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On Scaling Laws for the Transition to Turbulence in Uniform-Shear Flows.

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Abstract. – Numerical and experimental results indicate that the critical amplitude for non-linear growth of disturbances in a uniform shear flow follows two scaling laws, depending on the shape of the initial disturbance: i) a Re^{-1} law for disturbances with zero streamwise dependence (streaks) (Kreiss *et al.*, submitted to *J. Fluid Mech.*); ii) a $Re^{-0.4\pm0.1}$ law for disturbances with non-zero streamwise dependence (spots) (Dauchot and Daviaud, submitted to *Phys. Fluid A*). It is shown that these laws can be explained by the competition between non-linear growth and viscous dissipation and that the difference in exponent simply reflects the difference in viscous-decay mechanism. Lower bounds for conditions for non-linear growth are derived analytically. They follow a Re^{-1} and $Re^{-1/3}$ law for streamwise and spotwise initial perturbations. The transition to a self-sustained turbulent state is also discussed. Upper bounds for this transition are derived. They follow, respectively, a $Re^{-1/2}$ and $Re^{-1/6}$ law for streak or spot initial perturbations.

Subcritical transition to turbulence has been observed in many uniform Couette, Poiseuille, wind tunnel and water channel shear flows. A detailed understanding of this transition is still missing. Over the past few years, however, important progress concerning the subcritical growth of initial disturbances has been made. Various mechanisms have been suggested. Dubrulle and Zahn [1] suggested that small but finite perturbations with zero streamwise and spanwise dependence could induce a substantial growth via creation of a local inflection point in the mean shear profile and development of a linear instability. A necessary condition for this mechanism to operate is that the relative amplitude of the disturbance, A, is larger than some power of the Reynolds number of the mean flow: $A > Re^{-2/3}$. Numerical simulations [2] have shown that such a mechanism may eventually lead to a self-sustained turbulent state in 3D, but not in 2D. More generally, it was shown that any large enough localized perturbation can be subject to an algebraic growth mechanism under subcritical conditions. There is not a clear consensus about the physical origin of this growth. Interesting properties of the linear dynamics follow from the fact that the linear operator of the shear flow problem is not self-conjugate [3-5]. Algebraic growth is then possible via a resonance mechanism between the pseudomodes [6] and a coupling of vertical vorticity and vertical velocity [6-9]. Detailed analysis of the pseudospectrum of the Couette and Poiseuille flow shows that the resonance (hence the energy amplification) is largest for perturbations with zero streamwise wave numbers, but non-zero spanwise dependence [5].

In the case of uniform-shear flows the linear problem can be solved analytically; this approach is known as the Rapid Distortion Theory (RDT) [10, 11]. The linear theory in this case fails in a finite time inversely proportional to the initial amplitude of perturbations. A weakly non-linear generalization of the RDT has been recently worked out by Nazarenko and Zakharov [12].

Work is also in progress investigating the scaling-law behavior of the critical perturbation amplitude for non-linear growth and/or transition to turbulence in plane Couette flow. Kreiss *et al.* [13] derive an exact lower bound for the amplitude below which all perturbations eventually decay. This bound varies like $Re^{-21/4}$. Using numerical simulations, they also study the threshold of transition towards a turbulent state when the initial condition is in the form of two streamwise, counter-rotating vortices (generating the so-called streaks with zero streamwise dependence in the streamwise velocity field). The threshold varies like Re^{-1}. The transition is related to the onset of a secondary instability in the spanwise components, due to a large shear in that direction. The behavior of initial perturbations with non-zero streamwise dependence (spots) has been studied recently in laboratory experiments by Daviaud and collaborators [14, 15]. The threshold amplitude for non-linear growth of the initial perturbation follows a scaling law: $A \propto (Re - Re_*)^{-0.4 \pm 0.1}$, where $Re_* = 350 \pm 20$ is the critical Reynolds number below which any initial spot decays. The non-linear growth regime was detected by the spanwise spreading of the spots (in the linear regime, only streamwise growth is allowed). The behavior of the spots was not studied for sufficiently long time to check whether the non-linear growth is automatically followed by a transition to a self-sustained turbulent state. However, preliminary results [16] show that for the smallest perturbations the non-linear growth does not lead to turbulence. This is an indication of the existence of a higher threshold for transition towards a turbulent state. This observation was also made by Klingman [9] in the case of the Poiseuille flow.

The coexistence of different scaling laws depending on the shape of the initial condition is an indication of the complexity of the transition to turbulence. It seems also to imply that several roads to turbulence are possible. In fact, we would like to show here that these laws simply can be explained by two different viscous-decay mechanisms and that a universal transition to turbulence is still possible. Our study will focus on the case of the plane Couette flow, a parallel shear flow of constant vorticity. Its velocity profile is given by

$$U = \sigma y e_x . \tag{1}$$

The streamwise and spanwise direction, therefore, respectively refer to the x- and z-direction. The boundary conditions compatible with such velocity law are irrelevant to the various mechanisms discussed in this paper. They shall henceforth be ignored.

