + A') = p(b) \quad \text{for } b \geq cm + a.$$

Then

$$p(b+1) - p(b) = \frac{1}{2}\Big((D')^2(b+1) + D'A' + D'(bD' + A' - K_B)\Big).$$

The right-hand side is strictly positive for $b \geq cm + a$. Indeed, D' is nef and A' is effective. so that the first two terms are ≥ 0; furthermore,

$$bD' + A' - K_B = (\lceil N \rceil)' = N' + (\lceil N \rceil - N)' = \begin{pmatrix} \text{ample} \\ \mathbb{Q}\text{-divisor} \end{pmatrix} + \begin{pmatrix} \text{effective} \\ \mathbb{Q}\text{-divisor} \end{pmatrix}$$

so that $D' \overset{\text{num}}{\not\sim} 0$ implies that the third term is strictly positive. Hence $p(b)$ is a strictly increasing function from $cm + a$ onwards. Q.E.D.

1.6 End of the proof

If $h^0(mD) \neq 0$ and $\operatorname{Bs}|mD| \neq \emptyset$ then I claim that for every $a \gg 0$, $\operatorname{Bs}|amD| \subsetneq \operatorname{Bs}|mD|$; Theorem 0.0 then follows by an easy Noetherian induction. For the claim, set $M = |mD|$ in 1.2. The argument of 1.3–1.5 shows that there is a component F_0 of the base locus of $f^*|mD|$ for which

$$H^0(Y, bf^*D) = H^0(Y, bf^*D + A) \twoheadrightarrow H^0(F_0, (bf^*D + A)_{F_0}) \neq 0$$

for every $b \gg 0$, so that $F_0 \not\subset \operatorname{Bs}|bf^*D|$, and hence $f(F_0) \not\subset \operatorname{Bs}|bD|$. In particular, taking $b = am$ with $a \gg 0$,

$$\operatorname{Bs}|amD| \subsetneq \operatorname{Bs}|mD|. \quad \text{Q.E.D.}$$

1.7 Proof of Proposition 0.1

(a) is "relative vanishing". Let $H \in \operatorname{Pic} \mathbb{Z}$ be an ample divisor such that $D = \varphi^* H$; consider the Leray spectral sequence for $H^i(X, \mathcal{O}_X(mD))$, using $R^i\varphi_*\mathcal{O}_X(mD) \cong R^i\varphi_*\mathcal{O}_X \otimes \mathcal{O}_Z(mH)$:

$$E_2^{p,q} = H^p(Z, R^q\varphi_*\mathcal{O}_X \otimes \mathcal{O}_Z(mH)) \implies H^i(X, \mathcal{O}_X(mD)).$$

Since H is ample on Z, Serre vanishing gives that for $m \gg 0$, $E_2^{p,q} = 0$ if $p \neq 0$, and hence $H^0(R^q\varphi_*\mathcal{O}_X \otimes \mathcal{O}_Z(mH)) = H^q(X, \mathcal{O}_X(mD))$. But by vanishing,

$H^q(X, \mathcal{O}_X(mD)) = 0$ for $m \geq a$ (see Proposition 3.1), and hence $R^q\varphi_*\mathcal{O}_X = 0$ for $q > 0$. Finally, for every $m \geq a$, $H^p(Z, \mathcal{O}_Z(mH)) = H^p(X, \mathcal{O}_X(mD)) = 0$ for $p > 0$.

For (b), set r = index of X, and choose $m \geq a(r+1)$; then $D' = mD - rK_X \in \operatorname{Pic} X$, and both D' and $D' - K_X$ are nef and big. Applying Theorem 0.0 to D' gives the morphism g; it contracts exactly the curves $C \subset X$ with $DC = K_XC = 0$, so φ factors through g.

There are only 2 nontrivial assertions in (c): when $\dim Z = 2$, $X \to Z$ is birational to a standard conic bundle by Sarkisov [11]: I have

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{f_1} & X & \xrightarrow{\varphi} & Z \\ & {}_g\searrow & & & \uparrow f_2 \\ & & Y & \xrightarrow{h} & S \end{array}$$

where f_1 and f_2 are resolutions, g is a birational morphism and h is a standard conic bundle. Then by (a) above,

$$\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_X);$$

since X has rational singularities, and g is a birational morphism of smooth varieties, $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\tilde{X}} = \chi(\mathcal{O}_Y)$; and h is a standard conic bundle, so that $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_S)$.

Hence $\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_S)$, proving that Z has rational singularities.

Finally, if $Z = \mathrm{pt}$, then $\operatorname{Pic} X$ is reduced because $H^1(\mathcal{O}_X) = 0$; if $D \in \operatorname{Pic} X$ is a torsion element then Theorem 0.0 applies to D to give $D = 0$, hence $\operatorname{Pic} X$ is torsion free. Q.E.D.

1.8

The rest of this section is concerned with the proof of Corollary 0.4; the reader who is more interested in the rest of the proof of Theorem 0.0 should proceed to §2.

Lemma $h^0(mD) > 0$ *for* $m \geq 2a + 2$.

Proof As seen in Lemma 1.1, $h^0(mD) = p(m)$ is a polynomial in m of degree ≤ 3 for $m \geq a$; if $\deg p \leq 1$ then obviously $h^0(mD) > 0$ for $m \geq a+1$. If $\deg p = 2$ or 3 then p has at most 2 integer zeros $\geq a+1$, since if p is cubic, $p(a) \geq 0$ implies that one real root of p is $\leq a$; furthermore if there are 2 integer zeros $\geq a+1$ these must be consecutive, since $p(x) < 0$ between them.

Now the set $\{m \mid h^0(mD) \neq 0\}$ is a semigroup, and if p has no zeros in $[a+1, \dots, 2a]$ is certainly contains every integer $\geq 2a+2$. The alternative is that some $b \leq 2a$ is a zero, and then possibly $b+1$ is also a zero, but $p(m) > 0$ for $m \geq 2a+2$. Q.E.D.

1.9 Proof of Corollary 0.4

Let $m \geq 2a+2$; if $\Gamma \subset X$ is a prime divisor appearing as base component of multiplicity ≥ 2 of $M = |mD|$, then making the construction of 1.2, the proper transform of Γ is an F_j with $a_j = 0$, $r_j \geq 2$. Then by definition of c (in 1.4), $c \leq \frac{1}{2}$. Now the argument of 1.3–1.5 shows that the base component F_0 of $|mf^*D|$ of maximal multiplicity in the sense of 1.4 is not a base component of $|bf^*D$ for $b \geq cm + a + 1$. But m itself satisfies $m \geq cm + a + 1$, which is a contradiction.

The argument for the other statements of Corollary 0.4 is similar, and I only sketch it: if $C \subset \operatorname{Sing} X$ is a 1-dimensional component then by [8], Theorem 1.14, X has a Du Val singularity at the generic point $\eta \in C$. Above η, the resolution $f\colon Y \to X$ dominates the minimal resolution, and so contains a number of components F_j with $a_j = 0$, which by the argument just given must have $r_j \leq 1$. Using easy facts about the resolution of Du Val singularities (see Lemma 4.3, (iii)), it is then easy to see that X has an A_n point at η, and M an ordinary double point.

If $C \subset X$ is a curve with $C \not\subset \operatorname{Sing} X$ appearing in the general element of M with multiplicity ≥ 3, the blowup of C gives an F_j with $a_j = 1$, $r_j \geq 3$, so that $c \leq \frac{2}{3}$, which by the same argument is impossible if $m \geq 3a+3$. Finally, if the general element of M has a non-ordinary double locus along C, then after 3 blowups I get a component F_j with $a_j = 4$, $r_j \geq 6$: for example, a curve of ordinary cusps gives the embedded resolution of Figure 1. Then

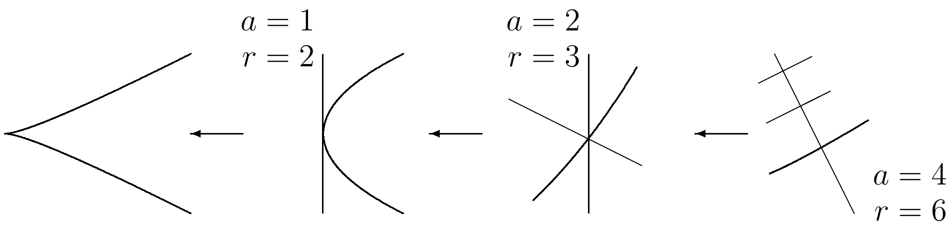

Figure 1: Embedded resolution of cuspidal curve $y^2 = x^3$

$c \leq \frac{5}{6}$ and by the same argument this is impossible if $m \geq 6a+6$. Q.E.D.

The following result is exactly similar to Corollary 0.4, and will be used in the proof of Theorem 0.5 in §4.

