When the Hasselmann equation fails: "Fast" nonlinear evolution of water wave spectra

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Motivation

The cornerstone of the established view on nonlinear evolution random wave fields due to nonlinear quartet interactions is that the spectra of any such random wave field (e.g. water waves) evolve on the ε^{-4} timescale or slower. ε is the typical wave steepness.

The wave frequency of the spectral peak is presumed to be $O(1)$.

Then the ε^{-4} evolution of the wave spectra is described by the *kinetic* equation or, in the context of water waves, the *Hasselmann equation*.

Awkward Facts for Wind Waves

Along with vast anecdotal evidence suggesting occurrence of a faster field evolution, there are also a few well documented measurements:

(i) in the field:

- In situations of sudden change of wind direction the wave field adjusts faster than predicts the kinetic equation [*van Vledder & Holthuijsen, 1993*]
- The relaxation time scale of short gravity waves perturbed by internal waves is *much shorter* than one would expect from the kinetic equation [*Hughes & Grant 1978*]

(ii) in the tank: the observations of short wind waves subject to abrupt change of wind by *Waseda, Toba, Tullin 2001* and *Caulliez, 2007* suggest existence of a very fast field evolution after a sudden change of wind.

We are unaware of any attempts to explore theoretically a possibility of a faster field evolution.

Here we show that:

(i) a much faster (up to ε^{-2}) evolution of wave spectra can indeed occur,

(ii) explain why and when this happens.

(iii) propose a generalized kinetic equation able to describe fast evolution.

Plan

1. Review of the basics of the established statistical approach to water waves and of classical derivation of the kinetic equation.
2. Identification of the crucial junction and derivation of the *generalized kinetic equation*.
3. Explanation and interpretation of the $O(\varepsilon^{-2})$ evolution.
4. Results of DNS demonstrating examples of the $O(\varepsilon^{-2})$ evolution will be shown separately.

Review: (i) Pre-history

Consider 3D potential gravity waves on the free surface of an incompressible fluid. Wave slopes are $O(\varepsilon)$. Then the eq-s of motion could be cast in the Hamiltonian form

$$\frac{\partial\zeta(\mathbf{x},t)}{\partial t} = \frac{\delta H}{\delta\psi(\mathbf{x},t)}, \qquad \frac{\partial\psi(\mathbf{x},t)}{\partial t} = -\frac{\delta H}{\delta\zeta(\mathbf{x},t)}$$

where $\psi(\mathbf{x},t) = \varphi(\mathbf{x},\zeta(\mathbf{x},t),t)$, and the Hamiltonian H is the total energy of the system.

Hamiltonian

$$H = \frac{1}{2}\int_{-\infty}^{\zeta}\left[(\nabla\varphi)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right]\mathrm{d}z\,\mathrm{d}\mathbf{x} + \frac{1}{2}g\int\zeta^2\,\mathrm{d}\mathbf{x}.$$

Zakharov's complex amplitudes:

$$a(\mathbf{k}) = \frac{1}{\sqrt{2}}\left\{\sqrt{\frac{\omega(\mathbf{k})}{k}}\zeta(\mathbf{k}) + \mathrm{i}\sqrt{\frac{k}{\omega(\mathbf{k})}}\psi(\mathbf{k})\right\},$$

where $\omega(\mathbf{k}) = \sqrt{gk}$ is the linear dispersion relation, $k = |\mathbf{k}|$.

Equations for $a(\mathbf{k})$

$$\mathrm{i}\frac{\partial a(\mathbf{k})}{\partial t} = \frac{\delta H}{\delta a^*(\mathbf{k})},$$

Expansion in ε

Compact notation: $a_0 = a(\mathbf{k_0}), \delta_{\mathbf{0-1-2}} = \delta(\mathbf{k_0} - \mathbf{k_1} - \mathbf{k_2})$, etc.

$$
\begin{aligned}
H \quad = \quad & \int \omega_0 a_0^* a_0 \, \mathrm{d}\mathbf{k}_0 \\
& + \int U^{(1)}_{012} \left(a_0^* a_1 a_2 + a_0 a_1^* a_2^* \right) \delta_{0-1-2} \, \mathrm{d}\mathbf{k}_{012} \\
& + \frac{1}{3} \int U^{(3)}_{012} \left(a_0^* a_1^* a_2^* + a_0 a_1 a_2 \right) \delta_{0+1+2} \, \mathrm{d}\mathbf{k}_{012} \\
& + \int V^{(1)}_{0123} \left(a_0^* a_1 a_2 a_3 + a_0 a_1^* a_2^* a_3^* \right) \delta_{0-1-2-3} \, \mathrm{d}\mathbf{k}_{0123} \\
& + \frac{1}{2} \int V^{(2)}_{0123} a_0^* a_1^* a_2 a_3 \delta_{0+1-2-3} \, \mathrm{d}\mathbf{k}_{0123} \\
& + \frac{1}{4} \int V^{(4)}_{0123} \left(a_0^* a_1^* a_2^* a_3^* + a_0 a_1 a_2 a_3 \right) \delta_{0+1+2+3} \, \mathrm{d}\mathbf{k}_{0123} \\
& + \ldots
\end{aligned}
$$

"Reduced" Zakharov equation

Canonical transformation to nonlinear normal variables $a_0 = b_0 + O(\varepsilon)$ retains only essential nonlinear interactions (shown in red).

