FROM NEWTON'S MECHANICS TO EULER'S EQUATIONS

How the equations of fluid dynamics were born 250 years ago

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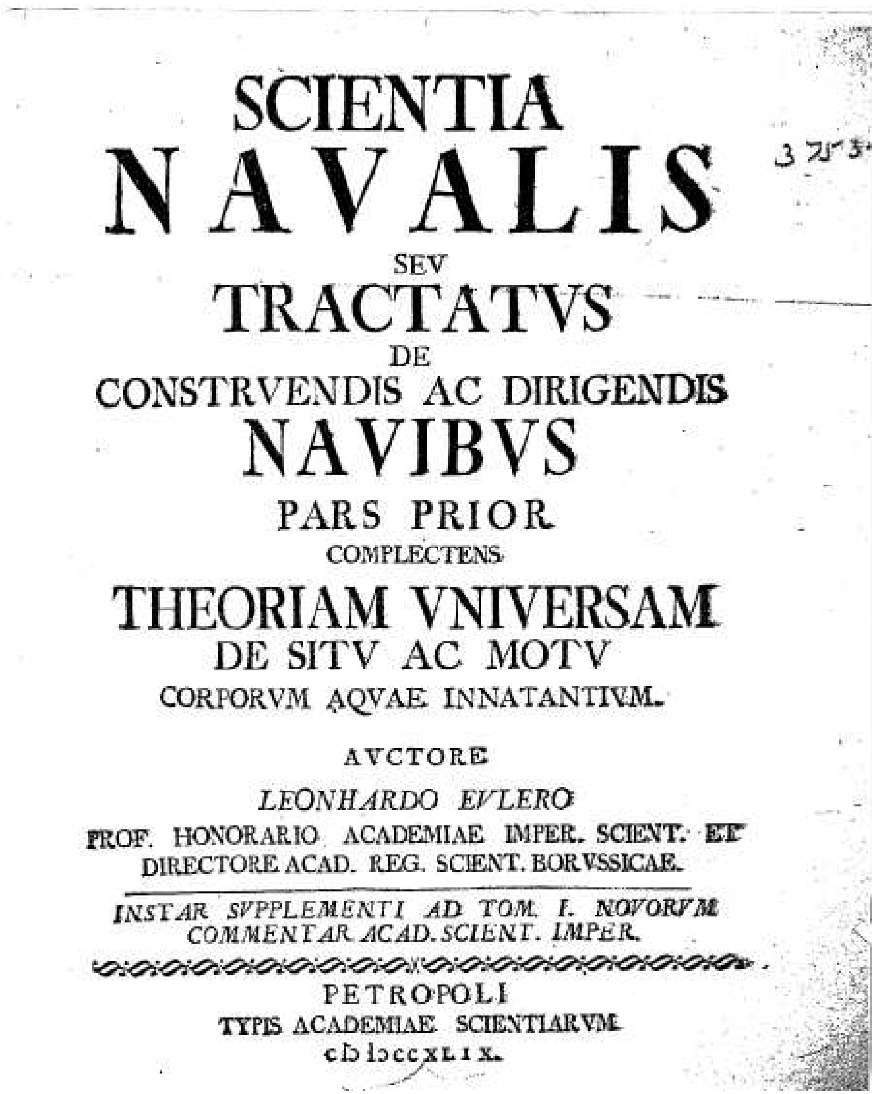
O. Darrigol and U. Frisch, Physica D in press.