Lemma 1.10 *Let X be a weak Fano 3-fold; then the general element $D \in |-K_X|$ is reduced and has only ordinary double curves.*

Proof As in 1.2, there exists a resolution $f\colon Y \to X$, a divisor with normal crossings $\sum F_j$ and constrants a_j, r_j, p_j and q such that

(1) $K_Y = f^*K_X + \sum a_jF_j$, where $a_j \in \mathbb{Z}$, $a_j \geq 0$ and $a_j > 0$ only if F_j is exceptional for f;

(2) $f^*|-K_X| = L + \sum r_jF_j$ with $|L|$ a free linear system, $r_j \in \mathbb{Z}$ and $r_j \geq 0$;

(3) $qf^*(-K_X) - \sum p_jF_j$ is an ample $\mathbb{Q}$-divisor, where $p_j, q \in \mathbb{Q}$, $0 \leq p_j \ll 1$ and $0 < q < \min\{1/r_j\}$, the minimum being taken over j with $r_j > 0$.

Claim *For every j, $r_j \leq a_j + 1$.*

As in the proof of Corollary 0.4, this implies that the general element $D \in |-K_X|$ is reduced, with ordinary double curves, proving Lemma 1.10.

To prove the claim, suppose that $r_j \geq a_j + 2$ for some j. Then setting

$$c = \min\left\{\frac{a_j + 1 - p_j}{r_j}\right\},$$

it follows that $c \leq 1 - 1/r_j$, and hence $1 - c \geq q$. As in Method 1.3, set

$$\begin{aligned} N = N(b,c) &= bf^*(-K_X) + \sum(-cr_j + a_j - p_j)F_j - K_Y \\ &\overset{\text{num}}{\sim} cL + (b + 1 - c)f^*(-K_X) - \sum p_jF_j; \end{aligned}$$

by (3) and the fact that $1 - c \geq q$, this is an ample $\mathbb{Q}$-divisor for $b \geq 0$. The argument of Method 1.3 and Proposition 1.5 now gives a contradiction: the component $B = F_0$ which is the base component of $f^*|-K_X|$ of maximal multiplicity is not a base component of $|bf^*(-K_X)|$ for $b \geq 1$. This proves the claim, and hence Lemma 1.10.

2 Proof of $\kappa(D) \geq 0$

2.1

Let X, D and a be as in Theorem 0.0, and $f\colon Y \to X$ any resolution for which the exceptional locus is a divisor with normal crossings; then for any $m \geq a$ and any $D_m \in \operatorname{Pic} X$, with $D_m \overset{\text{num}}{\sim} mD$,

$$h^0(D_m) = \frac{1}{6}D^3m^3 - \frac{1}{4}D^2K_Xm^2 + \frac{1}{12}(DK_X^2 + f^*Dc_2(Y))m + \chi(\mathcal{O}_X). \quad (*)$$

This is proved in Corollary 3.2 below. The right-hand side is a polynomial in m, and the purpose of this section is to prove that it is not identically zero.

Note first that this is trivial if $\kappa_{\text{num}}(D) \neq 1$. Indeed, if $\kappa_{\text{num}} = 3$ then $D^3 > 0$; if $\kappa_{\text{num}} = 2$ then by Lemma 0.9, $-D^2K_X = D^2(aD - K_X) > 0$; finally, if $D \overset{\text{num}}{\sim} 0$ then I can take $D_m = 0$ for every m, and $h^0(D_m) = 1$.

Note then that Theorem 0.0 is proved in case $\kappa_{\text{num}}(D) \geq 2$, and I'm entitled to use it in the proof for $\kappa_{\text{num}}(D) = 1$.

Remark By Lemma 0.9, $DK_X^2 = D(aD - K_X)^2 > 0$ in case $\kappa_{\text{num}}(D) = 1$, and as conjectured in Problem 0.7, (c), we have a right to expect that $f^*Dc_2(Y) < 0$ should lead to some very strong restriction on Y; unfortunately, I don't know how to exploit this, so I don't get any pleasure out of the linear term in $h^0(D_m)$. A posteori, if $\varphi\colon X \to Z$ is a weak fibre space of del Pezzo surfaces of degree d (as defined in Proposition 0.1), and if $D = \varphi^*H$ then $f^*Dc_2(Y) = (12 - d)\deg H$ with $1 \leq d \leq 9$, so that in fact $f^*Dc_2(Y) > 0$.

Proposition 2.2 *If* $\kappa_{\text{num}}(D) = 1$ *then* $\kappa(X) = -\infty$*, and in particular* $p_g = 0$*. Hence if* $\chi(\mathcal{O}_X) = 0$ *then* $q = h^1(\mathcal{O}_X) > 0$*.*

Proof $aD - K_X$ is nef and big, so that by Lemma 0.9, (ii)

$$(-K_X)(aD - K_X)D = (aD - K_X)^2D > 0;$$

hence $H^0(mK_X) = 0$ for all $m > 0$. Q.E.D.

Proposition 2.3 *Let* X *be a normal variety having a resolution* $f\colon Y \to X$ *such that* $R^1f_*\mathcal{O}_Y = 0$*. Then* $f^*\colon \operatorname{Pic}^0 X \xrightarrow{\simeq} \operatorname{Pic}^0 Y$ *is an isomorphism, and the Albanese map of* Y *factors through* X*. In particular if* $h^1(\mathcal{O}_X) \neq 0$ *(and* $\operatorname{char} k = 0$*, of course), then there is a nontrivial morphism* $\alpha\colon X \to \operatorname{Alb} X$ *from* X *to an Abelian variety.*

Proof This is general nonsense. $R^1f_*\mathcal{O}_Y = 0$ implies that $f^*\colon H^1(\mathcal{O}_X) \xrightarrow{\simeq} H^1(\mathcal{O}_Y)$, and hence that $f^*\operatorname{Pic}^0 X \to \operatorname{Pic}^0 Y$ is etale. Now the morphism $\alpha\colon X \to (\operatorname{Pic}^0 X)^\vee$ is defined by the universal property of Pic: if P is the (Poincaré) universal line bundle over $X \times \operatorname{Pic}^0 X$ then $\alpha\colon X \to (\operatorname{Pic}^0 X)^\vee$ is defined on the level of points by taking $x \in X$ to P_X, the restriction of P to $x \times \operatorname{Pic}^0 X$, considered as a point of $(\operatorname{Pic}^0 X)^\vee$. Functoriality of Pic gives a commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\alpha_Y} & (\operatorname{Pic}^0 Y)^\vee & = \operatorname{Alb} Y \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle f^\vee} & \\ X & \xrightarrow{\alpha_X} & (\operatorname{Pic}^0 X)^\vee, & \end{array}$$

where f is birational and $f^\vee$ an isogeny of Abelian varieties. It is then obvious that any curve contracted by f is also contracted by α_Y, so that using the Zariski Main Theorem, the diagram splits as indicated by the oblique arrow, and $f^\vee$ is an isomorphism. Q.E.D.

2.4

If $\kappa_{\text{num}}(D) = 1$ and $\kappa(D) = -\infty$ then by $(*)$ in 2.1, $\chi(\mathcal{O}_X) = 0$, and $q(X) \neq 0$ by Proposition 2.2, so that by Proposition 2.3, X has a nontrivial morphism $\alpha\colon X \to \operatorname{Alb} X$ to an Abelian variety. Since $\kappa(X) = -\infty$, $\dim \alpha(X) \leq 2$. I prove later (Key Lemma 2.6) that even in the case that $\alpha(X) = F$ is a surface, X has a surjective morphism $h\colon X \to C$ to a curve of genus ≥ 1. First of all, I show how to complete the proof from this.

Proposition 2.5 *Let X, D and a be as in Theorem 0.0. Suppose that $\kappa_{\text{num}}(D) = 1$, and that X has a surjective morphism $h\colon X \to C$ to a curve of genus $g \geq 1$. Then there exists an $m \geq a$ and an effective divisor D_m with $D_m \stackrel{\text{num}}{\sim} mD$; hence by $(*)$ in 2.1, $h^0(mD) \neq 0$ for every $m \gg 0$.*

Proof Let A be a general fibre of $X \to C$. The easy case is when $D_{|A} \stackrel{\text{num}}{\sim} 0$; then $D^2 \stackrel{\text{num}}{\sim} DA \stackrel{\text{num}}{\sim} A^2 \stackrel{\text{num}}{\sim} 0$, so that by the Index Theorem 0.10, D is numerically equivalent to qA for $q \in \mathbb{Q}$. Proposition 2.5 is then obvious.

In the other case $D_{|A} \stackrel{\text{num}}{\not\sim} 0$, the proof proceeds by reducing to a similar looking problem over a surface.

