“Noisy” spectra, long correlations and intermittency in wave turbulence

Yuri V. Lvov[†] and Sergey Nazarenko[*]
[†] *Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, NY 12180*
[*] *Mathematics Institute, The University of Warwick, Coventry, CV4-7AL, UK*

We study the k-space fluctuations of the waveaction about its mean spectrum in the turbulence of dispersive waves. We use a minimal model based on the Random Phase Approximation (RPA) and derive evolution equations for the arbitrary-order one-point moments of the wave intensity in the wavenumber space. The first equation in this series is the familiar Kinetic Equation for the mean waveaction spectrum, whereas the second and higher equations describe the fluctuations about this mean spectrum. The fluctuations exhibit a nontrivial dynamics if some long coordinate-space correlations are present in the system, as it is the case in typical numerical and laboratory experiments. Without such long-range correlations, the fluctuations are trivially fixed at their Gaussian values and cannot evolve even if the wavefield itself is non-Gaussian in the coordinate space. Unlike the previous approaches based on smooth initial k-space cumulants, the RPA model works even for extreme cases where the k-space fluctuations are absent or very large and intermittent. We show that any initial non-Gaussianity at small amplitudes propagates without change toward the high amplitudes at each fixed wavenumber. At each fixed amplitude, however, the PDF becomes Gaussian at large time.

I. INTRODUCTION

The concept of Wave Turbulence (WT), which describes an ensemble of weakly interacting dispersive waves, significantly enhanced our understanding of the spectral energy transfer in complex systems like the ocean, the atmosphere, or in plasmas [1–5]. This theory also became a subject of renewed interest recently, (see, e.g. [6–9]). Traditionally, WT theory deals with derivation and solutions of the Kinetic Equation (KE) for the mean waveaction spectrum (see e.g. [1]). However, all experimentally or numerically obtained spectra are “noisy”, i.e. exhibit k-space fluctuations which contain a complimentary to the mean spectra information. These k-space fluctuations always develop in numerical experiments even though, typically, most numerical experiments (e.g. [6,7]) start with initial waveaction fields in the k-space which have random phases but which have no amplitude fluctuations. How fast and why do these amplitude fluctuations get developed? Are they a numerical artifact or a real physical phenomenon? Are they Gaussian or some intermittent bursts of Fourier amplitudes can be expected? These questions remain unanswered because the waveaction fluctuation have not been studied before. Such a study involves description of the higher one-point moments of the Fourier amplitudes and it will be the main focus of the present work. We will show that when these one-point moments are not Gaussian the coordinate space fields are long-correlated. Such fields are very common in WT, and the numerical initial conditions discussed above is a typical example. Thus, we will have to generalize WT to include such long correlated fields.

II. RANDOM PHASES VS GAUSSIAN FIELDS.

The random phase approximation (RPA) has been popular in WT because it allows a quick derivation of KE [1,3]. We will use RPA in this paper because it provides a minimal model for for description of the k-space fluctuations of the waveaction about its mean spectrum, but we will also discuss relation to the approach of [2] which does not assume RPA.

Let us consider a wavefield $a(\mathbf{x}, t)$ in a periodic box[1]

[1]Periodic box is an essential intermediate step for formulating RPA and for defining the new correlators $M_k^{(p)}$ later in this paper. This is related to the fact that, strictly speaking, the infinite-space Fourier transform is a distribution, rather than a smooth function, for the class of functions corresponding to statistically homogeneous fields. The previous theory considered a class of correlators which are box-size independent and which could be formally obtained via a direct manipulation with the

of volume $\mathcal{V}$ and let the Fourier transform of this field be a_k. Later, we take the large box limit in order to consider homogeneous wave turbulence. Let us write the complex function a_k as $a_k = A_k \psi_k$ where A_k is a real positive amplitude and ψ_k is a phase factor which takes values on the unit circle centered at zero in the complex plane. By definition, RPA for an ensemble of complex fields a_k means that

1. The phase factors ψ_k are uniformly distributed on the unit circle in the complex plane and are statistically independent of each other

$$\langle \psi_{k_1} \psi_{k_2} \rangle = \delta_2^1,$$

where δ_2^1 is the Kronecker symbol.

2. The phases are statistically independent from the amplitudes A_k,

$$\langle \psi_{k_1} A_{k_2} \rangle = 0$$

Thus, the averaging over the phase and over the amplitude statistics can be performed independently.