Euler: 56 years of fluid dynamics

- 1727: Dissertatio physica de sono $\Longrightarrow$ [1784]: Calculs sur les ballons aérostatiques.
- Over 40 papers or books on fluids (theory and applications)



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CORPORVM AQVAE INNATANTIVM.
AVCTORE
LEONHARDO EVLERO
PROF. HONORARIO ACADEMIAE IMPER. SCIENT. ET
DIRECTORE ACAD. REG. SCIENT. BORVSSICAE.
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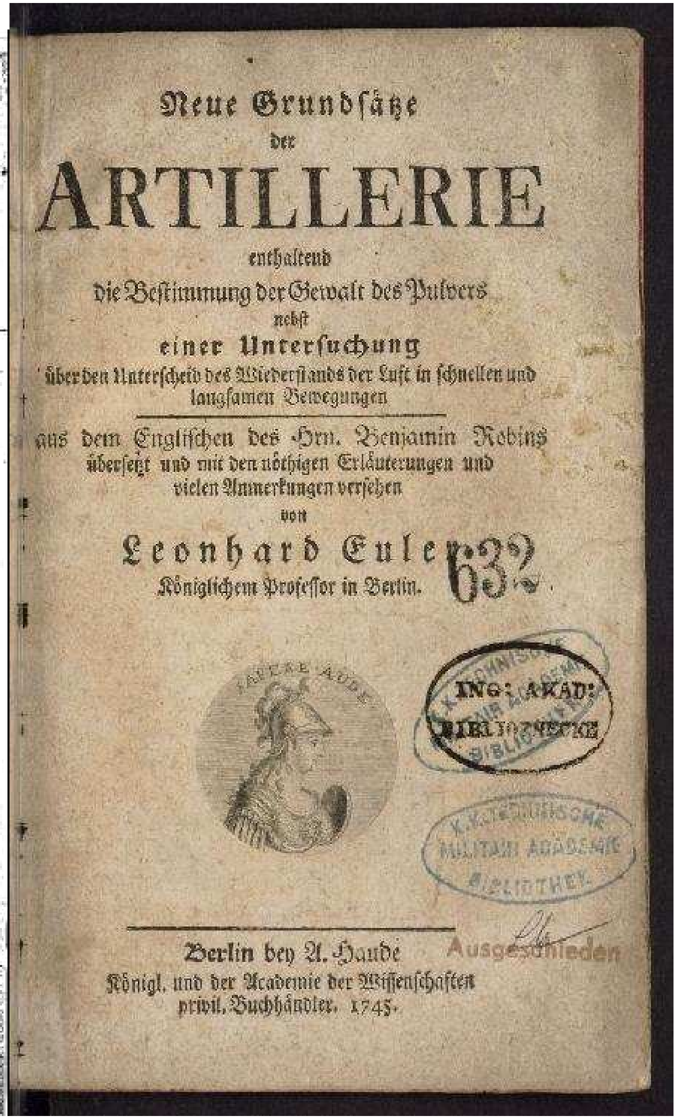


Neue Grundsätze
der
ARTILLERIE
enthaltend
die Bestimmung der Gewalt des Pulvers
nebst
einer Untersuchung
über den Unterscheid des Wiederstands der Luft in schnellen und
langsamen Bewegungen
aus dem Englischen des Hrn. Benjamin Robins
übersetzt und mit den nöthigen Erläuterungen und
vielen Anmerkungen versehen
von
Leonhard Euler
Königlichem Professor in Berlin.
Berlin bey A. Haude
Königl. und der Academie der Wissenschaften
privil. Buchhändler. 1745.

250 years ago: Euler’s equations

- “Principes généraux du mouvement des fluides”, presented to the Académie Royale des Sciences et Belles-Lettres (Berlin) on 4 September 1755 and published in 1757.

HISTOIRE
DE
L’ACADEMIE ROYALE
DES
SCIENCES
ET
BELLES LETTRES.

ANNÉE MDCCLV.

A BERLIN,
CHEZ HAUDE ET SPENER,
Libraires de la Cour & de l’Académie Royale.
MDCCLVII.

PRINCIPES GÉNÉRAUX
DU MOUVEMENT DES FLUIDES.
PAR M. EULER.

I.