STEP 1 h factors as

$$\begin{array}{ccc} X & \xrightarrow{h} & C \\ & {\scriptstyle\varphi}\searrow \quad \nearrow{\scriptstyle g} & \\ & S & \end{array}$$

where

(i) S is a surface with rational singularities;

(ii) there exists $L \in \operatorname{Pic} S$ which is relatively ample for g, and such that $D = \psi^* L$ with $L^2 = 0$;

(iii) $\varphi_* \mathcal{O}_X = \mathcal{O}_S$, $R^i \varphi_* \mathcal{O}_X = 0$ for $i > 0$ and $H^i(S, mL) = 0$ for all $m \geq a$ and $i > 0$.

Proof This is a relative form of Theorem 0.0, and comes by noting that for $i \geq 1$, $D + iA$ is a divisor on X satisfying the hypotheses of Theorem 0.0, and with $\kappa_{\text{num}}(D + iA) = 2$. The morphism φ contracts exactly the curves of X with $DC = AC = 0$, so h factors through S.

Step 2 L is relatively ample for g, so for $m \gg 0$, $R^1 g_* L^{\otimes m} = 0$ by Serre vanishing. Thus for $m \gg 0$, $g_* L^{\otimes m} = \mathcal{E}_m$ is a vector bundle on C of rank $r > 0$ with

$$0 \leq h^0(S, L^{\otimes m}) = \chi(S, L^{\otimes m}) = \chi(C, \mathcal{E}_m).$$

The following statement implies that for $m \gg 0$ and for suitable $\mathcal{L} \in \operatorname{Pic}^0 C$,

$$0 \neq H^0(C, \mathcal{E}_m \otimes \mathcal{L}) = H^0(S, \mathcal{L}^{\otimes m} \otimes g^*\mathcal{L}) = H^0(X, \mathcal{O}_X(mD) \otimes h^*\mathcal{L}).$$

proving Proposition 2.5:

Easy Exercise Let $\mathcal{E}$ be a vector bundle of rank $r > 0$ over a curve C with $\chi(C, \mathcal{E}) \geq 0$. Then

either $C \cong \mathbb{P}^1$ and $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r}$,

or for every $P \in C$ there exists $Q \in C$ such that $H^0(\mathcal{E} \otimes \mathcal{O}_C(P - Q)) \neq 0$.

Proposition 2.5 is proved. Q.E.D.

Now comes the hard part.

Key Lemma 2.6 *Let X, D and a be as in Theorem 0.0, with $\kappa_{\text{num}}(D) = 1$, and assume that $\alpha(X) = F \subset \operatorname{Alb} X$ is a surface. Then F is a fibre bundle $F \to C$ over a curve C of genus $g \geq 1$ (with fibre an elliptic curve); in particular, there exists a surjective morphism $h\colon X \to C$ to a curve of genus $g \geq 1$.*

Sublemma 2.7 *(i) If S is any effective Weil divisor on X which is nef and big, then one component of S maps surjectively to F.*

(ii) If $S_0 \subset X$ is any surface for which $\alpha(S_0) = F$ then for $m > a$, we have $(mD - K_X)^2 S_0 > 0$.

Proof Applying Lemma 0.9 to α^*M, where M is ample on F, (i) is trivial. For (ii), setting r = index of X, $r(mD - K_X) \in \operatorname{Pic} X$ obviously satisfies the hypotheses of Theorem 0.0, with $\kappa_{\text{num}}(mD - K_X) = 3$, so that there is a birational morphism $\varphi\colon X \to Z$ such that $mD - K_X = \varphi^*H$ for H an ample $\mathbb{Q}$-divisor on Z. By Proposition 0.1, Z has only rational singularities, so that using Proposition 2.3 above, I get that α factors through Z: that is, $\alpha\colon X \to Z \to F \subset \operatorname{Alb} X$. Now S_0 must map to a surface in Z, which gives the result. Q.E.D.

Proof of Key Lemma 2.6 It is shown in Corollary 3.3 below that for $m \gg 0$, $h^0(mD - K_X) \neq 0$; let $f\colon Y \to X$ be a resolution which induces the minimal resolution along the Du Val locus, so that $K_Y = f^*K_X + \Delta$, where $f(\Delta)$ is a finite set (f is 0-minimal in the sense of [8], §5). Now it follows directly from the definition of canonical singularities that, for $i \geq 0$, there is a map $f'\colon f^{-1}\omega_X^{[i]} \to \omega_Y^{\otimes i}$ (where f^{-1} is the sheaf theoretic inverse image), defined by viewing $s \in H^0(U, \omega_X^{[i]})$ as a rational i-fold canonical differential, which then remains regular on $f^{-1}U$. This gives a map ("proper transform")

$$\begin{aligned} f'\colon H^0(mD - K_X) &= H^0(\mathcal{O}_X(mD - rK_X) \otimes \omega_X^{[r-1]}) \\ &\longrightarrow H^0(\mathcal{O}_Y(f^*(mD - rK_X) + (r-1)K_Y) \\ &= H^0(\mathcal{O}_Y(f^*mD - K_Y + r\Delta). \end{aligned}$$

Let $S \in |mD - K_X|$ and $T = f'S \in |mD - K_Y + r\Delta|$; write $T = \sum a_iT_i$. By Sublemma 2.7 applied to $S \subset X$, there is a component T_0 of T mapping surjectively to F, and such that $f^*(mD - K_X)^2T_0 > 0$. Write $g\colon \widetilde{T} \to T_0$ for the minimal resolution; since T_0 is Gorenstein, $K_{\tilde{T}} = g^*K_{T_0} - Z$, with Z an effective divisor on $\widetilde{T}$. Now by adjunction

$$\begin{aligned} a_0K_{T_0} &= \Big(a_0K_Y + mf^*D - K_Y + r\Delta - \sum\nolimits_{i\neq 0} a_iT_i\Big)_{|T_0} \\ &= \Big(a_0mf^*D - (a_0 - 1)f^*(mD - K_X) - \sum_{i\neq 0} a_iT_i + (r + a_0 - 1)\Delta\Big)_{|T_0}, \end{aligned}$$

so that, writing $'$ for the pullback of a divisor on X or Y to $\widetilde{T}$, we get

$$\begin{aligned} &a_0mD' + (r + a_0 - 1)\Delta' \\ &\qquad = a_0K_{\tilde{T}} + (a_0 - 1)f^*(mD - K_X)' + (a_0Z + \sum\nolimits_{i\neq 0} a_iT_i)'. \end{aligned}$$

Now restricting $f\colon Y \to X$ to T_0, f induces a birational map $\widetilde{f}\colon \widetilde{T} \to S_0$, where S_0 is a component of S, and Δ' is contracted by $\widetilde{f}$. It follows that the left-hand side of this formula is a $\mathbb{Q}$-divisor with $\kappa \leq 1$. On the other hand, if $a_0 \neq 1$, or if $\widetilde{T}$ is a surface of general type, then the right-hand side has $\kappa = 2$: indeed, $h^0(K_{\tilde{T}}) > 0$ because $\widetilde{T}$ has a generically finite morphism to $F \subset \operatorname{Alb} X$, $(mD - K_X)'$ is nef and big on $\widetilde{T}$, and the third term is effective. Hence $a_0 = 1$, and $\kappa(\widetilde{T}) = 0$ or 1. The above adjunction formula simplifies to

$$mD' + r\Delta' = K_{\tilde{T}} + (Z + \textstyle\sum_{i\neq 0} a_i T_i)'. \qquad (**)$$

Case $\kappa(\widetilde{T}) = 1$ This is the easy case: $\widetilde{T}$ has a generically finite morphism to $F \subset \operatorname{Alb} X$, so that the elliptic structure of the minimal model of $\widetilde{T}$ is a fibre bundle; the image of any fibre is an elliptic curve $E \subset \operatorname{Alb} X$ such that F is invariant under translations by E.