3. the fluctuations of the amplitudes A_k must also be decorrelated at different k's.

$$\langle A_{k_1}^n A_{k_2}^m \rangle = \langle A_{k_1}^n \rangle \langle A_{k_2}^m \rangle \ \ (m, n = 1, 2, 3, ...).$$

Properties 2 and 3 have typically not been mentioned explicitly before. The name RPA itself does not refer to the amplitudes but to the phases only. However, this important assumption about the amplitude statistics has always been made implicitely when using RPA, often without even realizing it.

To illustrate the relation between the random phases and Gaussianity, let us consider the fourth-order moment for which RPA gives

$$\langle a_{k_1} a_{k_2} \overline{a}_{k_3} \overline{a}_{k_4} \rangle = n_{k_1} n_{k_2} (\delta_3^1 \delta_4^2 + \delta_4^1 \delta_3^2) + Q_{k_1} \delta_2^1 \delta_3^1 \delta_4^1, \quad (1)$$

where

$$n_k = \langle A_k^2 \rangle$$

is the waveaction spectrum and

$$Q_k = \langle A_k^4 \rangle - 2 \langle A_k^2 \rangle$$

is a cumulant coefficient. The last term in this expression appears because the phases drop out for $\mathbf{k_1} = \mathbf{k_2} = \mathbf{k_3} = \mathbf{k_4}$ and their statistics poses no restriction on the value of this correlator at this point. This cumulant part of the correlator can be arbitrary for a general random-phased field whereas for Gaussian fields Q_k must be zero. Such a difference between the Gaussian and the random-phased fields occurs only at a vanishingly small set of modes with $\mathbf{k_1} = \mathbf{k_2} = \mathbf{k_3} = \mathbf{k_4}$ and it has been typically ignored before because its contribution to KE is negligible. Therefore, if the mean waveaction spectrum was the only thing we were interested in, we could safely ignore contributions from all (one-point) moments

$$M_k^{(p)} = \langle |a_k|^{2p} \rangle \ \ (p = 1, 2, 3, ..).$$

However, it is precisely moments $M_k^{(p)}$ that contain information about fluctuations of the waveaction about its mean spectrum. For example, the standard deviation of the waveaction from its mean is

$$\xi_k = (\langle |a_k|^4 \rangle - \langle |a_k|^2 \rangle^2)^{1/2} = (M_k^{(2)} - n_k^2)^{1/2} \quad (2)$$

This quantity can be arbitrary for a general random-phased field whereas for a Gaussian wave field the fluctuation level ξ_k is fixed, $\xi_k = n_k$. Note that different values of moments $M_k^{(p)}$ can correspond to hugely different typical wave field realizations. In particular, if $M^{(p)} = n^p$ then there is no fluctuations and A_k is deterministic, $\xi_k = 0$. For the opposite extreme of large fluctuations we would have $M^{(p)} \gg n^p$ which means that the typical realization is sparse in the k-space and is characterized by few intermittent peaks of A_k and close to zero values in between these peaks. Such an information about the spectral fluctuations of the waveaction contained in the one-point moments $M^{(p)}$ is completely erased from the multiple-point moments by the random phases and it is precisely why these new objects play a crucial role for the description of the fluctuations.

Will the waveaction fluctuations appear if they were absent initially? Will they saturate at the Gaussian level

infinite-box a_k's.

$\xi_k = n_k$ or will they keep growing leading to the k-space intermittency? To answer these questions, we will use RPA to derive and analyze equations for the moments $M_k^{(p)}$ for arbitrary orders p and thereby describe the statistical evolution of the spectral fluctuations. Note that RPA, without a stronger Gaussianity assumption, is totally sufficient for the WT closure at any order. This allows us to study wavefields with moments $M_k^{(p)}$ very far from their Gaussian values, which may happen, for example, because of the choice of initial conditions or a non-Gaussianity of the energy source in the system.

In [2] non-Gaussian fields of a rather different kind were considered. Namely, statistically homogeneous wave fields were considered in an infinite space which initially have decaying correlations in the coordinate space and, therefore, smooth cumulants in the k-space, e.g.

$$\langle a_{k_1} a_{k_2} \overline{a}_{k_3} \overline{a}_{k_4} \rangle = n_{k_1} n_{k_2} (\delta^{k_1}_{k_3} \delta^{k_2}_{k_4} + \delta^{k_1}_{k_4} \delta^{2}_{3}) + C_{123} \delta^{k_1+k_2}_{k_3+k_4},$$