Ayant établi dans mon Mémoire précedent les principes de l’équilibre des fluides le plus généralement, tant à l’égard de la diverſe qualité des fluides, que des forces qui y puiſſent agir ; je me propoſe de traiter ſur le même pied lè mouvement des fluides, & de rechercher les principes géneraux, ſur lesquels toute la ſcience du mouvement des fluides eſt fondée. On comprend aiſément que cette matiere eſt beaucoup plus difficile, & qu’elle renferme des recherches incomparablement plus profondes : cependant j’eſpère d’en venir auſſi heureuſement à bout, de ſorte que s’il y reſte des difficultés, ce ne ſera pas du côté du méchanique, mais uniquement du côté de l’analytique: cette ſcience n’étant pas encore portée à ce degré de perfection, qui ſeroit néceſſaire pour développer les formules analytiques, qui renferment les principes du mouvement des fluides.

II. Il s’agit donc de découvrir les principes, par lesquels on puiſſe déterminer le mouvement d’un fluide, en quelque état qu’il ſe trouve, & par quelques forces qu’il ſoit ſollicité. Pour cet effet examinons en détail tous les articles, qui conſtituent le ſujet de nos recherches, & qui renferment les quantités tant connues qu’inconnues. Et d’abord la nature du fluide eſt ſuppoſée connue, dont il faut conſidérer les diverſes eſpeces : le fluide eſt donc, ou incompreſſible, ou compreſſible. S’il n’eſt pas ſuſceptible de compreſſion, il faut diſtinguer deux cas, l’un où toute la maſſe eſt compoſée de parties homogenes, dont la denſité eſt partout & demeure toujours la même, l’autre

How Euler's equations were written

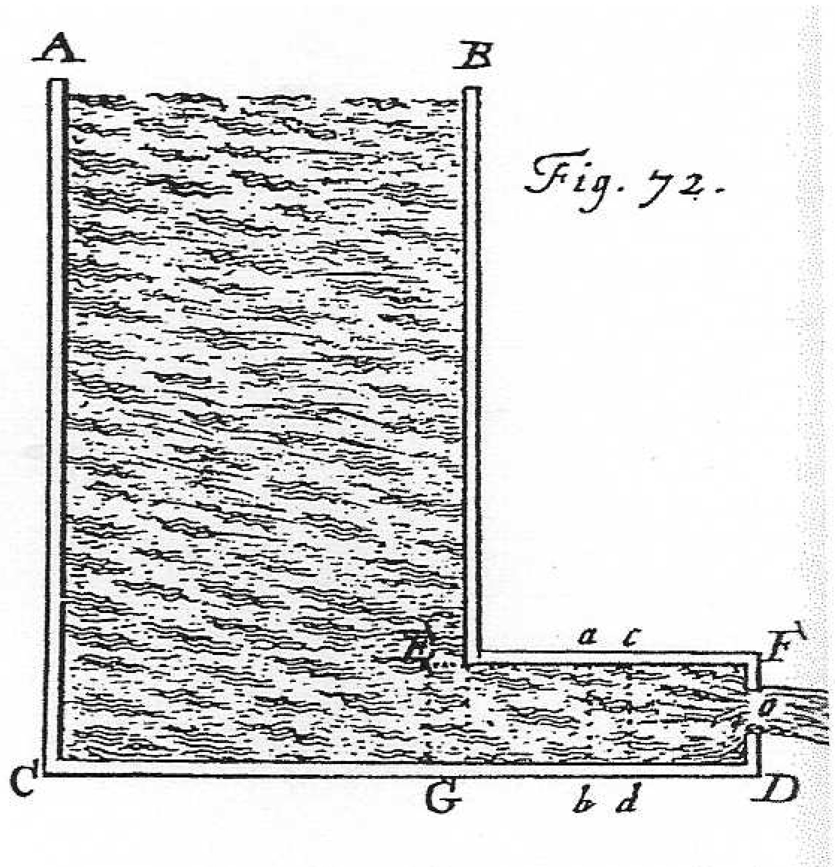
$$\left.\begin{aligned}
&\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)+\left(\frac{\mathrm{d}.qu}{\mathrm{d}x}\right)+\left(\frac{\mathrm{d}.qv}{\mathrm{d}y}\right)+\left(\frac{\mathrm{d}.qw}{\mathrm{d}z}\right)=0\\
&P-\frac{1}{q}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)=\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)+u\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)+v\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)+w\left(\frac{\mathrm{d}u}{\mathrm{d}z}\right)\\
&Q-\frac{1}{q}\left(\frac{\mathrm{d}p}{\mathrm{d}y}\right)=\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)+u\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)+v\left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)+w\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)\\
&R-\frac{1}{q}\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)=\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)+u\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)+v\left(\frac{\mathrm{d}w}{\mathrm{d}y}\right)+w\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right).
\end{aligned}\right\}$$

