

Strong effect of weak diffusion on scalar turbulence at large scales

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Passive scalar turbulence forced steadily is characterized by the velocity correlation scale, L , injection scale, l , and diffusive scale, r_d . The scales are well separated if the diffusivity is small, $r_d \ll l, L$, and one normally says that effects of diffusion are confined to smaller scales, $r \ll r_d$. However, if the velocity is single scale one finds that a weak dependence of the scalar correlations on the molecular diffusivity persists to even larger scales, e.g. $l \gg r \gg r_d$ [9]. We consider the case of $L \gg l$ and report a counter-intuitive result – the emergence of a new range of large scales, $L \gg r \gg l^2/r_d$, where the diffusivity shows a strong effect on scalar correlations.

Studies of passive scalar advection in a random smooth flow were pioneered by Batchelor [1] and Kraichnan [2] who considered the opposite extremes of almost frozen and short-correlated in time random velocity gradients. The two approaches were later extended into a unified theory describing the statistics of scalar correlations in a general smooth flow [3–7]. These theoretical studies were originally motivated by interest in explaining the so-called viscous-inertial interval of advection at the scales smaller than the viscous, Kolmogorov, scale. However, the theory, which is nowadays often called Batchelor flow theory, also applies to many cases of non-turbulent but chaotic smooth flows, e.g. of the type discovered recently in polymer solutions [8].

Main theoretical efforts in the field were focused on the analysis of scalar correlations within the convective range, $r_d \ll r \ll l$, i.e. at the scales smaller than the injection scale l but larger than the diffusive scale r_d [4, 9, 10]. The range of scales above the pumping scale, even though very nontrivial with highly intermittent correlations [11], attracted much less attention. In this work, we continue to discuss the domain of large scales. Complementary to our general interest in understanding multi-point correlations in turbulence, this study was additionally motivated by our recent interest in the condensate regime of $2d$ turbulence [12], where small scale vorticity is advected passively by the large scale, coherent part of the flow.

On a superficial level studying correlations of a fluctuating quantity upscales from the injection scale may seem akin to many problems in equilibrium statistical mechanics, e.g. of the type considered in the field of critical phenomena where one studies fluctuations of an order parameter driven by thermal noise at small scales. However, the essential difference here is that our problem is off-equilibrium due to the scalar is advected by the prescribed velocity field. A particularly important consequence of this fact is an intermittent, strongly non-Gaussian statistics observed for the problem both at the scales smaller [6, 9] and larger [11] than the pumping length l .

In this work we extend the analysis of Ref. [11] accounting for the effects of molecular diffusivity which were ignored in [11]. A surprising result of our study is

that *diffusion, even though small, dominates correlation functions of the scalar at large scales, $r \gg l^2/r_d$* . This result is *Batchelor flow specific* and it can be explained in dynamical, Lagrangian, terms. The collinear anomaly, established in [9] and later discussed in [6, 10, 14, 15], states that Lagrangian particles released along a line in a Batchelor flow stay aligned, unless weak diffusive effects are accounted for. The anomaly reveals itself in an angular singularity of the passive scalar multi-point correlation functions observed near the parallel alignment of the points [9, 10]. Translation of the collinear anomaly from the dynamical to statistical language goes as follows. If diffusivity is neglected, a blob of freshly injected passive scalar is deformed by a smooth flow into a strip of the same density. The strip contributes to correlation function of the passive scalar provided it covers all the points where the correlations are measured. Thus the strip should have the length r of the order of the separation between the points and it should also be oriented in a way that all the points are covered. Since the flow is chaotic and orientation of the stripe is random, the probability to cover the points is determined by the angular size of the stripe. In incompressible flow, the blob volume is conserved. The volume can be estimated as l^d , where d is the space dimensionality, thus the cross-section of the stripe can be estimated as l^d/r and the angular size of the stripe is $\sim l^d/r^d$. This results in the $\propto r^{-d}$ scaling for the n -th order correlation function of the scalar, $K^{(n)} = \langle \theta_1 \cdots \theta_n \rangle$, in the collinear geometry, i.e. when the points $\mathbf{r}_1, \dots, \mathbf{r}_n$ lie on a straight line, and r is the size of the most separated pair of points [11]. Volume preserving stretching of the scalar blob, injected at the pumping scale l , should be modified when the blob size in the contracting direction reaches $\sim r_d$, since the diffusion blocks further contraction of the blob beyond r_d .