The reduced Hamiltonian

$$\tilde{H} = \int \omega_0 b_0 b_0^* \,\mathrm{d}\mathbf{k}_0 + \frac{1}{2} \int T_{0123} b_0^* b_1^* b_2 b_3 \delta_{0+1-2-3} \,\mathrm{d}\mathbf{k}_{123} + \ldots .$$

yields the "reduced" Zakharov equation

$$\mathrm{i}\frac{\partial b_0}{\partial t} \;=\; \omega_0 b_0 + \int T_{0123} b_1^* b_2 b_3 \delta_{0+1-2-3} \,\mathrm{d}\mathbf{k}_{123} + \ldots$$

(This is the starting point of our analysis.

Notation: $\delta_{0+1-2-3} = \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$, $\mathrm{d}\mathbf{k}_{123} = \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2 \mathrm{d}\mathbf{k}_3$.

(ii) Statistical description

Now we consider ensembles of random wave fields (each being governed by the deterministic Zakharov eq-n). We are interested in the ensemble averaged characteristics of the wave field.

Assumption of spatial homogeneity yields

$$< b_0^* b_1 >= n_0 \delta_{0-1}$$

brackets mean ensemble averaging, the second-order correlator, n_0 is the spectral density of wave action at wavevector $\mathbf{k} = \mathbf{k}_0$.

The classical problem is to find (and solve) a closed equation in terms of $n(k)$, i.e. to find evolution of wave action spectral density $n(k)$ with time.

The classical solution to this problem is the kinetic or Hasselmann equation

$$\frac{\partial n_0}{\partial t} = 4\pi \int T_{0123}^2 f_{0123} \delta_{0+1-2-3} \delta(\Delta\omega) \, \mathrm{d}\mathbf{k}_{123}$$

$$f_{0123} = n_2 n_3 (n_0 + n_1) - n_0 n_1 (n_2 + n_3),$$

$$\Delta\omega = \omega(\mathbf{k}_0) + \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2) - \omega(\mathbf{k}_3)$$

The key steps and assumptions

On multiplying the reduced Zakharov equation by b_0^*, and its c.c. by b_0, upon ensemble averaging we immediately find

$$\frac{\partial n_0}{\partial t} = 2\text{Im} \int T_{0123} J_{0123} \delta_{0+1-2-3} \, \mathrm{d}\mathbf{k}_{123}$$

$$J_{0123}^{(0)} \delta_{0+1-2-3} =< b_0^* b_1^* b_2 b_3 >$$

Assumption of gaussianity yields

$$< b_0^* b_1^* b_2 b_3 >= n_0 n_1 \left(\delta_{0-2}\delta_{1-3} + \delta_{0-3}\delta_{1-2} \right).$$

which is a real quantity and, since T_{0123} is also real, does not contribute to evolution of n_0.

Completely random phases provide no spectral evolution!

Non-gaussian effects

Assuming quasi-gaussianity find non-gaussian correction $J^{(1)}_{0123}$.
The cumulant $J^{(1)}_{0123}$ is specified by an evolution equation containing on the right-hand-side the sixth-order correlator I_{012345}. which by invoking the quasi-Gaussianity assumption is replaced by the corresponding free-field Gaussian correlator $I^{(0)}_{012345}$ representable in terms of the products of pair correlators. As a result we have

(1) $$\left(\mathrm{i}\frac{\partial}{\partial t} + \Delta\omega\right) J^{(1)}_{0123} = 2T_{0123} f_{0123},$$

where $\Delta\omega = \omega_0 + \omega_1 - \omega_2 - \omega_3$, and
$f_{0123} = n_2 n_3(n_0 + n_1) - n_0 n_1(n_2 + n_3)$

The crucial assumption

It is usually assumed that n_0 and, hence, f_{0123} depends on slow time μt, such that $\mu/\Delta\omega \ll 1$.

Then neglecting $\frac{\partial}{\partial t}$ in $\left(\mathrm{i}\frac{\partial}{\partial t} + \Delta\omega\right) J^{(1)}_{0123} = 2T_{0123} f_{0123} \qquad \Rightarrow$

$$J^{(1)}_{0123}(t) \simeq \frac{2T_{0123}}{\Delta\omega} f_{0123}.$$

This solution represents a large t asymptotics and is understood in terms of generalized functions

$$J^{(1)}_{0123}(t) = 2T_{0123}\left[\frac{P}{\Delta\omega} + \mathrm{i}\pi\delta(\Delta\omega)\right] f_{0123}(t),$$ (P is ‘principal val

This asymptotic derivation yields the classic kinetic equation and is valid as long as our interest is confined to slow $O(\varepsilon^{-4})$ evolution.