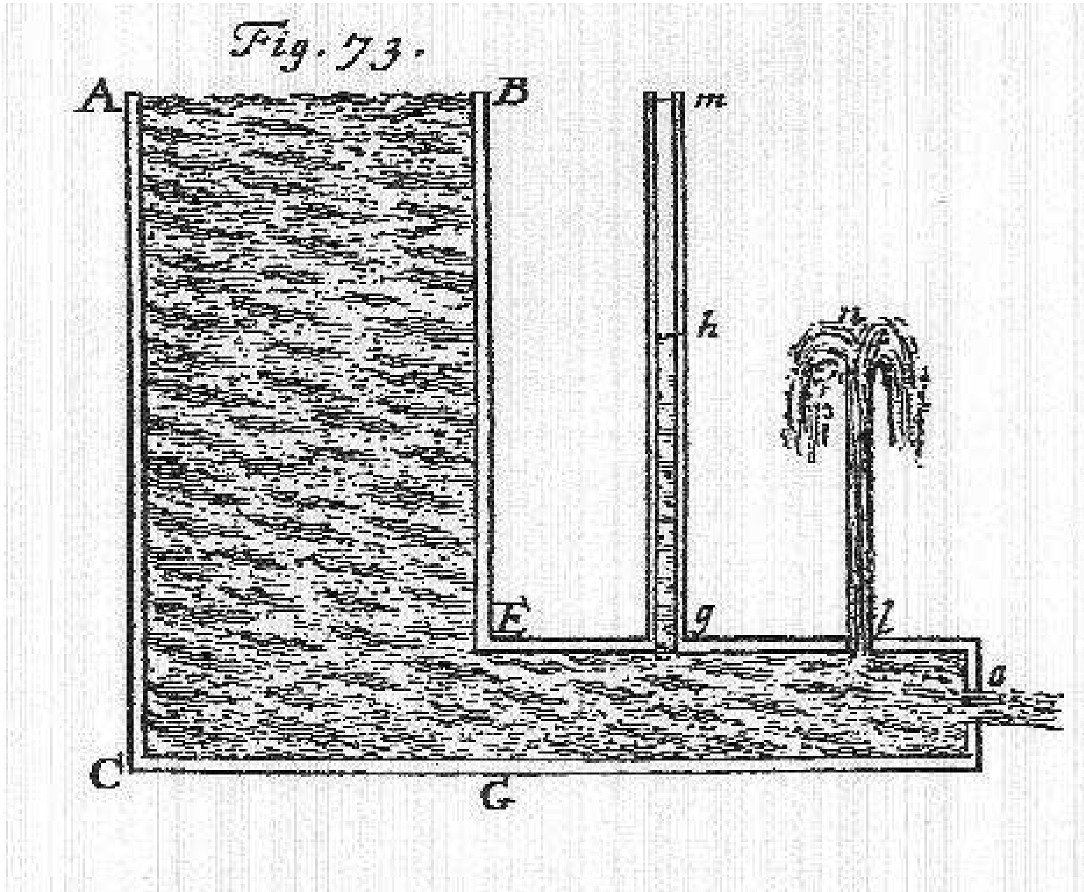
Flashback

Newton $\Rightarrow$ Johann and Daniel Bernouilli $\Rightarrow$ D'Alembert $\Rightarrow$ Euler