where C_{123} is a smooth function of $\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}$ and δ's now mean Dirac deltas. On the other hand, by taking the large box limit it is easy to see that our expression (1) corresponds to a *singular* cumulant $C_{123} = Q_{k_1}/\mathcal{V}\, \delta^{k_1}_{k_2} \delta^{k_1}_{k_3}$. It tends to zero when the box volume $\mathcal{V}$ tends to infinity and yet it gives a finite contribution to the waveaction fluctuations in this limit.[2] This singular cumulant corresponds to a small component of the wavefield which is long-correlated, - the case not covered by the approach of [2]. On the other hand, it would be straightforward to go beyond our RPA by adding a cumulant part of the initial fields which tends to a smooth function of $\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}$ in the infinite box limit (like in [2]). However, such cumulants would give a box-size dependent contribution to the waveaction fluctuations which vanishes in the infinite box limit (e.g. it would change ξ_k^2 by $C_{kkk}/\mathcal{V}$). Thus, in large boxes the waveaction fluctuation for the fields with smooth cumulants is fixed at the same value as the for Gaussian fields, $\xi_k = n_k$, and introduction of the singular cumulant is essential to remove this restriction on the level of fluctuations. On the other hand, the smooth part of the cumulant has no bearing on the closure (as shown in [2]) and on the large-box fluctuation and, therefore, will be omitted in this manuscript for brevity and clarity of the analysis.

III. TIME-SCALE SEPARATION ANALYSIS

Consider weakly nonlinear dispersive waves in a periodic box. Here we consider quadratic nonlinearity and the linear dispersion relations ω_k which allow three-wave interactions. Example of such systems include surface capillary waves [4] and internal waves in the ocean [9]. In Fourier space, the general form for the Hamiltonian systems with quadratic nonlinearity looks as follows,[3]

$$\begin{aligned}
\mathcal{H} &= \sum_{n=1}^{\infty} \omega_n |c_n|^2 + \epsilon \sum_{l,m,n=1}^{\infty} \left(V^l_{mn} \bar{c}_l c_m c_n \delta^l_{m+n} + c.c. \right), \\
i\dot{c}_l &= \frac{\partial \mathcal{H}}{\partial \bar{c}_l}, \quad c_l = a_l e^{-i\omega_l t}, \\
i\,\dot{a}_l &= \epsilon \sum_{m,n=1}^{\infty} \Big(V^l_{mn} a_m a_n e^{i\omega^l_{mn} t}\, \delta^l_{m+n} \\
&\qquad + 2\bar{V}^m_{ln} \bar{a}_n a_m e^{-i\omega^m_{ln} t}\, \delta^m_{l+n} \Big), \quad (3)
\end{aligned}$$

where $a_n = a(k_n)$ is the complex wave amplitude in the interaction representation, $k_n = 2\pi n/L$, L is the box side length, $n = (n_1, n_2)$ for 2D, or $n = (n_1, n_2, n_3)$ in 3D, (similar for k_l and k_m), $\omega^l_{mn} \equiv \omega_{k_l} - \omega_{k_m} - \omega_{k_m}$ and $\omega_l = \omega_{k_l}$ is the wave linear dispersion relation. Here, $V^l_{mn} \sim 1$ is an interaction coefficient and ϵ is introduced as a formal small nonlinearity parameter.

In order to filter out fast oscillations at the wave period, let us seek for the solution at time T such that $2\pi/\omega \ll T \ll 1/\omega\epsilon^2$. The second condition ensures that T is a lot less than the nonlinear evolution time. Now let us use a perturbation expansion in small ϵ,

$$a_l(T) = a_l^{(0)} + \epsilon a_l^{(1)} + \epsilon^2 a_l^{(2)}.$$

Substituting this expansion in (3) we get in the zeroth order $a_l^{(0)}(T) = a_l(0)$, i.e. the zeroth order term is time

[2]Thus, assuming a finite box is an important intermediate step when introducing the relevant to the fluctuations objects like Q_k.

[3]We will follow the RPA approach as presented by Galeev and Sagdeev [3] but deal with a slightly more general case where the wave field is not restricted by the condition $\overline{a}(k) = a(-k)$. We will also use elements of the technique and notations of [2].

independent. This corresponds to the fact that the interaction representation wave amplitudes are constant in the linear approximation. For simplicity, we will write $a_l^{(0)}(0) = a_l$, understanding that a quantity is taken at $T = 0$ if its time argument is not mentioned explicitly. The first order is given by

$$a_l^{(1)}(T) = -i \sum_{m,n=1}^{\infty} \left(V_{mn}^l a_m a_n \Delta_{mn}^l \delta_{m+n}^l + 2\bar{V}_{ln}^m a_m \bar{a}_n \bar{\Delta}_{ln}^m \delta_{l+n}^m \right), \quad (4)$$

where $\Delta_{mn}^l = \int_0^T e^{i\omega_{mn}^l t} dt = (e^{i\omega_{mn}^l T} - 1)/i\omega_{mn}^l$. Here we have taken into account that $a_l^{(0)}(T) = a_l$ and $a_k^{(1)}(0) = 0$.