Let us concentrate on the $2d$ case. For a seriously stretched stripe, with the spatial extent $r > l^2/r_d$, the stripe is $\sim r_d$ -wide in cross-section and therefore $\alpha_d \sim r_d/r$ gives an estimate for the angular size of the stripe. The temporal dynamics of the stripe is as follows. The stripe grows in size (along the stretching direction) while the scalar density inside the stripe, estimated as $\sim r_d/r$ fraction of the initial density, decreases. Besides,

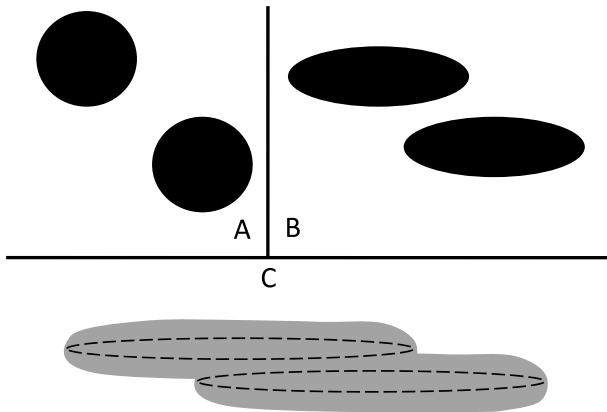


FIG. 1: Figure Caption. Schematic plot illustrating Lagrangian (temporal) evolution of two blobs of scalar. (A) Initial injection. The blobs are of size $\sim l$ separated by the distance $\sim l$. (B) Result of diffusion-less deformation. The blobs grow in size along the expanding direction of the flow and decrease in size along the contracting direction. Volumes and initial concentrations of the scalar inside the blobs are preserved. This phase terminates when the width of the blobs reaches r_d . (C) Further deformation keeps the width of the blobs $\sim r_d$, while the lengths of the blobs continue to increase. Volumes of the blobs grow and the density of the scalar inside the blobs decreases. Blobs will eventually overlap. Dashed lines show projected shape of the blobs realized if the diffusivity is ignored (naive extension of the stage B).

different stripes stretched simultaneously start to overlap along the contracting direction. Due to the random character of the passive scalar injection, the sign of the density in the overlapping stripes alternates, thus leading to destructive interference. This additional effect leads to further suppression of the scalar correlations by the factor $1/\sqrt{N}$ where, $N \sim rr_d/l^2$, is an estimate for the number of stripes (that were initially of size l and separated by the distance $\sim l$) which contribute to the overlapped conglomerate. Combining the pieces, one derives the following scaling for the $2n$ -th order correlation function of the passive scalar measured at $r \gg l^2/r_d$ within the collinear geometry:

$$K^{(2n)} \propto \alpha_d N^{-n} \propto r^{-n-1} r_d^{1-n}. \quad (1)$$

Eq. (1) is the main result of the paper, that will be confirmed below with a proper rigor. The result shows a strong dependence of the high order correlations of the scalar at the scales beyond the injection scales on the diffusivity, and it should thus be contrasted with a much weaker dependence on diffusion observed in the passive scalar correlations at the scales smaller than the injection scale [9].

Even though Eq. (1) is derived in $2d$, the qualitative result, stating a strong sensitivity to diffusivity of the scalar fluctuations at large scales, also extends to $3d$ (and higher dimensions). In general, the simultaneous corre-

lations are expressed in terms of the Lagrangian evolution of a fluid blob, while diffusivity stops contraction of the blobs at the diffusion scale, r_d , thus making the blob globally sensitive to the small scale, diffusion-related physics. Obviously, the $3d$ picture of the phenomenon is more evolved due to existence of an additional, third, dimension in the blob dynamics that can be either contracting or expanding. As a result, the $3d$ generalization of Eq. (1) becomes sensitive to the sign of the second Lyapunov exponent of the flow [13]. This sensitivity is similar to effects discussed in [14] and [15] in the contexts of kinematic dynamo, and decaying scalar turbulence, respectively.