Case $\kappa(\widetilde{T}) = 0$ Then $\widetilde{T}$ is itself birational to an Abelian surface, and I have the following set-up:

$$\begin{array}{lll} T & \in & |mf^*D - K_Y + r\Delta|, \quad T = \sum a_i T_i \\ \downarrow & & \\ S & \in & |mD - K_X| \end{array} \qquad \begin{array}{ccccc} Y & \supset & T_0 & \xleftarrow{\;g\;} & \widetilde{T} \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle h} \\ X & \supset & S_0 & \xleftarrow{\;\nu\;} & \widetilde{S} \\ & & & & \downarrow{\scriptstyle j} \\ & & & & G \\ & & & & \downarrow \\ & & & \operatorname{Alb} X = & F \end{array}$$

where $\nu\colon \widetilde{S} \to S_0$ is the normalisation of S_0, and in the left-hand column,

$$G = \operatorname{Alb} \widetilde{T} = \text{minimal model of } \widetilde{T}$$

is an etale cover of F. Now $\widetilde{S}$ has rational singularities, and $K_{\tilde{S}}$ is an effective Weil divisor containing every exceptional curve of j with strictly positive coefficient. $(**)$ gives

$$m\nu^* D = K_{\tilde{S}} + h_*((Z + \textstyle\sum a_i T_i)'). \qquad (***)$$

Subcase $\nu^* D \overset{\text{num}}{\sim} 0$ The right-hand side of $(***)$ is an effective $\mathbb{Q}$-divisor, so that $h_*((\sum_{i\neq 0} a_i T_i)') = 0$; it is clear that this implies that S_0 does not meet $S - S_0$ in curves, and then by the connectivity result Lemma 0.9, (iii), that $S = S_0$. Then $\nu^* D \overset{\text{num}}{\sim} 0$ is impossible: by Lemma 0.9, (i)

$$0 < (mD - K_X)^2 D = \nu^*(mD - K_X)\nu^* D.$$

Subcase $\nu^*D \overset{\text{num}}{\not\sim} 0$ In this case ν^*D is nef and $(\nu^*D)^2 = 0$, so that (***) gives $(\nu^*D)\Gamma_i = 0$ for every exceptional curve Γ_i of j; using $j_*\mathcal{O}_S = \mathcal{O}$, it follows that $\nu^*D = j^*D_G$, where D_G is an effective $\mathbb{Q}$-divisor on G; $(\nu^*D)^2 = 0$ implies $D_G^2 = 0$, so that G is not a simple Aelian variety, hence F has a surjective morphism to an elliptic curve. Q.E.D.

3 Computing $h^0(mD)$ and $h^0(mD - K_X)$

Write $r =$ index of X; for $q \in \mathbb{Z}$, write $q = pr + i$ with $0 \le i \le r-1$. Let $f\colon Y \to X$ be a resolution which coincides with the minimal resolution above the Du Val locus, and such that the exceptional locus of f is a divisor with normal crossings.

Proposition 3.1 *(i) Suppose that $A \in \operatorname{Pic} X$ is such that $A - K_X$ is nef and big. Then $H^k(X, A) = 0$ for $k > 0$, and*

$$\begin{aligned} h^0(X,A) &= \chi(X,A) = \chi(Y, f^*A) \\ &= \frac{1}{6}A^3 - \frac{1}{4}A^2K_X + \frac{1}{12}(AK_X^2 + f^*Ac_2(Y)) + \chi(\mathcal{O}_X). \end{aligned}$$

(ii) Suppose that $A \in \operatorname{Pic} X$ and $q \in \mathbb{Z}$ are such that $A + (q-1)K_X$ is nef and big; then

$$\begin{aligned} h^0(X, A + qK_X) &\ge h^0(f^*(A + prK_X) + iK_Y + \lceil -(i-1)\Delta \rceil) \\ &= \chi(f^*(A + prK_X) + iK_Y + \lceil -(i-1)\Delta \rceil); \end{aligned}$$

if we set $R_i = i\Delta + \lceil -(i-1)\Delta \rceil$, this is equal to

$$\begin{aligned} &= \frac{1}{6}(A + qK_X)^3 - \frac{1}{4}(A + qK_X)^2K_X \\ &\quad + \frac{1}{12}\Big((A + qK_X)K_X^2 + f^*(A + qK_X)c_2(Y)\Big) \\ &\quad + \frac{1}{6}R_i^3 - \frac{1}{4}R_i^2K_Y + \frac{1}{12}R_i(K_Y^2 + c_2(Y)) + \chi(\mathcal{O}_X). \end{aligned}$$

Proof The $\mathbb{Q}$-divisor

$$\begin{aligned} N &= f^*(A + prK_X) + (i-1)K_Y - (i-1)\Delta \\ &= f^*(A + (q-1)K_X) \end{aligned}$$

is nef and big on Y, so that vanishing gives $H^k(\lceil N \rceil + K_Y) = 0$ for $k > 0$; now

$$\lceil N \rceil + K_Y = f^*(A + prK_X) + iK_Y + \lceil -(i-1)\Delta \rceil .$$

For (i), $p = i = 0$, so that $\lceil N\rceil + K_Y = f^*A + \lceil\Delta\rceil$. Now from the exact sequence

$$0 \to \mathcal{O}_Y(f^*A) \to \mathcal{O}_Y(f^*A + \lceil\Delta\rceil) \to \mathcal{O}_{\lceil\Delta\rceil}(\lceil\Delta\rceil) \to 0,$$

we get

$$H^k(\mathcal{O}_{\lceil\Delta\rceil}(\lceil\Delta\rceil)) = H^{k+1}(\mathcal{O}_Y(f^*A)) \quad \text{for } k \geq 0.$$

Since $R^k f_*\mathcal{O}_Y = 0$ for $k > 0$,

$$H^k(\mathcal{O}_{\lceil\Delta\rceil}(\lceil\Delta\rceil)) = H^{k+1}(\mathcal{O}_X(A)) \quad \text{for } k \geq 0.$$

The left-hand side does not depend on the particular $A \in \operatorname{Pic} X$, and by taking A to be a large multiple of an ample divisor the right-hand side is zero by Serre vanishing. Hence $H^k(\mathcal{O}_{\lceil\Delta\rceil}(\lceil\Delta\rceil)) = 0$, and

$$H^k(X, A) = H^k(Y, f^*A) = H^k(Y, f^*A + \lceil\Delta\rceil) \quad \text{for } k \geq 0.$$

This proves (i).

For (ii), I can assume that $i \geq 1$, so that $\lceil -(i-1)\Delta\rceil$ is minus an effective divisor, and

$$\begin{aligned} H^0(N + K_Y) = H^0(f^*(A + prK_X) + iK_Y + \lceil -(i-1)\Delta\rceil) \\ \subset H^0(f^*(A + prK_X) + iK_Y). \end{aligned}$$

Since by definition of canonical singularities $f_*\omega_Y^{\otimes i} = \omega_X^{[i]}$, the final group is equal to $H^0(X, A + qK_X)$. Finally,

$$h^0(\lceil N\rceil + K_Y) = \chi(\lceil N\rceil + K_Y);$$

substitute

$$\lceil N\rceil + K_Y = f^*(A + qK_X) + R_i$$

in the Riemann–Roch polynomial; using the fact that $f(\operatorname{Supp}\Delta)$ is a finite set, all terms involving $f^*(A + qK_X) \cdot \Delta$ or $f^*(A + qK_X) \cdot R_i$ vanish, giving the formula in (ii). Q.E.D.

Corollary 3.2 *Let X, D and a be as in Theorem 0.0; then for any $m \geq a$, and any $D_m \in \operatorname{Pic} X$ with $D_m \overset{\text{num}}{\sim} mD$,*

$$h^0(D_m) = \frac{1}{6}D^3m^3 - \frac{1}{4}D^2K_Xm^2 + \frac{1}{12}(DK_X^2 + f^*Dc_2(Y))m + \chi(\mathcal{O}_X).$$

Proof Substitute $A = D_m$ in (i).

Note also that the hypothesis in Proposition 3.1 that f coincides with the minimal resolution above the Du Val locus is a posteori not necessary, since $f^*Dc_2(Y)$ is independent of the model $f\colon Y \to X$.

Corollary 3.3 *Let X, D and a be as in Theorem 0.0; then if $D \overset{\text{num}}{\not\sim} 0$, $h^0(mD - K_X)$ tends to infinity with m.*

Proof For $m \geq 2a$, $mD - 2K_X$ is nef and big, so that Proposition 3.1, (ii) applies:

$$h^0(mD - K_X) \geq \frac{1}{6}(mD - K_X)^3 - \frac{1}{4}(mD - K_X)^2 K_X + \\ + \frac{1}{12}\Big((D - K_X)K_X^2 + f^*(mD - K_X)c_2 Y\Big) + \text{const. in } m.$$

If $D^2 \overset{\text{num}}{\not\sim} 0$, this grows at least like m^2. If $D^2 \overset{\text{num}}{\sim} 0$, the linear term in m is

$$\Big(DK_X^2 + \frac{1}{12}(DK_X^2 + f^*Dc_2(Y)\Big)m.$$

Now by Corollary 3.2, $\frac{1}{12}\big(DK_X^2 + f^*Dc_2(Y)\big)$ is the coefficient of m in $h^0(mD)$, and therefore

$$\frac{1}{12}(DK_X^2 + f^*Dc_2(Y) \geq 0;$$

also

$$DK_X^2 = D(D - K_X)^2 > 0$$

by Lemma 0.9. Q.E.D.

4 The base locus of $|-K_X|$ for a weak Fano 3-fold

In this section I prove Theorem 0.5 by polishing up Shokurov's ingenious proof [12]. The key points are Proposition 4.5 and 4.8–4.10 below, and the reader may like to jump forward to these while I unburden myself of some trivialities.