To calculate the second iterate, write

$$i\dot{a}_l^{(2)} = \sum_{m,n=1}^{\infty} \Big[V_{mn}^l \delta_{m+n}^l e^{i\omega_{mn}^l t} \left(a_m^{(0)} a_n^{(1)} + a_m^{(1)} a_n^{(0)} \right) + 2\bar{V}_{ln}^m \delta_{l+n}^m e^{-i\omega_{ln}^m t} \left(a_m^{(1)} \bar{a}_n^{(0)} + a_m^{(0)} \bar{a}_n^{(1)} \right) \Big]. \quad (5)$$

We now have to substitute (4) into (5) and integrate over time to obtain

$$\begin{aligned} a_l^{(2)}(T) = \sum_{m,n,\mu,\nu=1}^{\infty} \big[& 2V_{mn}^l \left(-V_{\mu\nu}^m a_n a_\mu a_\nu E[\omega_{n\mu\nu}^l, \omega_{mn}^l] \delta_{\mu+\nu}^m - 2\bar{V}_{m\nu}^\mu a_n a_\mu \bar{a}_\nu \bar{E}[\omega_{n\mu}^{l\nu}, \omega_{mn}^l] \delta_{m+\nu}^\mu \right) \delta_{m+n}^l \\ & + 2\bar{V}_{ln}^m \left(-V_{\mu\nu}^m \bar{a}_n a_\mu a_\nu E[\omega_{\mu\nu}^{ln}, -\omega_{ln}^m] \delta_{\mu+\nu}^m - 2\bar{V}_{m\nu}^\mu \bar{a}_n a_\mu \bar{a}_\nu E[-\omega_{n\nu l}^\mu, -\omega_{ln}^m] \delta_{m+\nu}^\mu \right) \delta_{l+n}^m \\ & + 2\bar{V}_{ln}^m \left(\bar{V}_{\mu\nu}^n a_m \bar{a}_\mu \bar{a}_\nu \delta_{\mu+\nu}^n E[-\omega_{l\nu\mu}^m, -\omega_{ln}^m] + 2V_{n\nu}^\mu a_m \bar{a}_\mu a_\nu E[\omega_{\nu m}^{\mu l}, -\omega_{ln}^m] \delta_{n+\nu}^\mu \right) \delta_{l+n}^m \big], \end{aligned} \quad (6)$$

where we used $a_k^{(2)}(0) = 0$ and introduced $E(x, y) = \int_0^T \Delta(x - y) e^{iyt} dt$.

IV. STATISTICAL DESCRIPTION

Let us now develop a statistical description applying RPA to the fields $a_k^{(0)}$. Since phases and the amplitudes are statistically independent in RPA, we will first perform average over the random phases (denoted as $\langle ... \rangle_\psi$) and then we average over amplitudes (denoted as $\langle ... \rangle_A$) to calculate the moments,

$$M_k^{(p)}(T) \equiv \langle |a_k(T)|^{2p} \rangle_{\psi,A}. \quad p = 1,\ 2,\ 3\ ...,$$

First, let us calculate $|a_l(T)|^{2p}$ as

$$\begin{aligned} |a_l(T)|^{2p} = \left(a_l^{(0)} + \epsilon a_l^{(1)} + \epsilon^2 a_l^{(2)} \right)^p \left(\bar{a}_l^{(0)} + \epsilon \bar{a}_l^{(1)} + \epsilon^2 \bar{a}_l^{(2)} \right)^p = |a_l^{(0)}|^{2p} + \epsilon p |a_l^{(0)}|^{2p-2} \left(a_l^{(0)} \bar{a}_l^{(1)} + \bar{a}_l^{(0)} a_l^{(1)} \right) + \\ \epsilon^2 |a_l|^{2p-4} \Big[C_p^2 (a_l^{(0)} \bar{a}_l^{(1)})^2 + C_p^2 (\bar{a}_l^{(0)} a_l^{(1)})^2 + p^2 |a_l^{(0)}|^2 |a_l^{(1)}|^2 + p |a_l^{(0)}|^2 \left(a_l^{(0)} \bar{a}_l^{(2)} + \bar{a}_l^{(0)} a_l^{(2)} \right) \Big] + ..., \end{aligned}$$

where C_p^2 is the binomial coefficient.