The dynamic equation for a passively advected scalar field, θ , is

$$\partial_t \theta + \mathbf{u} \nabla \theta = \phi + \kappa \Delta \theta, \quad (2)$$

where $\mathbf{u}(t, \mathbf{r})$ is the flow velocity field, κ is the diffusion coefficient and $\phi(t, \mathbf{r})$ is the pumping term. The velocity \mathbf{u} and the forcing ϕ are assumed to be independent and random functions in space/time with prescribed statistics, spatio-temporally homogeneous and spatially isotropic. The forcing is correlated at the scale, l . We consider Batchelor, that is spatially smooth, flow where the velocity field is correlated at the scale L , the largest scale in the problem. We also assume that the velocity fluctuations are sufficiently intense to guarantee that the diffusive range, $r \ll r_d$, where the effects of advection are strongly suppressed by diffusion, is realized at the scales smaller than the pumping length, i.e. $r_d < l$. In Batchelor flow, the velocity difference between points separated by a distance much smaller than L is given by the first term of the Taylor expansion in the inter-point separation, $u_\alpha(\mathbf{r}_1) - u_\alpha(\mathbf{r}_2) \approx \sigma_{\alpha\beta}(r_{1;\beta} - r_{2;\beta})$. Therefore, the velocity derivatives matrix, $\hat{\sigma}$, is the only velocity related characteristic entering the problem at the scales smaller than L . In an incompressible flow, discussed here, the velocity gradient matrix is traceless, $\text{tr} \hat{\sigma} = 0$. We also assume that $\hat{\sigma}$, followed in the reference frame of a fluid parcel, is finitely correlated in time.

Representing solution of Eq. (2) in the Lagrangian frame (see [5] for derivation details) one arrives at the following formal expression for the scalar field

$$\theta(t, \mathbf{r}) = \int_{-\infty}^t dt' \exp\left(\kappa \int_{t'}^t d\tau \left[\nabla \hat{W}(t, \tau)\right]^2\right) \phi(t', \mathbf{R}). \quad (3)$$

Here $\mathbf{R} = \hat{W}(t', t) \mathbf{r}$ and $\hat{W}(t', t) = \text{T exp} \left(\int_{t'}^t d\tau \hat{\sigma}(\tau) \right)$ is the ordered exponential. Note that in an incompressible flow, $\det \hat{W}(t) = 1$. The argument $\mathbf{R}(t')$ of the function ϕ in Eq. (3) traces back in time the Lagrangian trajectory arriving at the position \mathbf{r} at the moment of time t . The κ -dependent exponential on the right-hand side of Eq. (3) represents effects of diffusion. Therefore, Eq. (3) is nothing but a formal way to express the aforementioned qualitative arguments concerning the Lagrangian evolution of a passive scalar blob. Since ϕ is spatially correlated at

the scale l , the temporal integral on the right-hand side of Eq. (3) is formed at $t - t' \sim \bar{\lambda}^{-1} \ln(r/l)$ where $\bar{\lambda}$ is the principal Lyapunov exponent of the flow, defined as the average logarithmic rate of Lagrangian trajectories divergence. This stretching time diverges as $r \rightarrow \infty$, which allows us to consider the forcing as short-correlated in time. Then the forcing field is effectively Gaussian and thus fully described by the pair correlation function

$$\chi(\mathbf{r} - \mathbf{r}') = \int dt \langle \phi(t, \mathbf{r}) \phi(t', \mathbf{r}') \rangle, \quad (4)$$

decaying sufficiently fast with increase in r at $r > l$. We assume that $\chi(\mathbf{r})$ is a function of $|\mathbf{r}|$ only.

For the short-correlated forcing any correlation function of the scalar field can be calculated in two steps. First, one averages over times larger than the pumping correlation time but smaller than $\bar{\lambda}^{-1} \ln(r/l)$. This is formally equivalent to averaging over the statistics of forcing for a given realization of the velocity field. Averaging over velocity, corresponding to longer times and larger spatial scales, follows. This scheme gives the following expression for the simultaneous pair correlation function of the scalar, $\mathcal{K}(\mathbf{r}) = \langle \theta(t, \mathbf{r}) \theta(t, \mathbf{0}) \rangle$,

$$\mathcal{K}(\mathbf{r}) = \int_0^\infty dt \int d\mathbf{k} \langle \exp(J) \chi_{\mathbf{k}} \rangle, \quad (5)$$