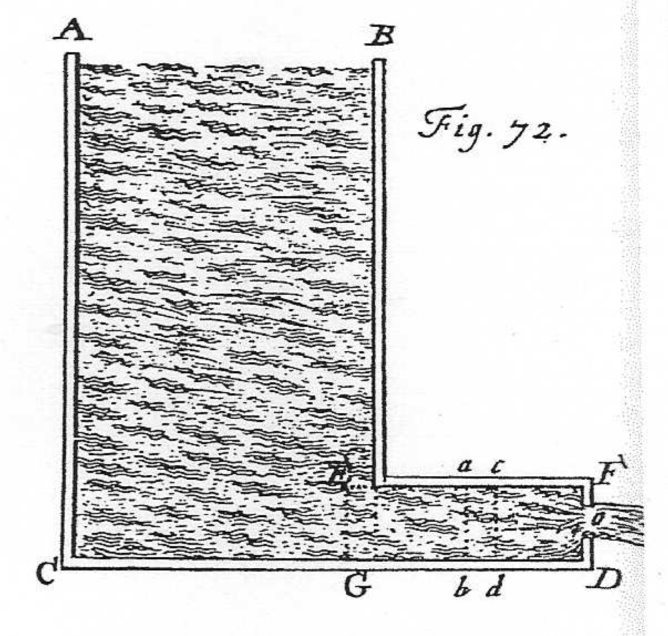
- Newton's law of "resistance" (drag): isolated particles individually impacting the head of the moving body; thus: $\mathsf{drag} \propto (\mathsf{velocity})^2$ (first scaling argument).
- Newton's theory of the propagation of sound: elasticity of the air is assumed and the (normal) pressure between successive layers of the air is taken into account.
- This is the first instance of *internal* pressure action.
- Many years needed before Euler's reintroduction of internal pressure as a means to derive the motion of fluid elements.

The derivation of (Daniel) Bernoulli's pressure law in 1734-1738





- According to Toricelli's law of efflux, the water escapes from the hole o with the velocity $u = \sqrt{2gh}$.
- Mass conservation implies that, within the appended tube EFDG the water flows with the velocity $v = (\varepsilon/s)u$, where ε and s are the crosssections of the hole and of the appended tube.



If in truth there were no barrier FD, the final velocity of the water in the same tube would be [s/ε times greater]. Therefore, the water in the tube tends to a greater motion, but its pressing [*nisus*] is hindered by the applied barrier FD. By this pressing and resistance [*nisus et renisus*] the water is compressed [*comprimitur*], which compression [*compressio*] is itself kept in by the walls of the tube, and thence these too sustain a similar pressure [*pressio*]. Thus it is plain that the pressure [*pressio*] on the walls is proportional to the acceleration. . . that would be taken on by the water if every obstacle to its motion should instantaneously vanish, so that it were ejected directly into the air.

- Principle of live forces (kinetic + potential energy conservation):

$$hsv\mathrm{d}t = \frac{v^2}{2g} sv\mathrm{d}t + bs\mathrm{d}\left(\frac{v^2}{2g}\right) \ , \quad b = Ea \ .$$

- Wall pressure $P \propto \mathrm{d}v/\mathrm{d}t = (gh - v^2/2)/b$ must reduce to gh for $v = 0$. Thus

$$P = gh - \frac{1}{2}v^2 \ .$$

D'Alembert: "Eulerian" coordinates and the use of PDEs

- In 1747 Jean le Rond d'Alembert submitted a prize-winning memoir "The cause of winds" to the Berlin Academy. The prescribed subject was:

  …to determine the order & the law wind should follow, if the Earth were surrounded on all sides by the Ocean; so that one could at all times predict the speed & direction of the wind in all places.