4.1 Preliminaries: 0-minimal resolution

Let X be a 3-fold with canonical singularities and $\mathcal{I} \subset \mathcal{O}_X$ an ideal (in application, $\mathcal{I}$ is the ideal defining the base locus of a linear system). If $C \subset X$ is any irreducible curve, $P \in C$ a general point and $P \in X' \subset X$ a local general hyperplane section through P, $P \in X'$ will be a Du Val singularity or nonsingular point. Let $\mathcal{I}' \subset \mathcal{O}_{X',P}$ be the ideal induced by $\mathcal{I}$. A *good resolution* $f\colon Y \to X$ of X and $\mathcal{I}$ is a resolution having a normal crossing divisor $\sum F_j$ which includes the exceptional locus of f, and such that

$$\mathcal{I} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-\sum r_j F_j);$$

by Bertini's theorem, f induces a good resolution f' of X' and $\mathcal{I}'$:

$$\begin{array}{ccccccc} G_k & \subset & Y' & \subset & Y & \supset & F_j \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \\ & & X' & \subset & X; & & \end{array}$$

here each G_k is a connected component of some $F_j \cap Y'$ and $r_k = r_j$ (that is, $r(G_k) = r(F_j)$). Say that f is a 0-*minimal* good resolution if f' is the minimal good resolution of X' and $\mathcal{I}'$ for all X'. It is easy to construct this by successively blowing up 1-dimensional components of $\operatorname{Sing} X$ and of the locus where $\mathcal{I}$ is not invertible, and then making an arbitrary resolution which is an isomorphism except over a finite set of X.

Lemma 4.2 *Let $P \in X'$ be a Du Val singularity or nonsingular point, and $\mathcal{I}' \subset \mathcal{O}_{X',P}$ an ideal; suppose that $f'\colon Y' \to X'$ is a good resolution of $P \in X'$ and $\mathcal{I}'$, and set*

$$\mathcal{I} \cdot \mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-\sum r_k G_k); \quad K_{Y'} = f'^* K_{X'} + \sum a_k G_k.$$

Then f' is the minimal good resolution of X' and $\mathcal{I}'$ if and only if there does not exist a -1-curve $G_k \subset {f'}^{-1} P \subset Y'$ which meets at most two other components G_{k_i} such that $r_k = \sum r_{k_i}$.

Lemma 4.3 *Furthermore, if f' is the minimal good resolution, the following hold:*

(i) $r_j \geq a_j$ for all j.

(ii) Except for cases (a–b) below, $r_j > a_j$ for all j.

(iii) $r_j \leq 1$ for all j is only possible in one of the cases (a–d) below.

Here the exceptional cases are:

(a) *$P \in X'$ is nonsingular, $\mathcal{I}' = m_P$ and f' is the blowup of P;*

(b) *$\mathcal{I}' = \mathcal{O}_{X',P}$ and f' is the minimal resolution of $P \in X'$;*

(c) *$P \in X'$ is nonsingular, $\mathcal{I}' = \mathcal{I}_H$ where $H \subset X'$ is a curve with normal crossing at P (either nonsingular or a node), and $f' = \mathrm{id}_{X'}$;*

(d) *$P \in X'$ is an A_n point for $n \geq 1$ and $\mathcal{I}'$ contains an element h defining a curve $H \subset X'$ having a node at P.*

The proof is an easy exercise.

4.4

Now let X be a weak Fano 3-fold, that is, a projective 3-fold with canonical singularities and $-K_X \in \operatorname{Pic} X$ nef and big. It follows from Riemann–Roch and vanishing (as in Proposition 3.1) that $h^0(-K_X) = g + 2$, where $g \in \mathbb{Z}$, $g \geq 2$ is defined by $-K_X^3 = 2g-2$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal defining the base locus of $|-K_X|$, that is, $\mathcal{I} \cdot \mathcal{O}_X(-K_X)$ is the $\mathcal{O}_X$-submodule of $\mathcal{O}_X(-K_X)$ generated by the H^0.

Let $f\colon Y \to X$ be a 0-minimal good resolution of X and $\mathcal{I}$, and let $\sum F_j$ be as usual; set

$$\left.\begin{aligned} K_Y &= f^*K_X + \sum a_j F_j, \\ f^*|-K_X| &= |L| + \sum r_j F_j, \end{aligned}\right\} \qquad (*)$$

where $a_j, r_j \in \mathbb{Z}$, $a_j, r_j \geq 0$ and $|L|$ is a free linear system. I start by proving Theorem 0.5 assuming that $|L|$ is not composed of a pencil, that is, by Bertini's theorem, the general $L \in |L|$ is irreducible, nonsingular and $\kappa_{\mathrm{num}}(L) \geq 2$.

Proposition 4.5 *Under the hypotheses of 4.4, suppose that $|L|$ is not composed of a pencil. Then $\chi(\mathcal{O}_L) \geq 2$.*

Proof L is a nonsingular surface, and $f^*(-K_X)_{|L}$ is nef and big by 0.9, (ii). Thus vanishing gives

$$H^i(L, \mathcal{O}_L(f^*(-K_X) + K_L)) = 0 \quad \text{for } i \geq 0.$$

Using $(*)$,

$$K_Y + L + f^*(-K_X) = L + \sum a_j F_j;$$

hence

$$\begin{aligned} g+1 &\le h^0(L,\mathcal{O}_L(L)) \le h^0(L,\mathcal{O}_L(L+\sum a_jF_j)) \\ &= \chi(\mathcal{O}_L) + \frac{1}{2}(L+\sum a_jF_j)f^*(-K_X)L, \end{aligned}$$

by Riemann–Roch on L. However,

$$\begin{aligned} 2g-2 = f^*(-K_X)^3 \ge f^*(-K_X)^2L &= f^*(-K_X)(L+\sum r_jF_j)L \\ &\ge f^*(-K_X)(L+\sum a_jF_j)L, \end{aligned}$$

using the fact that $r_j \ge a_j$ unless $fF_j = \mathrm{pt} \in X$ (Lemma 4.3, (i)). Q.E.D.

4.6 Proof of Theorem 0.5

Using $(*)$ again,

$$K_L = \Big(\sum (a_j - r_j)F_j\Big)_{|L};$$

Lemma 4.3, (i) gives that $r_j \ge a_j$ unless $fF_j = \mathrm{pt} \in X$. Hence

$$K_L = A - B,$$

with $A \ge 0$ a divisor on L contracted by the birational map $f_{|L}$, and $B \ge 0$. In addition, Proposition 4.5 says that $p_g(L) \ne 0$; it follows that $B = 0$ and that a minimal model of L has trivial canonical class. This also proves

$$a_j \ge r_j \quad \text{if } F_j \cap L \ne \emptyset. \tag{$**$}$$

On the other hand, assuming that L is not composed with a pencil, L is nef with $\kappa_{\mathrm{num}}(L) \ge 2$; hence I can apply vanishing in the form Kawamata [5], Corollary on p. 45, to the cohomology exact sequence of $\mathcal{O}_Y \twoheadrightarrow \mathcal{O}_L$ to deduce that $H^1(\mathcal{O}_L) = 0$, and L is birational to a K3.

Pushing down $(*)$ in 4.4,

$$-K_X = S + \sum r_j f_*F_j,$$

where $S = fL$, and f_*F_j is the cycle theoretic image, that is,

$$f_*F_j = \begin{cases} \overline{F}_j & \text{if } F_j \text{ maps birationally to } \overline{F}_j \subset X, \\ 0 & \text{otherwise.} \end{cases}$$

If F_j is not contracted by f then $a_j = 0$, so that by $(**)$ either $r_j = 0$ or $F_j \cap L = \emptyset$. But now I claim that S and $\sum r_j f_*F_j$ do not intersect along curves of X; if $S = fL$ intersects some $\overline{F}_j$ in a mobile curve (as L moves in $|L|$) then $F_j \cap L \neq \emptyset$ and $r_j = 0$ by $(**)$; on the other hand, if all S pass through some fixed curve $C \subset X$ then $f^{-1}C$ contains at least one component F_j with $F_j \cap L \neq \emptyset$, hence $a_j \geq r_j$ by $(**)$. Applying Lemma 4.3, (ii) gives $C \not\subset \operatorname{Sing} X$, and the general element of $|-K_X|$ has multiplicity 1 along C, hence $C \subset S$, $C \not\subset \sum r_j f_* F_j$.

It follows from what I have just proved and from the connectedness lemma 0.9, (iii) that $\sum r_j f_* F_j = 0$ and that $S \in |-K_X|$; hence the irreducible surface S has $K_S = 0$. Since the resolution $f_{|L} \colon L \to S$ has $K_L \geq 0$, S has canonical singularities, that is, Du Val singularities. This proves Theorem 0.5 in this case.

4.7

The next result is the first step in proving that $|L|$ cannot be composed of a pencil.