$$J = i\mathbf{k}\hat{W}(-t, 0)\mathbf{r} - 2\kappa\mathbf{k}\hat{I}(t)\mathbf{k}, \quad (6)$$

$$\hat{I}(t) = \int_0^t d\tau \hat{W}(-t, -\tau)\hat{W}^T(-t, -\tau), \quad (7)$$

where $\chi_{\mathbf{k}}$ is the Fourier transform of $\chi(\mathbf{r})$ and T indicates matrix transposition. The only averaging left to be done in Eq. (5) is over statistics of $\hat{\sigma}$.

Consider the $d = 2$ case and introduce an Iwasawa-like decomposition for the ordered exponential

$$\hat{W}(-t, 0) = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} e^\varrho & 0 \\ 0 & e^{-\varrho} \end{pmatrix} \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}. \quad (8)$$

This representation is useful as the three governing fields, φ, ϱ, ζ decouples in the asymptotic limit of large time, $t \gg \bar{\lambda}^{-1}$. Moreover, at the large times, the orientation angle φ becomes random uniformly distributed over the range $(0; 2\pi)$, the distribution function of ζ freezes to a non-universal stationary shape and the typical ζ becomes a fluctuating $O(1)$ value, while the probability distribution of the finite time Lyapunov exponent, $\lambda = \varrho/t$, attains the following self-similar form [17]

$$\mathcal{P}(t, \lambda) \propto \sqrt{t} \exp[-tS(\lambda)]. \quad (9)$$

Here $S(\lambda)$ is the so-called Crámer function which is concave and achieves its minimum at $\lambda = \bar{\lambda}$, and the condition $S(\bar{\lambda}) = 0$ together with the \sqrt{t} factor account for accurate normalization of the total probability to unity.

In the limit $\bar{\lambda}t \gg 1$, the main contribution to the integral (7) originates from τ close to t , thus leading to

$$\hat{I}(t) = \frac{c}{\bar{\lambda}} e^{2\varrho(t)} \hat{O}(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \hat{O}^{-1}(t), \quad (10)$$

where c is a fluctuating factor of order unity and \hat{O} is the φ -dependent part of the decomposition (8). (See [5] for detailed discussion of the c -field statistics.) Averaging over homogeneous random orientations φ (reflecting the assumed isotropy of the velocity fluctuations) one derives the following expression for J from Eq. (6)

$$J = ir (k_1 \zeta e^\varrho + k_2 e^{-\varrho}) - 2cr_d^2 k_1^2 e^{2\varrho}, \quad (11)$$

where $r_d^2 = \kappa/\bar{\lambda}$ and k_1, k_2 are components of the wave vector \mathbf{k} in the reference frame fixed by the decomposition (8).

A comparison of the two terms in Eq. (11) suggests that the outer scale interval, $r \gg l$, splits in two distinct sub-intervals: $l \ll r \ll l^2/r_d$ and $r \gg l^2/r_d$. To describe the first interval of relatively small scales one may ignore the last term in Eq. (11). Then, direct integration of Eq. (5) results in the diffusionless scaling

$$\mathcal{K}(r) \propto \frac{1}{r^2} \int d^2x \chi(\mathbf{x}), \quad (12)$$

already derived in [11]. However, evaluating the integrals in Eq. (5) in the regime where the second term in Eq. (11) dominates the first one does not actually change the final answer for the pair correlation function (12). Indeed, in this limit, integration over k_1 in Eq. (5) is determined by the diffusive exponential, that allows simply to replace k_1 by zero in the integrand of Eq. (5). Integrating the result over t one arrives at a factor $\delta(k_2)/r$, while subsequent integrations over k_2 and over the domain of small ζ , $\zeta \ll 1$, leads to the same expression for $\mathcal{K}(r)$, independent of the diffusion coefficient.