Up to the second power in ϵ terms, we have

$$\langle |a_l(T)|^{2p} \rangle_\psi = |a_l|^{2p} + \epsilon^2 |a_l|^{2p-2} \left(p^2 \langle |a_l^{(1)}|^2 \rangle_\psi + p \, \langle a_l^{(0)} \bar{a}_l^{(2)} + \bar{a}_l^{(0)} a_l^{(2)} \rangle_\psi \right) \quad (7)$$

Here, the terms proportional to ϵ dropped out after the phase averaging. Further, we assume that there is no coupling to the $k = 0$ mode, i.e. $V_{k_1 k_2}^{k=0} = V_{k_1 k=0}^{k1} = 0$, so that there is no contribution of the term like $\langle (a_l^{(0)} \bar{a}_l^{(1)})^2 \rangle_\psi$. We now use (4) and (6) and the averaging over the phases to obtain

$$\langle |a_l^{(1)}|^2\rangle_\psi =$$

$$2\sum_{m,n}^{\infty}\big[|V^l_{mn}|^2\delta^l_{m+n}|\Delta^l_{mn}|^2|a_m|^2|a_n|^2 +2|V^n_{lm}|^2|\delta^n_{l+m}|^2\Delta^n_{lm}|a_n|^2|a_m|^2\big],$$

$$\langle a_l^{(0)}\bar{a}_l^{(2)}+\bar{a}_l^{(0)}a_l^{(2)}\rangle_\psi =$$

$$-8|a_l|^2\sum_{m,n}^{\infty}\big[|V^l_{mn}|^2\delta^l_{m+n}E(0,\omega^l_{mn})|a_m|^2 +|V^n_{lm}|^2\delta^n_{l+m}E(0,\omega^n_{lm})(|a_m|^2-|a_n|^2)\big].$$

Let us substitute these expressions into (7), perform amplitude averaging, take the large box limit[4] and then large T limit ($T\gg 1/\omega$)[5]. We have

$$M_k^{(p)}(T)=M_k^{(p)}(0)+T\left(-p\gamma_k M_k^{(p)}+p^2\eta_k M_k^{(p-1)}\right), \tag{8}$$

with

$$\eta_k = 4\pi\epsilon^2\int d\mathbf{k_1}d\mathbf{k_2}\, n_1n_2(|V^k_{12}|^2\delta^k_{12}\delta(\omega^k_{12}) +2|V^2_{k1}|^2\delta^2_{k1}\delta(\omega^2_{k1}))\,, \tag{9}$$

$$\gamma_k = 8\pi\epsilon^2\int d\mathbf{k_1}d\mathbf{k_2}(|V^k_{12}|^2\delta^k_{12}\delta(\omega^k_{12})n_2 +|V^2_{k1}|^2\delta^2_{k1}\delta(\omega^2_{k1})(n_1-n_2)). \tag{10}$$

Now, assuming that T is a lot less than the nonlinear time ($T\ll 1/\omega\epsilon^2$) we finally arrive at our main result,

$$\dot{M}_k^{(p)}=-p\gamma_k M_k^{(p)}+p^2\eta_k M_k^{(p-1)}. \tag{11}$$

In particular, for the waveaction spectrum $M_k^{(1)}=n_k$ (11) gives the familiar kinetic equation (KE)

$$\dot{n}_k=-\gamma_k n_k+\eta_k=\epsilon^2 J(n_k), \tag{12}$$

where $J(n_k)$ is the "collision" term [1,3],

$$J(n_k)=\int d\mathbf{k_2}d\mathbf{k_1}(R^k_{12}-R^1_{k2}-R^2_{1k}),$$

with

$$R_{k12}=4\pi|V^k_{12}|^2\delta^k_{12}\delta(\omega^k_{12})\Big(n_2n_1-n_k(n_2+n_1)\Big). \tag{13}$$

The second equation in the series (11) allows to obtain the r.m.s. $\xi_k^2=M_k^{(2)}-n_k^2$ (2) of the fluctuations of the waveaction $\langle|a_k|^2\rangle$. We emphasize that (11) is valid even for strongly intermittent fields with big fluctuations.

V. ANALYSIS OF SOLUTIONS: GAUSSIANITY VS INTERMITTENCY

Let us now consider the stationary solution of (11), $\dot{M}_k^{(p)}=0$ for all p. Then for $p=1$ from (12) we have $\eta_k=\gamma n_k$. Substituting this into (11) we have

$$M_k^{(p)}=pM_k^{(p-1)}n_k,$$

with the solution $M_k^{(p)}=p!\ n_k^p$. Such a set of moments correspond to a Gaussian wavefield a_k. To see how such a Gaussian steady state forms in time, let us rewrite (11) in terms of relative deviations of $M_k^{(p)}$ from their Gaussian values,