Lemma *If $|-K_X|$ is composed of a pencil then $L = (g+1)E$ with $|E|$ a free pencil, in particular $\mathcal{O}_E(E) \cong \mathcal{O}_E$; $f^*(-K_X)^2 E = 1$, and there is a unique component F_0 of $\sum F_j$ such that*

$$\begin{aligned} &f^*(-K_X)F_0E = 1, \quad r_0 = 1, \quad a_0 = 0 \\ &\textit{and} \quad r_j f^*(-K_X)F_jE = 0 \quad \textit{for } j \neq 0. \end{aligned}$$

Proof

$$2g - 2 = f^*(-K_X)^3 \geq (g+1)f^*(-K_X)^2 E,$$

and by Lemma 0.9, (ii), $f^*(-K_X)^2 E > 0$. This proves $f^*(-K_X)^2 E = 1$. For the rest, set

$$D = f^*(-K_X)_{|E} = \Big(\sum r_j F_j\Big)_{|E};$$

D is nef and $D^2 = 1$, so it has a component Γ with $D\Gamma = 1$, and all the others have $D\Gamma = 0$.

To prove that $a_0 = 0$, note that by Lemma 4.3, (i), $a_0 \leq r_0 = 1$; on the other hand, a_0 is even, since

$$K_E + D = \Big(\sum a_j F_j\Big)_{|E}$$

and

$$(K_E + D)D = \Big(\sum a_j F_j\Big)_{|E} D = f^*(-K_X)\big(\sum a_j F_j\big)E = a_0. \quad \text{Q.E.D.}$$

4.8

For the remainder of the proof, I want to work on a different model: using Theorem 0.0 and Proposition 0.1, (b), there is no loss of generality in assuming that $-K_X$ is ample; now let X_1 be the normalised graph of the rational map $\varphi_{-K_X}\colon X \dashrightarrow \mathbb{P}^1$. Then there is a diagram

$$\begin{array}{ccccc} & & Y & & \\ & {}^{f}\swarrow & \downarrow_h & \searrow^{\varphi_E} & \\ X & \stackrel{p}{\leftarrow} & X_1 & \stackrel{q}{\rightarrow} & \mathbb{P}^1 \end{array}$$

in which p and q are the projections, $f\colon Y \to X$ is as in 4.4, φ_E is the morphism defined by $|E|$, and h the diagonal morphism.

Claim *(i)* $-K_{X_1} = p^*(-K_X)$, *so that* X_1 *has canonical singularities,* $-K_{X_1} \in \operatorname{Pic} X_1$, *and* $-K_{X_1}$ *is relatively ample for* q*;*

(ii) $|-K_{X_1}| = |(g+1)E_1| + F_1$, *where* F_1 *is an irreducible surface,* $|E_1|$ *a free pencil, and for every* $E_1 \in |E_1|$, E_1 *is a reduced irreducible surface and* $F_1 \cap E_1$ *a reduced irreducible curve.*

Proof Every curve $C \subset X_1$ contracted by p maps isomorphically to $\mathbb{P}^1$; it follows that if p contracts any surface $F \subset X_1$, this has to meet every fibre of q in a curve, and hence F corresponds birationally to $F_0 \subset Y$, the component of Lemma 4.7; then $a_0 = 0$, and hence $-K_{X_1} = p^*(-K_X)$. (ii) follows because as in Lemma 4.7,

$$(-K_{X_1})^2 E_1 = (-K_{X_1}) F_1 E_1 = 1. \quad \text{Q.E.D.}$$

4.9

Now F_1 is a Gorenstein surface, having a free pencil $|E'|$ every fibre of which is reduced and irreducible, and such that

$$K_{F_1} = -(g+1)E'; \quad p_a E' = 1.$$

The long exact cohomology sequence of

$$0 \to \mathcal{O}_{F_1}(-(g+1)E') \to \mathcal{O}_{F_1} \to \mathcal{O}_{(g+1)E'} \to 0$$

implies at once that $h^1(\mathcal{O}_{F_1}) \geq g$.

On the other hand, Lemma 1.10 applied to X_1 gives that F_1 has at worst ordinary double curves in codimension 1. I can now appeal to the following result to deduce a contradiction.

Lemma 4.10 *Let F be an irreducible projective Cohen–Macaulay surface having a morphism $q\colon F \to \mathbb{P}^1$ with reduced irreducible fibres of arithmetic genus* 1*; suppose that F has at worst ordinary double curves in codimension* 1*; then $h^1(\mathcal{O}_F) \le 1$.*

Proof If F has isolated singularities and $f\colon G \to F$ is a resolution, then $h^1(\mathcal{O}_G) \le 1$ from the classification of surfaces, and $h^1(\mathcal{O}_F) \le h^1(\mathcal{O}_G)$ follows from the Leray spectral sequence for f:

$$0 \to H^1(\mathcal{O}_F) \to H^1(\mathcal{O}_G) \to R^1 f_*\mathcal{O}_G \to \\ \to H^2(\mathcal{O}_F) \to H^2(\mathcal{O}_G) \to 0.$$

Suppose then that F has a double curve; the hypothesis implies that F is not singular along a fibre, so that there is just one double curve C, and the general fibre of $q\colon F \to \mathbb{P}^1$ is a rational curve with a node at its intersection with C. Obviously $q_{|C}\colon C \to \mathbb{P}^1$ is an isomorphism. Let $\pi\colon G \to F$ be the normalisation; then by the classification of surfaces, $H^1(\mathcal{O}_G) = 0$. If $\mathcal{C}$ is the conductor ideal of the normalisation then $\mathcal{C} \subset \mathcal{O}_G$ defines a reduced curve $D \subset G$ with $D \to C$ a double cover. It follows that there is an isomorphism $\pi_*\mathcal{O}_G/\mathcal{O}_F \cong \pi_*\mathcal{O}_D/\mathcal{O}_C$, and that $H^0(\pi_*\mathcal{O}_G/\mathcal{O}_F)$ is 0- or 1-dimensional depending on whether D has 1 or 2 connected components. The lemma follows from the exact sequence

$$0 \to H^0(\pi_*\mathcal{O}_G/\mathcal{O}_F) \to H^1(\mathcal{O}_F) \to H^1(\mathcal{O}_G).$$

This completes the proof of Theorem 0.5.

Counterexample 4.11 Lemma 4.10 is false without the hypothesis of ordinary double curves: let $\mathbb{F}_n$ be the standard rational scroll with a section B having $B^2 = -n$; the divisor $2B$ is naturally a subscheme of $\mathbb{F}_n$ and has a morphism $\pi\colon 2B \to \mathbb{P}^1$ induced by the projection of $\mathbb{F}_n$. Take F to be the surface obtained by pinching $\mathbb{F}_n$ along π; that is, F has the same underlying space as $\mathbb{F}_n$, but has sheaf of rings in such a way that $\mathcal{O}_{\mathbb{F}_n}/\mathcal{O}_F \cong \pi_*\mathcal{O}_{2B}/\mathcal{O}_{\mathbb{P}^1}$; in other words, replace coordinate neighbourhoods $\operatorname{Spec} k[X,Y]$ of $\mathbb{F}_n$, where $X = 0$ defines B, by $\operatorname{Spec} k[X^2, X^3, Y]$.

Then it is immediate that F has a morphism $F \to \mathbb{P}^1$ with every fibre a cuspidal rational curve, and $K_F = -(n+2)E$, $H^1(\mathcal{O}_F) = n+1$.

5 Weak Theorem on the Cone

Definition 5.1 A normal variety X is $\mathbb{Q}$-*factorial* if every Weil divisor of X is $\mathbb{Q}$-Cartier.

Remarks (a) This is a local condition: every Weil divisor near $P \in X$ is the restriction of a global one, and the condition for a Weil divisor to be Cartier or $\mathbb{Q}$-Cartier is local.

(b) The condition is not invariant under local analytic equivalence. For example, an ordinary double point of a 3-fold is analytically $(xy = zt)$, which is the typical example of a nonfactorial variety. However, it is easy to show that a hypersurface $X_d \subset \mathbb{P}^4$ of degree $d \geq 3$ having an ordinary double point $P \in X$ as its only singularity has class group $\operatorname{Cl} X \cong \mathbb{Z}$, with the hyperplane section as generator. (Proof: Blowing up $P \in X \subset \mathbb{P}^4$ leads to a smooth very ample divisor $\widetilde{X} \subset \widetilde{\mathbb{P}}$; we know the divisors of $\widetilde{\mathbb{P}}$, and the result follows from the Lefschetz theorem.)

(c) If X is $\mathbb{Q}$-factorial and nonsingular in codimension 2, and $D \subset X$ is a prime divisor, then D is Gorenstein in codimension 1, so that the $\mathbb{Q}$-divisor K_D is well defined and equal to $(K_X + D)_{|D}$.