We will see now that the cancelation of the r_d dependence in the pair correlation function is incidental, and it does not actually extend to the general case of higher order correlation functions. Consider, for example, the fourth order simultaneous correlation function, $K^{(4)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = \langle \theta(t, \mathbf{r}_1) \theta(t, \mathbf{r}_2) \theta(t, \mathbf{r}_3) \theta(t, \mathbf{r}_4) \rangle$, which is decomposed into the following sum, $K^{(4)} = \mathcal{C}(\mathbf{r}_{12}, \mathbf{r}_{34}) + \mathcal{C}(\mathbf{r}_{23}, \mathbf{r}_{14}) + \mathcal{C}(\mathbf{r}_{13}, \mathbf{r}_{24})$, in the case of a Gaussian pumping, where $\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$. Being interested in establishing scaling law for the special case of collinear geometry, one focuses on analysis of $\mathcal{C}(\mathbf{r}, \mathbf{r})$. Generalizing evaluations resulted in Eqs. (5,11), one arrives at the following expression valid at $r \gg l$,

$$\begin{aligned} \mathcal{C}(\mathbf{r}, \mathbf{r}) \propto & \int_0^\infty dt_1 \int_0^\infty dt_2 \int d^2k \int d^2q \\ & \left\langle \exp \left[ir (k_1 \zeta e^{\varrho_1} + k_2 e^{-\varrho_1} + q_1 \zeta e^{\varrho_2} + q_2 e^{-\varrho_2}) \right. \right. \\ & \left. \left. - 2cr_d^2 (k_1^2 e^{2\varrho_1} + q_1^2 e^{2\varrho_2}) \right] \chi_{\mathbf{k}} \chi_{\mathbf{q}} \right\rangle, \quad (13) \end{aligned}$$

where $\varrho_1 = \varrho(t_1)$ and $\varrho_2 = \varrho(t_2)$. If $l \ll r \ll l^2/r_d$ then the diffusive exponent in (13) can be neglected and one arrives at the diffusionless expression

$$\mathcal{C}(\mathbf{r}, \mathbf{r}) \propto \int_0^\infty dt_1 dt_2 \langle \chi(r\zeta e^{\varrho_1}, r e^{-\varrho_1}) \chi(r\zeta e^{\varrho_2}, r e^{-\varrho_2}) \rangle,$$

leading to the scaling $\mathcal{C}(\mathbf{r}, \mathbf{r}) \propto 1/r^2$ derived in [11]. Note that the main contribution to the above time integrals comes from the region $\exp(\varrho_1) \sim \exp(\varrho_2) \sim r/l \gg 1$. In the $r \gg l^2/r_d$ limit, the diffusive exponential in Eq. (13) cannot be replaced by unity. On the contrary, it dominates integration over k_1 and q_1 resulting in emergence of the $\chi_{(0,k_2)}\chi_{(0,q_2)}$ term in the integrand. Then, integrations over t_1 and t_2 decouple from each other and one arrives at $\mathcal{C}(r, r) \propto (r_d r^3)^{-1} (\int d^2x \chi(\mathbf{x}))^2$, in accordance with the general formula (1).

The strong dependence of the correlation function on the diffusion, observed for the collinear geometry, does not extend to a general off-collinear case, where thus the diffusionless consideration of [11] applies. These distinct collinear and off-collinear results are asymptotically matched in the r_d/r -small angular vicinity of the collinear geometry. Note also that if the Corrsin integral $\int d^2r \chi(\mathbf{r})$ is equal to zero then the leading terms (12) and (??) in the correlation functions cancel. In this case the behavior of the correlation functions is determined by non-universal features of the flow statistics [18].

Summarizing, we have shown in this work that weak molecular diffusion does control the large scale correlations in scalar turbulence steered by the Batchelor incompressible flow. The main logical points of this work are: (a) Correlation of the passive scalar within the collinear geometry are much stronger than in an off collinear case. The angular extent of the collinear anomaly domain is controlled by the fact that a scalar stripe injected and stretched by the flow cannot get thinner than r_d . (b) The effect of diffusivity leads to a faster decay of scalar correlations with the scale r at the largest scales than in the domain of smaller scales, where diffusion is irrelevant. (c) Scaling in the diffusion-controlled regime becomes sensitive to the order of correlation function and in higher dimensions on details of the flow statistics [13].

The work at Los Alamos National Laboratory was carried out under the auspices of the National Nuclear Security Administration of the U.S. Department of Energy under Contract No. DE-AC52-06NA25396. IK and VL also acknowledge partial support of the RFBR grant 06-02-17408-a.

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