The new kinetic equation

If we allow for faster variability of statistical moments of wave field, we can use the exact solution for $J^{(1)}$ in the form

$$J^{(1)}_{0123}(t) = -2\mathrm{i}T_{0123}\int_0^t \mathrm{e}^{-\mathrm{i}\Delta\omega(\tau-t)} f_{0123}\,\mathrm{d}\tau + J^{(1)}_{0123}(0)\mathrm{e}^{\mathrm{i}\Delta\omega t}.$$

$J^{(1)}_{0123}(0)$ is specified by initial conditions.

The resulting "generalized" kinetic equation reads

$$\frac{\partial n_0}{\partial t} = 4\mathrm{Re}\int T^2_{0123}\left[\int_0^t \mathrm{e}^{-\mathrm{i}\Delta\omega(\tau-t)} f_{0123}\,\mathrm{d}\tau\right]\delta_{0+1-2-3}\,\mathrm{d}\mathbf{k}_{123}$$

$$+2\mathrm{Re}\int\left[\mathrm{i}T_{0123}J^{(1)}_{0123}(0)\mathrm{e}^{\mathrm{i}\Delta\omega t}\right]\delta_{0+1-2-3}\,\mathrm{d}\mathbf{k}_{123}.$$

The GKE tends to the classic KE for large times

Initial stages

In general setting the evolution of spectral density n depends not only on the initial distribution of n, but also on the initial distribution of $J^{(1)}_{0123}(0)$.

"Cold start". Zero value of $J^{(1)}_{0123}(0)$ corresponds to the situations where the wave field is initially free, so that the wave components are not correlated, and waves begin to interact only after $t = 0$. Then the GKE reads

$$\frac{\partial n_0}{\partial t} = 4 \int T^2_{0123} \left[\int_0^t \cos[\Delta\omega(\tau - t)] f_{0123}\, \mathrm{d}\tau \right] \delta_{0+1-2-3}\, \mathrm{d}\mathbf{k}_{123}$$

$$\frac{\partial n_0}{\partial t}|_{t=0} = 0, \qquad \frac{\partial^2 n_0}{\partial t^2}|_{t=0} = 4 \int T^2_{0123} f_{0123}\, \mathrm{d}\delta_{0+1-2-3}\, \mathrm{d}\mathbf{k}_{123}$$

The Timescales

$$\frac{\partial n_0}{\partial t}|_{t=0} = 0, \qquad \frac{\partial^2 n_0}{\partial t^2}|_{t=0} = 4\int T_{0123}^2 f_{0123}\,\mathrm{d}\delta_{0+1-2-3}\,\mathrm{d}\mathbf{k}_{123}$$

Since $n \sim \varepsilon^2$ and the RHS is $\sim n^3 \sim \varepsilon^6$, then the timescale of initial evolution is $O(\varepsilon^{-2})$.

For generic initial conditions with non-zero $J_{0123}^{(1)}(0)$ the second term on the RHS should be taken into account

$$2\mathrm{Re}\int \left[\mathrm{i}T_{0123}J_{0123}^{(1)}(0)\mathrm{e}^{\mathrm{i}\Delta\omega t}\right]\delta_{0+1-2-3}\,\mathrm{d}\mathbf{k}_{123}|_{t=o}$$

$$= -2\int T_{0123}\mathrm{Im}J_{0123}^{(1)}(0)\delta_{0+1-2-3}\,\mathrm{d}\mathbf{k}_{123}$$

which implies: $\frac{\partial n_0}{\partial t}|_{t=0} \sim \varepsilon^4$, and, hence, the $O(\varepsilon^{-2})$ timescale.

Fast evolution: Why and When?

Our choice of the initial moment was special. Usually the wave field was evolving due to nonlinear interactions for quite a long time before the moment we choose as initial. Therefore, $J^{(1)}_{0123}(0)$ cannot be prescribed arbitrarily, but is a result of preceding evolution. With time the phases get adjusted in such a way that fast evolution doesn't occur.

However, if, for example, there is also a wind forcing which varies on $O(\varepsilon^{-2})$ scale or faster, then the phases do not have time to get adjusted and the wave field undergoes rapid evolution.

If the initial spectrum has been strongly perturbed and significantly deviates from the the solutions of the classic KE, then again a rapid adjustment is likely.

Examples

The wave field could be pushed out of "equilibrium" in many different ways, e.g. by:

- Rapid changes of wind;
- Interaction of several systems of waves:

  (i) Wind waves + swell

  (ii) Interaction with obstacles, e.g. interaction between incident and reflected waves;
- Any sufficiently strong spatial inhomogeneities (bottom features, currents, internal waves);
- Falling/sinking of a body

How fast is "fast"?

When the perturbation is strong enough to result in the "fast" evolution of wave field and how fast will be the "fast" evolution?

To answer these questions a dedicated study of each particular situation is required.

Depending on the specific circumstances the "fast" evolution could be anything between $O(\varepsilon^{-2})$ and $O(\varepsilon^{-4})$.

Results of direct numerical simulation of two archetypical situations (an abrupt increase of wind and interaction of waves with swell) are presented in the second part of this talk.