The Couette flow is linearly stable and, therefore, is generically in the subcritical regime. Small initial perturbations can then only grow algebraically. This growth results in an increase of the energy of the perturbation, and an amplification of the initial disturbance. It may lead, via non-linear interactions, to a redistribution of energy between all the velocity components, and further amplification via secondary instability mechanisms. The typical time scale for non-linear energy redistribution is of the order of $\tau_{\mathrm{nl}} = l/v$, where v and l are typical velocity and scale of the initial perturbation. The growth is, however, limited by viscous dissipation, which produces an exponential decay of the perturbation. The typical time scale for viscous decay depends on the shape of the initial disturbance. Indeed, continual tilting in the shear flow of surfaces of constant x results in an increase of the wave number k_y, proportional to the x-component k_x, while the latter remains unaffected:

$$k_y = k_y^* + k_x^* \sigma t, \qquad k_x = k_x^* . \tag{2}$$

k_x^* and k_y^* are the initial x and y wave numbers. Physically, the wave number k_x remains unchanged due to the invariance of the system with respect to translations along x. The linear growth of k_y is a consequence of the vortex-tube volume conservation in the process of its shearing by the mean flow. Of course, the described tilting mechanism only takes into account the transfer of energy in the spectral space due to the mean flow. Besides the tilting, a spectral cascade may result from the mutual interactions of the perturbations, which are not mediated by the mean flow. Such a spectral transfer is ineffective at the initial stage of the growth of perturbations. Further, we will assume that the tilting by the mean flow remains the dominant mechanism even near the transition to turbulence, when the perturbations cannot be treated as weak. An alternative assumption, that the turbulence structure is weakly affected by the shearing and determined only by the local interactions of the turbulent scales, would be more appropriate to the quasi-isotropic turbulent systems, where the source of the turbulent energy is not associated with the mean shear flow.

Using, for example, shearing coordinates [17], it can be proved that the tilting mechanism produces, in the linear regime, an exponential viscous decay proportional to

$$\exp\left[-\nu\int_0^t (k_x^2(\tau) + k_y^2(\tau))\,\mathrm{d}\tau\right]. \tag{3}$$

This leads to two typical viscous time scales, according to the shape of the initial perturbation. Streamwise perturbations ($k_x^* \ll k_y^*$) are insensitive to the tilting mechanism and decay in a time scale $\tau_{\text{visc}} \simeq (k_y^* H)^{-2} Re\, \sigma^{-1}$, as in ordinary non-sheared flow; spotwise perturbations ($k_x^* \gtrsim k_y^*$) decay on a faster time scale $\tau_{\text{visc}} \simeq (k_x^* H)^{-2/3} Re^{-1/3} \sigma^{-1}$. In these estimates, H is the typical scale of the basic flow (half the channel width) and Re its Reynolds number.

The possibility of non-linear growth (and transition to turbulence) results in a competition between the algebraic growth and energy redistribution, and the viscous decay. It can only occur provided the amplitude of the initial disturbance satisfies

$$\tau_{\text{nl}} = \frac{l}{v} < \tau_{\text{visc}} . \tag{4}$$

This translates into two different scaling laws for the relative amplitude of the perturbation $A = v/(\sigma H)$:

$$\begin{cases} A > (k_y^* H) Re^{-1}, & \textit{for streamwise perturbations } (l^{-1} \sim k_y^*); \\ A > (k_x^* H)^{-1/3} Re^{-1/3}, & \textit{for spotwise perturbations } (l^{-1} \sim k_x^*). \end{cases} \tag{5}$$

These conditions are necessary (lower bounds) for *the non-linear growth* of the perturbation. Their different scaling, close to the experimental values, is a direct consequence of the difference in the viscous-decay mechanism. This provides, therefore, the simplest explanation of the experimental data.

Conditions (5) are, however, probably not a sufficient condition for the existence of a *self-sustained* turbulent state, because they do not guarantee that the non-linear interaction will modify sufficiently the mean shear to stop completely the viscous decay. The latter occurs as long as the total mean shear can produce a significant increase of the x-component of the wave number of the perturbation (see eq. (2)). The mechanism that can stop this process is, therefore, the non-linear flattening of the mean-velocity profile, resulting in a zero rate of shear ($\sigma \approx 0$), except in narrow regions near the boundaries. Such flattening is indeed observed generically in many turbulent shear flows, such as plane Couette flow [18], plane Poiseuille flow, rotating Couette flow [19], and inside the turbulent spots [20, 21]. It is favored by two mechanisms: transfer of energy between the mean shear and the turbulent component, and modification of the mean-velocity profile by the streamwise turbulent motions. Such motions have a low x-wave number and decay on a longer time scale than the other motions. Moreover, they induce a modification of the mean profile by transport of low-velocity fluid (near the center of the channel) to regions of higher mean velocities (near the boundaries) and vice versa. This transport mechanism has been observed in Couette flow experiments thanks to small tracer particles [15]. Overall, this produces a flattening of the mean-shear profile, so that according to (2) the growth of k_x stops and the system reaches the stationary state.