$$F_k^{(p)}=\frac{M_k^{(p)}-p!\ n_k^p}{p!\ n_k^p},\ \ p=1,2,3,....$$

By definition, $F_k^{(1)}$ is always zero. For $p=2$, this expression measures the flatness of the distribution of Fourier amplitudes at each k. This quantity determines the r.m.s. $\xi_k^2=M_k^{(2)}-n_k^2$ of the fluctuations of the waveaction $\langle|a_k|^2\rangle$, (2) or the mean level of "noisyness",

$$\xi_k^2=n_k^2(2F_k^{(2)}+1).$$

Using (11), we obtain

$$\dot{F}_k^{(p)}=\frac{p\eta_k}{n_k}(F_k^{(p-1)}-F_k^{(p)}), \tag{14}$$

[4]The large box limit implies that sums will be replaced with integrals, the Kronecker deltas will be replaced with Dirac's deltas, $\delta^l_{m+n}\to\delta^l_{mn}/\mathcal{V}$, where we introduced short-hand notation, $\delta^l_{mn}=\delta(k_l-k_m-k_n)$. Further we redefine $M_k^{(p)}/\mathcal{V}^p\to M_k^{(p)}$.

[5]Note that $\lim_{T\to\infty}E(0,x)=T(\pi\delta(x)+iP(\frac{1}{x}))$, and $\lim_{T\to\infty}|\Delta(x)|^2=2\pi T\delta(x)$ (see e.g. [2]).

for $p = 2, 3, 4, ...$ This results has a particularly simple form of a decoupled equation for $p = 2$,

$$\dot{F}_k^{(2)} = -\frac{2\eta_k}{n_k} F_k^{(2)}.$$

Taking into account that $\eta_k > 0$, we see from this equation that deviations of the *mean* level of fluctuations from Gaussianity always decay. In fact, deviations $F^{(p)}$ decay at each *fixed* p. This is easy to see from the general solution of (14) (obtained recursively):

$$F_k^{(p)}(t) = e^{-p\theta} \sum_{j=2}^{p} \frac{\theta^{p-j} p!}{j!(p-j)!} F_k^{(j)}(t=0), \qquad (15)$$

where $\theta = \int_0^t \frac{\eta_k}{n_k} dt'$ is a "renormalized" time variable. One can see that this expression decays exponentially as $t \to \infty$ for any fixed p.

However, an interesting picture emerges at high p corresponding to high wave amplitudes. Although the deviations $F_k^{(p)}$ eventually decay at each *fixed* p, their initial values propagate in p without decay toward the larger values of p. Indeed, one can approximate (14) for $p \gg 1$ by a first-order PDE,

$$\partial_t F_k^{(p)} + \frac{p\eta_k}{n_k} \partial_p F_k^{(p)} = 0.$$

According to this equation, $F_k^{(p)}$ propagates toward high p's as a wave. This wave does not change shape with respect to coordinate $x = \ln p$ and, therefore, it spreads in p without change in amplitude. The speed of this wave (in x) is time independent for statistically steady states (i.e. when n_k and η_k are time independent). Note that this dynamics occurs at each k practically independently, i.e. the only coupling of different k's occurs in the propagation speed via η_k.

These solutions allow us to establish the character of intermittency in wave turbulence systems, i.e. to describe how high-amplitude "bursts" occur with greater than Gaussian probabilities. In terms of the PDF, the wave of non-Gaussianity $F_k^{(p)}$ toward high values of p corresponds to a wave propagating from low-amplitude "bulk" part to the high-amplitude "tail" on the PDF profile. Indeed, a Gaussian PDF for a_k corresponds to a distribution of $\lambda = |a_k|^2$ of form $P(\lambda) = n^{-1} e^{-\lambda/n}$, and moment $M_k^{(p)}$ "probes" this distribution in a range of λ around $\lambda_p = pn$ with a characteristic width $\delta\lambda \sim n$. Lifting $P(\lambda)$ in this range by a certain factor will result in an increase of moment $M_k^{(p)}$ by the same factor. Thus, the wave propagating from small to large p's corresponds to a wave from low to high λ's. This wave is such that the relative deviation from distribution $P(\lambda) = n^{-1} e^{-\lambda/n}$ remains unchanged, but the range of λ's at which such non-Gaussianity occurs moves into the tail (with speed η) and spreads (proportionally to its position in λ). Note however that at each *fixed* λ deviations from $P(\lambda) = n^{-1} e^{-\lambda/n}$ decay, which corresponds to decay of $F^{(p)}$ at each fixed p at large time.