- For a special case involving spherical coordinates he gave, the first time the complete equations of motion of an incompressible fluid in a genuinely two-dimensional case.
- Thus emerged the ("Eulerian") velocity field and partial derivatives with respect to two independent spatial coordinates.
- In 1749 d'Alembert submitted a memoir "The resistance of fluids" to another Berlin prize and failed to win (comparison with experimental results was lacking).
- Yet this memoir contains the first instance of what we now call the vorticity equation for a steady incompressible flow around an axially symmetric body; it is derived by demanding that the convective derivative be potential and thus curlfree; in modern terms:

$$\nabla \times [(\mathbf{v} \cdot \nabla)\mathbf{v}] = \mathbf{0}\,.$$

- Only zero-vorticity solutions are identified (a mistake Euler will repeat in 1752).

D'Alembert's first derivation of the incompressibility condition

- In the 1749 memoir d'Alembert derived the incompressibility condition

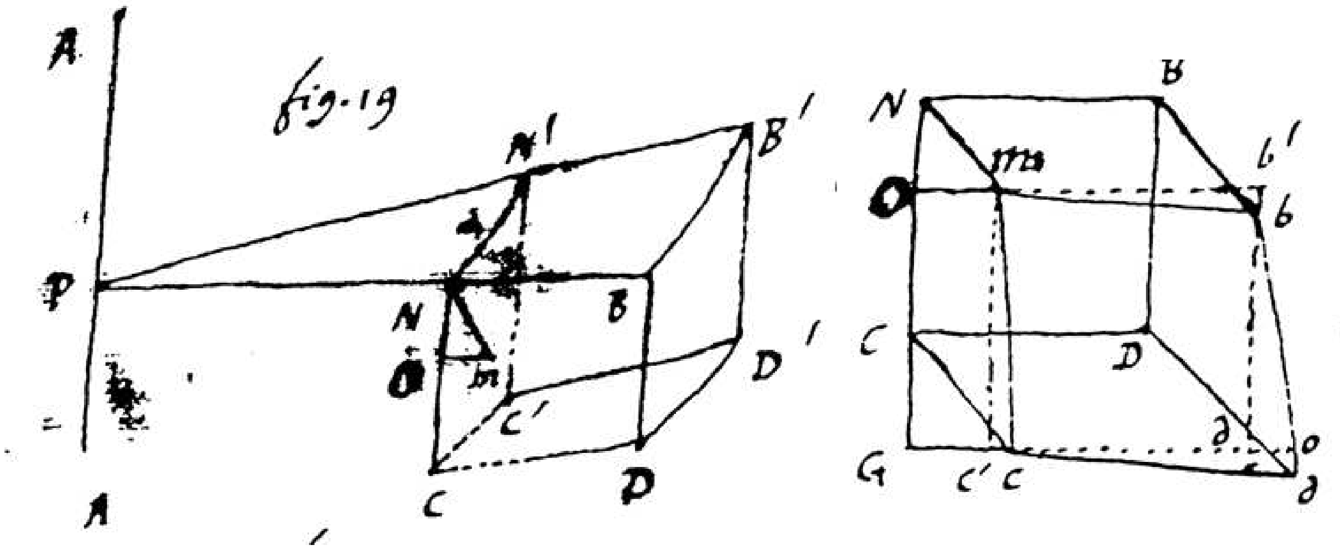
$$\nabla \cdot \mathbf{v} = \mathbf{0} \; .$$

- Actually, limiting himself to the *axially symmetric* case, he wrote literally:

$$\frac{\mathrm{d}q}{\mathrm{d}x} + \frac{\mathrm{d}p}{\mathrm{d}z} = \frac{p}{z} \; ,$$

  where z and x are the radial and axial coordinates and p and q the corresponding components of the velocity.

- For this, he considered the deformation of a small parallelepiped of fluid during an infinitesimal time interval.