5.2

Throughout this section X is a projective 3-fold with isolated $\mathbb{Q}$-factorial canonical singularities. The notation is as in 0.3; I make the following definitions: a ray R of $\overline{\mathrm{NE}}$ is an *extremal ray* if it's extremal in the sense of convexity (that is, $R \not\subset$ convex hull of $\overline{\mathrm{NE}} \setminus R$). An extremal ray R is *good* if $K_X R < 0$, and there exists an $H \in N^1_{\mathbb{Q}} X$ which is nef and such that $H^\perp \cap \overline{\mathrm{NE}} = R$. Let $\{R_i\}_{i \in I}$ be the set of good extremal rays; using Corollary 0.3 it is clear that each R_i is of the form $R_i = \mathbb{R}_+ C_i$ for some curve $C_i \subset X$. In particular each ray is rational in $N_1(X)$, and there are at most countably many.

Theorem 5.3 *Under the stated hypotheses,*

$$\overline{\mathrm{NE}}(X) = \Big(\overline{\mathrm{NE}}_{K_X} + \sum_{i \in I} R_i\Big)^{-},$$

where $^-$ *denotes closure in the usual real topology of* $N_1 X$*, and for* $D \in N^1 X$*,* $\overline{\mathrm{NE}}_D = \big\{ z \in \overline{\mathrm{NE}} \bigm| Dz \geq 0 \big\}$*. In particular if* K_X *is not nef then* X *has a good extremal ray.*

Remarks This is a weak version of the conjectured Theorem on the Cone; it is conjectured (and proved by Mori in the nonsingular case) that

(i) the rays R_i are discrete in the open halfspace $(K_X z < 0)$ of $N_1 X$ (so that there is no need to take closure in the theorem);

(ii) each ray R_i is spanned by a rational curve C_i;

(iii) the C_i can be chosen so that $-4 \le K_X C_i < 0$.

It is possible that these could be proved a posteori using Corollary 0.3 and Proposition 0.1; for example, (ii) can be checked in all cases except for that of a $\mathbb{Q}$-Fano 3-fold X, when it is required to prove that X contains a rational curve (conjecturally it is uniruled). Similarly, (iii) might be attacked on a case-by-case basis.[3] Part of (i) is implied by (iii), since assuming (iii) it is easy to see that the rays R_j are discrete in a neighbourhood of any fixed ray R_i.

I believe the hypotheses on the singularities of X can be weakened to allow any canonical singularities, using the methods of [9].

The next two results are the main steps in the proof of Theorem 5.3.

Kawamata's version 5.4 ([4], §2) *Let D be an effective $\mathbb{Q}$-divisor, and H an ample $\mathbb{Q}$-Cartier divisor. Then there exists a finite number of curves $l_j \subset X$ such that*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}_{K_X+H} + \overline{\mathrm{NE}}_D + \sum R_+ l_j.$$

Key rationality lemma 5.5 *Suppose that H is an ample $\mathbb{Q}$-divisor, and that K_X is not nef. Write $H_t = tH + K_X$, and set*

$$b = \inf\{t \in \mathbb{Q} \mid H_t \text{ is ample}\};$$

(that is $b \in \mathbb{R}$, and for $t \in \mathbb{Q}$, H_t is ample if $t > b$, and not nef if $t < b$). Then $b \in \mathbb{Q}$.

I start by deducing Theorem 5.3 from the key rationality lemma 5.5 and its relative form Lemma 5.11 below.

Definition 5.6 A *good supporting function* of $\overline{\mathrm{NE}}$ is an element $L \in N^1_{\mathbb{Q}}X$ such that L is nef and $F_L = L^\perp \cap \overline{\mathrm{NE}}$ is a nonzero face of $\overline{\mathrm{NE}}$ entirely contained in the open halfspace $(K_X z < 0) \subset N_1 X$; then F_L is a *good face* of $\overline{\mathrm{NE}}$. (Note that 0 is good if and only if $-K_X$ is ample, in which case $\overline{\mathrm{NE}}$ is itself a good face.)

By the argument given in 0.3, for suitable $a \gg 0$, $aL - K_X$ is ample, so that any such L is given by the construction of Lemma 5.5. Note also that a good extremal ray of $\overline{\mathrm{NE}}$ (as defined in 5.2) is the same thing as a good 1-face of $\overline{\mathrm{NE}}$.

[3](iii) $\Longrightarrow$ (i) is standard in Mori theory: for all ample H and $\varepsilon \ge 0$ the irreducible curves $C \subset X$ such that $HC < -(1/\varepsilon)K_X C \le 4/\varepsilon$ belong to a finite number of algebraic equivalence classes; hence (iii), together with Theorem 5.3 would imply $\overline{\mathrm{NE}} = \overline{\mathrm{NE}}_{K_X+\varepsilon H} + \sum R_i$, where the sum takes place over a finite number of rays representing these classes. (Note added in 1983–84.)

Lemma 5.7 *(i)* $\overline{\mathrm{NE}} = (\overline{\mathrm{NE}}_{K_X} + \sum_L F_L)^-$;

(ii) $\overline{\mathrm{NE}} \cap (K_X z < 0) = \bigcap_L (Lz \geq 0) \cap (K_X z < 0)$.

Here the sum and the intersection on the right-hand sides are taken over all good supporting functions $L \in N^1_{\mathbb{Q}} X$.

Proof Write B for the right-hand side of (i); then $B \cap (K_X z \geq 0) = \overline{\mathrm{NE}}_{K_X}$, and the inclusion $\overline{\mathrm{NE}} \supset B$ is trivial. The next statement, together with Kleiman's criterion, gives the opposite inclusion.

Claim *Let* $M \in N^1_{\mathbb{Q}} X$ *be such that* $M > 0$ *on* B*; then* M *is ample.*

To see this, note that $\overline{\mathrm{NE}}_{K_X}$ is the closed convex cone defined by the inequalities $Hz \geq 0$ for ample H and $K_X z \geq 0$. By convexity, $M > 0$ on $\overline{\mathrm{NE}}_{K_X}$ implies that M is a finite positive linear combination

$$M = \sum m_i H_i + m_0 K_X, \quad \text{with } m_i \in \mathbb{R},\ m_i \geq 0$$

where the H_i are ample. Since by 5.4 $\overline{\mathrm{NE}}$ has at least one face F_L in the $(K_X z < 0)$ halfspace, at least one $m_i > 0$, which implies that $M - m_0 K_X$ is ample for some $m_0 \geq 0$, and I can clearly take $m_0 \in \mathbb{Q}$. Now applying 5.4 to $H = M - m_0 K_X$, it follows that $L = M + aK_X$ is a good supporting function for some $a \in \mathbb{Q}$, $a > -m_0$. Since $F_L \subset B$ and $K_X < 0$ on F_L, necessarily $a > 0$. I've got $M - m_0 K_X$ ample with $m_0 \geq 0$, and $M + aK_X$ nef with $a > 0$, which implies that M is ample.

This proves (i); (ii) is left as an easy exercise.

5.8

Lemma 5.7 shows that $\overline{\mathrm{NE}}$ is the closed convex hull of its good faces, together with $\overline{\mathrm{NE}}_{K_X}$. The strategy from now on is to prove that each good face F_L of dimension ≥ 2 is in turn the closed convex hull of its proper faces (Lemma 5.12 below); Theorem 5.3 then follows by induction on the dimension.

Fix then a good face F_L of $\overline{\mathrm{NE}}$; by Lemma 0.3 there is a morphism $\varphi\colon X \to Z$ contracting exactly the curves $C \in F_L$; by construction $-K_X$ is relatively ample for φ. To carry out my strategy I need relative versions of the work so far, starting with the terminology (compare Kleiman [6], Chap. IV, §4). There are dual sequences (which will turn out to be exact in my case)

$$\begin{array}{ccccc} N_1(X/Z) & \hookrightarrow & N_1 X & \overset{\varphi_*}{\twoheadrightarrow} & N_1 Z, \\ N^1(X/Z) & \twoheadleftarrow & N^1 X & \overset{\varphi^*}{\hookleftarrow} & N^1 Z. \end{array} \qquad (*)$$

Here $N_1(X/Z) \subset N_1X$ is the subspace generated by curves C contracted by φ, and $N^1(X/Z)$ is its dual; the surjectivity of $N^1X \to N^1(X/Z)$ is standard in the theory of vector spaces. φ^* and φ_* are dual maps so that $\ker\varphi_* = (\operatorname{im}\varphi^*)^\perp$. Note also that

$$\mathrm{NE}(X) \cap L^\perp = \mathrm{NE}(X) \cap N_1(X/Z) = \mathrm{NE}(X/Z) \subset N_1(X/Z)$$

is the cone of effective 1-cycles contracted by φ.

5.9

It follows from the relative version of Kleiman's criterion that

$$F_L = \overline{\mathrm{NE}}(X) \cap N_1(X/Z) = (\mathrm{NE}(X/Z))^-. \qquad (*)$$

To see this, note that the inclusion $\supset$ is trivial; on the other hand, if $H \in N^1_{\mathbb{Q}}(X/Z)$ is strictly positive on $(\mathrm{NE}(X/Z))^-$ then by [6], p. 336, H is relatively ample for φ. Hence H comes from some ample $\widetilde{H} \in N^1X$, and so $H > 0$ on $\overline{\mathrm{NE}}(X) \cap N_1(X/Z)$.