Predictions (15) about the behavior of fluctuations of the waveaction spectra can be tested by modern experimental techniques which allow to produce surface water waves with random phases and a prescribed shape of the amplitude $|a_k|$ [11]. It is even easier to test (15) numerically. Consider for example capillary waves on deep water. If a Gaussian forcing at low k values is present, the steady state solution of the kinetic equation corresponds to the Zakharov-Filonenko (ZF) spectrum of Kolmogorov type [1,4]. It is given by

$$n_k = A k^{-17/4}, \qquad (16)$$

with $A = \sqrt{P} \rho^{3/2} C / \sigma^{1/4}$, where P is the value of flux of energy toward high wavenumbers, ρ and σ are the density and surface tension of water, and $C \simeq 13.98$. The simplest experiment would be to start with a zero-fluctuation (deterministic) spectrum and to compare the fluctuation growth with the predictions of (11). Note that such no-fluctuations initial conditions were used in [6,7].

Let us calculate the rate at which a fluctuations grow for such an initial conditions. Since n_k and η_k are time independent in this case, we have $\theta = \eta_k t / n_k = \gamma_k t$. Thus, the only quantity we need to calculate is γ_k. Let us take into account that the spectrum n_k is isotropic, that is it depends only on the modulus of the vector, not on its directions. We then can perform an angular averaging of (10) obtaining

$$\begin{aligned} \gamma_k = 8\epsilon^2 \int d\mathbf{k_1} d\mathbf{k_2} S^{-1}_{kk_1k_2} (|V^k_{12}|^2 \delta(\omega^k_{12}) n_2 \\ + |V^2_{k1}|^2 \delta(\omega^2_{k1})(n_1 - n_2)), \\ S_{k12} = \langle \delta(\mathbf{k} - \mathbf{k_1} - \mathbf{k_2}) \rangle \equiv \frac{1}{4\pi^2} \int \delta(\mathbf{k} - \mathbf{k_1} - \mathbf{k_2})\, d\theta_1 d\theta_2 \\ = \frac{1}{2} \sqrt{2\left((kk_1)^2 + (kk_2)^2 + (k_1k_2)^2\right) - k^4 - k_1^4 - k_2^4}\,. \end{aligned} \qquad (17)$$

Let us substitute ZF spectrum (16) into (17), take the values of ω_k and V^k_{12} appropriate for the capillary waves on deep water([1], eqs (5.2.1-2)). By changing the variables of integrations via $k_1 = k\xi_1, \quad k_2 = k\xi_2$ we can factor out the k dependence of γ_k. Performing one of ξ integrals analytically with the use of the delta function in ω's, we perform the remaining single integral numerically to obtain (all the integrals converge):

$$\gamma = \frac{4.30A\sqrt{\sigma}}{16\pi\rho^{3/2}}k^{3/4},$$

where the dimensionless constant 4.30 was obtained by numerical integration. Substituting the value of A we finally obtain

$$\gamma = 1.20\sqrt{P}\sigma^{1/4}k^{3/4},$$

Consequently, our prediction for the fluctuations growth is

$$\xi_k^2 = n_k^2(2F_k^{(2)} + 1) = A^2k^{-17/2}(1 - e^{-2\gamma_k t}). \tag{18}$$

Note that fluctuations stabilize at Gaussian values faster for high k values. One can also substitute γ_k calculated for the capillary waves into the solutions for the higher p's, (15). Again, the dynamics here is going to be faster at large k's because they correspond to higher values of $\theta = \gamma t$. In particular, at large k there will be a faster wave toward higher p's. For the particular type of initial conditions we have taken (no initial fluctuations), this wave will describe formation of a Rayleigh distribution (corresponding to the Gaussian statistics of a_k) behind a propagating front on the PDF profile. It a way, this dynamics is non-intermittent: zero initial fluctuations grow to the Gaussian level but never exceed it.

It is also interesting to test our predictions when the initial conditions, or forcing, are non-Gaussian, as in most practical situations. Our theory predicts that non-Gaussianity of the low-amplitude (bulk) part of the PDF will propagate without decay into the high-amplitude tail at each fixed k. The speed of this propagation is proportional to γ_k and, therefore, will be higher for large k's in the case of the capillary waves. This means higher intermittency in the low-k range in the case of stationary forced turbulence.

VI. DISCUSSION

In this manuscript, we derived a hierarchy of equations (11) for the one-point moments $M_k^{(p)}$ of the wave-action $|a_k|^2$. This system of equations has a "triangular" structure: the time derivative of the p-th moment depends only on the moments of order $p, p-1$ and 1 (spectrum). Their evolution is not "slaved" to the spectrum or any other low moments and it depends on the initial conditions. RPA allows the initial conditions to be far from Gaussian and deviation of n'th moment from its Gaussian. Among two allowed extreme limits are the wavefield with a deterministic amplitude $|a_k|$ (for which $M_k^{(p)} = n_k^p$) and the intermittent wavefields characterized by sparse k-space distributions of $|a_k|$ (for which $M_k^{(p)} \gg n_k^p$).