Euler's 1750 "discovery of a new principle of mechanics"

- In a 1750 memoir, mostly devoted to the motion of a solid, Euler claimed that the true basis of continuum mechanics was Newton's second law applied to the infinitesimal elements of bodies.
- Among the forces acting on the elements he included "connection forces" acting on the boundary of the elements. In the case of fluids, these internal forces were to be identified to the pressure.
- In a memoir written around 1750–1751 Euler applied this principle to the motion of rivers, treated as a fully two-dimensional (steady-state) problem.
- The Cartesian coordinates of a fluid particle were expressed as functions of time and of a fillet-labeling parameter.
- He wrote partial differential equations expressing the incompressibility condition and his new principle.
- He derived for the first time the Bernoulli law along the stream lines of an arbitrary steady incompressible flow.
- Euler appeared not very satisfied with this paper.

The 1752-1761 Latin memoir

- The 1750 principle is fully implemented for two- and three-dimensional incompressible flow, using what we now call (cartesian) Eulerian coordinates.
- The incompressibility condition is established, basically as in d'Alembert's 1749 memoir for the Berlin-prize, and written in Euler's own notation as

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}z} = 0\,.$$

- The momentum equation is written (in somewhat modernized form) as

$$\mathrm{d}p = \left(\mathbf{g} - 2\frac{\partial \mathbf{v}}{\partial t} - 2(\mathbf{v}\cdot\nabla)\mathbf{v}\right)\cdot \mathrm{d}\mathbf{r}\,.$$

- The full time-dependent vorticity equation is derived in 2D and 3D.
- The only solutions retained were potential $\mathbf{v} = \nabla\varphi$, a mistake not repeated in the 1755-1757 French memoir but which led to Laplace's equation for φ.
- Bernoulli's law is established for such potential flow

$$p = \mathbf{g}\cdot\mathbf{r} - \frac{1}{2}v^2 - \frac{\partial\varphi}{\partial t} + C\,.$$

The three 1755-1757 French memoirs

- The first memoir "Principes généraux de l'état d'équilibre des fluides" gives a PDE formulation of hydrostatics.

- The second memoir "Principes généraux du mouvement des fluides" is why we are here! It also contains the first formulation of the initial value problem for PDEs.

- The third memoir "Continuation des recherches sur la théorie du mouvement des fluides" contains an extension of Bernoulli's law to *vortical* steady flow: the constancy of $p + v^2/2$ is no more global but only along each line of flow.

- Summarizing, altogether 21 years where needed:

* D. Bernoulli: his pressure law derived from live forces (1734-1738).

* J. Bernoulli: theory of *gurges* applied to identifying the convective derivative (1742).

* J. d'Alembert: PDE for the multi-dimensional description of fluid flow (1746); incompressibility condition in modern notation (1749).

* L. Euler: Newton's second law + internal pressure to obtain the first instance of the incompressible 2D and 3D Euler equations but with a "vorticity mistake" (1752). Then he obtained in the clearest and most general manner an amazingly stable foundation for the science of fluid motion (1755).

A long way to go

- At the end of the second 1755 memoir Euler wrote:

  We see well enough …how far we still are from a complete knowledge of the motion of fluids, and that what I have explained here contains but a feeble beginning. However, all that the Theory of fluids holds, is contained in the two equations reported above, so that it is not the principles of Mechanics which we lack in the pursuit of these researches, but solely Analysis, which is not yet sufficiently cultivated for this purpose. Thus we see clearly what discoveries remain for us to make in this Science before we can arrive at a more perfect Theory of the motion of fluids.

- As we know, Euler's equation are the first instance of a *nonlinear* field theory and remain to this day shrouded in mystery, contrary for example to the heat equation introduced by Fourier in 1807 and the Maxwell equations, discovered in 1862.

- It soon became clear that their application to problems of resistance (drag) led to paradoxes (d'Alembert's zero-drag paradox).

- The discovery of viscous stresses in the 19th century led to a (very slow) erosion of interest in Euler's equations.

- We now observe a strong renewal of interest: weak solutions, Young measure solutions, generalized flows all give new life to our beloved Euler equations.