Obviously, the system can achieve this non-decaying state only if the time of mean flow modification t_{tr} is less than the time needed to reach the viscous scale, t_{visc}. Estimation of t_{tr} is difficult, because it involves strongly non-linear processes. An upper limit can however be given by considering that the flattening comes only from the transfer of energy between the mean flow and the turbulent motion. If the initial perturbation spectrum has a streamwise component (streaks with $k_x \approx 0$), then the turbulent energy growth appears already on the linear phase and can, therefore, be described by Rapid-Distortion Theory [10]:

$$E = C_1 E_0 \sigma t , \tag{6}$$

where C_1 is a numerical constant of order one, $E_0 = v_0^2$ is the initial energy of the perturbation. Non-linear flattening occurs when the energy of the mean flow approaches zero, or, in other words, when all initial mean-flow energy $(\sigma H)^2$ is transferred to the turbulence. The time required for this transfer, according to (6), is

$$t_{\mathrm{tr}} = \frac{H^2 \sigma}{C_1 v_0^2} . \tag{7}$$

This is only an upper limit, because strong non-linear interactions tend to accelerate energy exchanges between the mean flow and the turbulent motions. For example, the transport effect due to the streamwise component, the defect mechanism of Dubrulle and Zahn [1], or the secondary instability observed by Kreiss *et al.* [13] all accelerate energy transfer and flattening of the mean profile. Experiments on plane Couette flow indeed indicate that the transition to turbulence may be triggered by a defect mechanism [15]. If t_{tr} is indeed an upper limit on the energy transfer time, we obtain a sufficient, but not necessary condition for the existence of a self-sustained turbulent regime, via $t_{\mathrm{tr}} < t_{\mathrm{visc}}$. This translates into

$$A > \frac{k_y^* H}{C_1^{1/2}} Re^{-1/2} . \tag{8}$$

A slightly different approach should be used in case of spotwise initial perturbations. In fact, when the fraction of $k_x \approx 0$ perturbation is negligible in the initial spectrum, the turbulence growth cannot be described by the linear theory. Nazarenko and Zakharov [12]

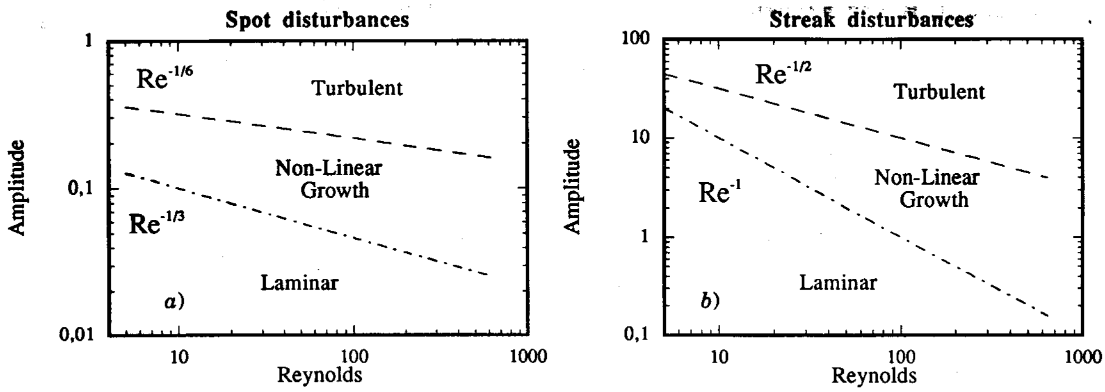

Fig. 1. – Critical amplitude in the Couette flow for spot (a)) and streak (b)) initial disturbances. The dot-dashed line corresponds to the laminar/non-linear–growth transition (also found in experiments), the dashed line to the non-linear-growth/turbulent transition. The axes values are only indicative, since the constants C_1 and C_2 have been taken equal to 1.

show that a t^2-growth appears, nonetheless, at the weakly non-linear stage:

$$E = C_2 \frac{E_0^2 t^2}{l^2}\,, \tag{9}$$

where C_2 is a numerical constant depending on the initial shape of the spectrum of perturbations. By equating (9) to the total energy of the mean flow, $(\sigma H)^2$, one obtains

$$t_{\mathrm{tr}} = C_2^{-1/2} \frac{(k_x^*)^{-1}}{v_0} \frac{\sigma H}{v_0}\,. \tag{10}$$

Condition $t_{\mathrm{tr}} < t_{\mathrm{visc}}$ with t_{visc} corresponding to the spotwise perturbations reads

$$A > C_2^{-1/4} (k_x^* H)^{-1/6} Re^{-1/6}\,. \tag{11}$$

Conditions (8), (11) are stronger than the conditions of the non-linear instability (Re^{-1} and $Re^{-1/3}$ laws) provided $Re > C_1$ or $Re > C_2^{3/2} (k_x^* H)^{-1}$, which is in general satisfied.

We have summarized this situation in fig. 1, for both spot and streak disturbances.

The necessary and sufficient condition for self-sustained turbulence can be bounded below by our set of conditions (5) and above by (8), (11). Since our bounds are not sharp, we are not able to prove the existence of a transition regime, in which initial perturbations experience a non-linear growth but decay eventually. It may, however, be possible to sharpen the bounds derived in this letter by a finer study of the mechanisms involved.

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