Proposition 5.10 *Let $\varphi\colon X \to Z$ be the contraction of a good face F_L of $\overline{\mathrm{NE}}$.*

- *(i) If $D \in N^1X$ is relatively nef for φ then there exists $H \in N^1Z$ such that $D + \varphi^*H$ is nef;*
- *(ii) the dual sequences $(*)$ are exact.*

(Note that although both statements here look formal, the proofs given below are ad hoc; probably the statements are false for general φ.)

Proof (i) If $Z = \mathrm{pt}$ there is nothing to prove. Suppose without loss of generality that $D \in \operatorname{Pic} X$.

Claim *Outside a finite number of fibres of φ, $\mathcal{O}_X(D)$ is relatively generated by its H^0, that is, $\varphi^*\varphi_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$ is surjective.*

This proves (i), since for any sufficiently ample $H \in \operatorname{Pic} Z$, the linear system $|D+\varphi^*H|$ is free outside a finite number of fibres of φ, and then $(D+\varphi^*H)C \geq 0$ for every curve $C \subset X$.

I prove the claim assuming $\dim Z = 2$; then since $-K_X$ is relatively ample, all but a finite number of fibres of φ are isomorphic to conics. A nef invertible sheaf on a conic is generated by its H^0, and $\varphi^*\varphi_*\mathcal{O}_X(D) \twoheadrightarrow \mathcal{O}_X(D)$ in a neighbourhood of such a fibre follows by an easy use of coherent base change.

The cases $\dim Z = 1$ or 3 are no harder, and are left to the reader.

(ii) follows from (i) and from Theorem 0.0: if $D \in N^1X$ maps to 0 in $N^1(X/Z)$ then by (i), for sufficiently ample $H \in N^1X$, $D + \varphi^*H$ satisfies the hypotheses of Theorem 0.0; the morphism corresponding to $D + \varphi^*H$ contracts the curves with $(D + \varphi^*H)C = 0$, and hence coincides with φ, so that $D + \varphi^*H \overset{\text{num}}{\sim} \varphi^*M$ for some $M \in N^1Z$. Q.E.D.

Lemma 5.11 *Suppose that $H \in N^1_{\mathbb{Q}}X$ is relatively ample; write $H_t = tH + K_X$, and set*

$$b = \inf\left\{t \in \mathbb{Q} \;\middle|\; H_t \textit{ is relatively ample for } \varphi\right\}.$$

Then $b \in \mathbb{Q}$.

This is a relative version of the rationality lemma 5.5, and will be proved together with it (see 5.14).

Lemma 5.12 *If $\dim N^1(X/Z) \geq 2$ then $\overline{\mathrm{NE}}(X/Z)$ is the closed convex hull of its proper good faces. In other words, defining a good supporting function $M \in N^1_{\mathbb{Q}}X$ in the obvious way,*

$$\overline{\mathrm{NE}}(X/Z) = \left(\sum\nolimits_{M\neq 0}(M^{\perp} \cap \overline{\mathrm{NE}}(X/Z)\right)^{-},$$

where the sum on the right-hand side is over all nonzero good supporting functions M.

Proof As before, write B for the right-hand side; the inclusion $\supset$ is trivial. If $z \in \overline{\mathrm{NE}}(X/Z) \setminus B$ with $z \neq 0$ then there exists a separating function $M \in N^1(X/Z)$ such that $Mz < 0$ but $M > 0$ on B; by the compactness of $B \cap$ (unit sphere), I can shift M very slightly if necessary to ensure that $M \in N^1_{\mathbb{Q}}X$ and that M is not a rational multiple of K_X (since $\dim \geq 2$).

Now Lemma 5.11 gives that $M + aK_X$ is a nonzero good supporting function of $\overline{\mathrm{NE}}(X/Z)$ for some $a \in \mathbb{Q}$. I now have a contradiction, since on the one hand $Mz < 0$ and $(M + aK)z \geq 0$ implies that $a < 0$, and on the other, since M is positive on the good face $(M + aK_X)^{\perp} \cap \overline{\mathrm{NE}}(X/Z)$, I get $a > 0$. This proves Lemma 5.12.

It is clear from Proposition 5.10, (i) that a good face of $\overline{\mathrm{NE}}(X/Z)$ is a good face of $\overline{\mathrm{NE}}(X)$; this proves Theorem 5.3.

5.13 Proof of Key Rationality Lemma 5.5

STEP 1 Suppose that H_t is an effective $\mathbb{Q}$-divisor for some $t \in \mathbb{Q}$ with $t \le b$; then by Kawamata's theorem 5.4 there are finitely many curves $l_j \subset X$ such that

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}_{H_t} + \sum \mathbb{R}_+ l_j.$$

Then clearly,

$$b = \max\left\{t, \frac{-K_X l_i}{H l_i}\right\} \in \mathbb{Q}.$$

STEP 2 Let t be an indeterminate, and consider the cubic polynomial

$$p(t) = H_t^3 = (tH + K_X)^3 \in \mathbb{Q}[t].$$

Then since $p'(t) = 3H(tH + K_X)^2$,

$$H_b^2 \overset{\text{num}}{\sim} 0 \iff b \text{ is a repeated root of } p \implies b \in \mathbb{Q}.$$

Thus I need only treat the case $H_b^2 \overset{\text{num}}{\not\sim} 0$.

STEP 3 If $H_b^3 > 0$ then there exists $q, m \in \mathbb{Z}$, $q, m > 0$ such that $m/q \le b$ and $H^0(mH + qK_X) \ne 0$, hence by Step 1, $b \in \mathbb{Q}$.

Proof For $m \in \mathbb{Z}$, $m > 0$, set $q = \lceil m/b \rceil$; then

$$q \ge \frac{m}{b} > q - 1;$$

by definition of b,

$$mH + (q-1)K_X$$

is an ample $\mathbb{Q}$-divisor, so that by Proposition 3.1, (ii),

$$h^0(mH + qK_X) = \frac{1}{6}(mH + qK_X)^3 - \frac{1}{4}(mH + qK_X)^2 K_X + O(m), \tag{1}$$

where $O(m)$ denotes terms bounded by a linear function of m. Write

$$\begin{aligned} mH + qK_X &= \frac{m}{b}(bH + K_X) + \left(q - \frac{m}{b}\right)K_X \\ &= \frac{m}{b}H_b + \left\{\frac{-m}{b}\right\}K_X, \end{aligned} \tag{2}$$

where $\{\,\}$ denotes "fractional part" of a real number. Then

$$h^0(mH + qK_X) = \frac{1}{6}H_b^3\left(\frac{m}{b}\right)^3 + O(m^2),$$

and tends to infinity with m. This proves this case.

Step 4 If $H_b^3 = 0$ but $H_b^2 \overset{\text{num}}{\not\sim} 0$ then $2/b \in \mathbb{Z}$.

Proof Substituting (2) into (1) and evaluating gives

$$0 \le h^0(mH + qK_X) = \Big(\frac{1}{2}\Big\{\frac{-m}{b}\Big\} - \frac{1}{4}\Big) H_b^2 K_X \Big(\frac{m}{b}\Big)^2 + O(m). \qquad (3)$$

Now $H_b^3 = 0$, $H_b^2 H > 0$ implies that $H_b^2 K_X < 0$. Furthermore, if b is irrational, or if $1/b$ is rational with denominator ≥ 3 then for infinitely many values of m, I have $\{-m/b\} \ge 2/3$. The right-hand side of (3) is then negative for large m, which is a contradiction. This completes the proof of Rationality Lemma 5.5.

5.14 Proof of Lemma 5.11

If $Z = \text{pt}$ then Lemma 5.11 is contained in 5.5. If $\dim Z = 1$ or 2, let

$$b' = \inf\Big\{ t \in \mathbb{Q} \;\Big|\; H_{t|A} \text{ is ample for a general fibre } A \text{ of } \varphi \Big\}.$$

The obviously $b' \le b$, and by the statement of Rationality Lemma 5.5 in dimension 1 or 2 (the proof of which I leave to the reader), $b' \in \mathbb{Q}$. If $b' < b$ then there is some $t < b$ such that H_t is relatively ample on the general fibre of φ; then for some sufficiently ample $D \in \operatorname{Pic} Z$, $H_t + \varphi^* D$ is effective, and then $b \in \mathbb{Q}$ follows from Kawamata's Theorem 5.4 as in Step 1 above.

If φ is birational, then $H_t + \varphi^* D$ is effective for any $t \in \mathbb{Q}$ and sufficiently ample $D \in \operatorname{Pic} Z$, so that I conclude in the same way.

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