Equations (14) for the deviations from Gaussianity have an interesting property that the nonlinear coupling between different modes k occurs only via a rate constant η/n. By removing this dependence into a "renormalized" time θ one gets a linear system of equation which can be easily solved in the general case, see (15). Analyzing these solutions we showed that the deviation from Gaussianity decreases as at each fixed amplitude $|a_k|$. At the same time, we showed that any initial non-Gaussianity at small amplitudes propagates as a non-decaying wave toward the high-amplitude tail of the PDF. This process describes the character of the wave turbulence intermittency when high-amplitude wave "bursts" occur in the system more frequently than predicted for Gaussian fields. On the other hand, the assumption about the weak nonlinearity breaks down when a high amplitude burst occurs in the system, leading to a failure of the RPA closure to describe the PDF tails. One can conjecture that the resulting phase coherence will lead to a nonlinear amplitude saturation which will stop the wave predicted by our theory which, in turn, will lead to a stagnation and accumulation in this region on the PDF tail. Thus, it is natural to expect even stronger intermittency when the higher order nonlinear effects are taken into account.

We would like to emphasize that the type of intermittency discussed in the present manuscript appears within the weakly nonlinear closure and not as a result of its breakdown as in [10]. This intermittency is quite subtle and it occurs only in the PDF tails and not in its core (which tends to a Gaussian state). As a result, the lower moments will not feel these rare "bursts" and they will evolve as predicted by the WT closure. We have showed that this kind of intermittency inevitably occurs at *all* wavenumbers k provided some initial non-Gaussianity is present in the PDF core. Paper [10] considers a different and a more dramatic kind of intermittency which occurs simultaneously with the strong nonlinearity of the typical wave from the PDF core. This kind of intermittency is more seldom and it takes place only in some special parts of the k-space (e.g. at very small scales). In particular, it never occurs for the capillary waves considered in this paper provided that only weakly nonlinear waves are produced at the forcing scale.

The present paper deals with the three-wave systems only. The four-wave resonant interactions are slightly more complicated in that the nonlinear frequency shift occurs at a lower order in nonlinearity parameter than the nonlinear evolution of the wave amplitudes. To build a consistent description of the amplitude moments one has to perform a renormalization of the perturbation series taking into account the nonlinear frequency shift. This will be done in a future publication.

Acknowledgments Authors thank Alan Newell for enlightening discussions. YL is supported by NSF CAREER grant DMS 0134955 and by ONR YIP grant N000140210528. SN thanks ONR for the support of his visits to RPI.

[1] V.E. Zakharov, V.S. L'vov and G.Falkovich, "Kolmogorov Spectra of Turbulence", Springer-Verlag, 1992.

[2] D.J. Benney and P.Saffman, Proc Royal. Soc, A(1966), 289, 301-320; B. J . Benney and A.C. Newell, Studies in Appl. Math. **48** (1) 29 (1969).

[3] A.A. Galeev and R.Z. Sagdeev, in "Reviews of Plasma Physics" Vol. 6 (Ed. M A Leontovich) (New York: Consultants Bureau, 1973)

[4] V.E.Zakharov and Filonenko, J. Appl. Mech. Tech. Phys. **4** 506-515 (1967).

[5] K. Hasselmann, J. Fluid Mech **12** 481 (1962). " Freely decaying weak turbulence for sea surface gravity waves "

[6] M. Onorato *et al*, PRL **89**, 144501, (2002).

[7] " Turbulence of capillary waves" A.N. Pushkarev, V.E. Zakharov, PRL **76**, 3320-3, (1996). Physica D, **135**, 98, (2000).

[8] "Generation of zonal flow and meridional anisotropy in two-layer weak geostrophic turbulence " T. Soomere, PRL **75**, 2440-3, (1995).

[9] Yu. V. Lvov and E.G. Tabak, PRL **87**, 168501, (2001)

[10] L. Biven, S.V. Nazarenko and A.C. Newell, "Breakdown of wave turbulence and the onset of intermittency", Phys Lett A, **280**, 28-32, (2001) A.C. Newell, S.V. Nazarenko and L. Biven, Physica D, **152-153**, 520-550, (2001).

[11] L. Shemer, H. Jiao, E. Kit and Y. Agnon "Evolution of a nonlinear wave field along a tank: experiments and numerical simulations based on the spatial Zakharov equation. " J. Fluid Mech. **427**, 107-129, (2001)