Triple Cascade Behavior in Quasigeostrophic and Drift Turbulence and Generation of Zonal Jets

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(Received 2 June 2009; published 11 September 2009)

We study quasigeostrophic (QG) and plasma drift turbulence within the Charney-Hasegawa-Mima (CHM) model. We focus on the *zonostrophy*, an extra invariant in the CHM model, and on its role in the formation of zonal jets. We use a generalized Fjørtoft argument for the energy, enstrophy, and zonostrophy and show that they cascade anisotropically into nonintersecting sectors in k space with the energy cascading towards large zonal scales. Using direct numerical simulations of the CHM equation, we show that zonostrophy is well conserved, and the three invariants cascade as predicted by the Fjørtoft argument.

DOI: 10.1103/PhysRevLett.103.118501 PACS numbers: 92.10.Lq, 52.35.Kt, 52.35.Ra, 92.10.Ty

Introduction.—Zonal jets are prominent features in geophysical fluids, e.g., the atmospheres of Jupiter, Saturn [1–3], and Earth [4,5] as well as the Earth's oceans [3,5,6]. They have also been observed in fusion plasmas [7] and are important because they can suppress the small-scale turbulence and block the transport in both geophysical settings [8] and in plasmas [7,9,10].

One of the main zonal jet generation mechanisms considered in the literature is the anisotropic inverse cascade [9,11–13] which brings the energy from initial small-scale turbulence to the large-scale zonal flows in a local in scale "step-by-step" transfer mechanism. Although the inverse cascade in both geophysical quasigeostrophic (QG) and plasma drift turbulence is similar to that of the 2D Navier-Stokes (NS) turbulence [14,15], the nonuniform rotation of geophysical fluids and plasma inhomogeneity make such a cascade anisotropic leading to condensation into large-scale zonal flows rather than round vortices [11–13]. In this Letter, we will follow the approach of [16,17] which is most relevant and asymptotically rigorous when the QG or drift turbulence is weak. In this case, the turbulence is dominated by waves which are involved in triad interactions and is shown to conserve an additional positive quadratic invariant [16–18], which, together with the other two quadratic invariants, the energy and the potential enstrophy, are involved in a triple cascade process which can be described via an argument similar to the standard Fjørtoft argument, originally developed for the 2D NS turbulence [14]. Considering its important role in the zonation process, hereafter we will label the extra invariant as *zonostrophy*.

Previous work [16,17] has been limited to considering either very large scales, those longer than the Rossby deformation or Larmor radius, or to the scales which are already anisotropic and are close to zonal. Here, we focus on the special case when the scales are much smaller than the deformation or Larmor radius, which is the most important and frequently considered limit. We will see below that the zonostrophy expression for such small-scale turbulence is positive and scale invariant. Thus, we can once again apply the generalized Fjørtoft argument developed in [16,17] which contains an additional information with respect to the argument presented in [19]: it predicts not only zonation but also k space paths of the three invariants during the zonation process. In this Letter, we numerically simulate the QG or drift turbulence for different levels of initial nonlinearity to confirm the conservation of zonostrophy and to detect these k space flow paths of the three invariants, thereby demonstrating zonostrophy's important role in directing the energy to the zonal jet scales.

Charney-Hasegawa-Mima model.—Geophysical QG flows and plasma drift turbulence are often mentioned together because some basic linear and nonlinear properties in these two systems can be described by the Charney-Hasegawa-Mima (CHM) equation [20,21] which in k space is

$$\partial_t \hat{\psi}_{\mathbf{k}} = i\omega_{\mathbf{k}} \hat{\psi}_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \hat{\psi}_{\mathbf{k}_1} \hat{\psi}_{\mathbf{k}_2}, \quad (1)$$

where $\hat{\psi}_{\mathbf{k}}$ is the Fourier transform of the stream function, $\omega_{\mathbf{k}} = -\frac{\beta k_x}{k^2}$ is the frequency of the linear waves (Rossby waves or drift waves in the geophysical and plasma contexts, respectively), β is a constant proportional to the gradient of the horizontal rotation frequency or of the plasma density, $k = |\mathbf{k}|$, and

$$T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = -\frac{(\mathbf{k}_1 \times \mathbf{k}_2)_z (k_1^2 - k_2^2)}{k^2} \quad (2)$$

is the nonlinear interaction coefficient.

Conservation of energy, enstrophy, and zonostrophy.—It is well known that the CHM equation (1) conserves the energy and the enstrophy which defined in terms of the wave action $n(\mathbf{k}) = \frac{k^4 |\hat{\psi}_k|^2}{2\beta k_x}$ are, respectively,

$$E = \int |\omega_{\mathbf{k}}| n_{\mathbf{k}} d\mathbf{k} \quad (3)$$

and

$$\Omega = \int k_x n_{\mathbf{k}} d\mathbf{k}. \tag{4}$$

E and Ω are *exact* invariants of the CHM model. Under the conditions of *weak nonlinearity* and *random phases*, the CHM model also conserves an extra invariant, zonostrophy,

$$Z = \int \zeta_{\mathbf{k}} n_{\mathbf{k}} d\mathbf{k}, \tag{5}$$

where function $\zeta_{\mathbf{k}}$ is the density of Z in the k space which satisfies the triad resonance condition $\zeta(\mathbf{k}) = \zeta(\mathbf{k}_1) + \zeta(\mathbf{k}_2)$ for all wave vectors $\mathbf{k}$, $\mathbf{k}_1$, and $\mathbf{k}_2$ which lie on the resonant surface given by the solutions of the wave vector and frequency resonance conditions, $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$ and $\omega(\mathbf{k}) = \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2)$. Expressions for $\zeta_{\mathbf{k}}$ were first found in [16,17] in the special cases of nearly zonal turbulence and large-scale turbulence and a general expression was found for all $\mathbf{k}$'s in [18]. In the short-wave limit considered here, we have

$$\zeta_{\mathbf{k}} = \frac{k_x{}^3}{k^{10}}(k_x{}^2 + 5k_y{}^2). \tag{6}$$

The integral (5) with the density (6) is an exact invariant of the kinetic equation [16,17], and thus it is an *approximate* invariant of the small-scale CHM equation. Expression (6) allows us to explicitly see that the invariant's density is strictly positive and scale-invariant, meaning that one can use the generalized Fjørtoft argument of [16,17] to find the cascade directions of the three invariants.

Dual and triple cascade behavior.—Let us first recall the classical Fjørtoft argument for 2D NS turbulence [14]. Consider 2D turbulence excited at some forcing scale $\sim k_0$ and dissipated at very large ($\sim k_- \ll k_0$) and at very small scales ($\sim k_+ \gg k_0$). The conservative ranges between the forcing scale and the dissipation scales are called the inertial ranges. The two conserved quantities in this case, in the absence of forcing and dissipation, are the energy E and the enstrophy Ω which are given by the same expressions as for the small-scale CHM model, see (3) and (4). In the presence of forcing and dissipation in steady-state turbulence, the rate of production of E and Ω by forcing must be exactly the same as the dissipation rates, ε and η, respectively. Note that the k space densities of E and Ω differ by a factor of k^2, and therefore the energy dissipation rate ε is related to the enstrophy dissipation rate η as $\eta \sim k_0^2\varepsilon$. Now, let us suppose *ad absurdum* that at $\sim k_+$ the energy is dissipated at a rate comparable to the rate of production at the forcing scales, i.e., $\sim\varepsilon$. This would mean that the enstrophy would be dissipated at the rate $\sim k_+^2\varepsilon$, but this amount greatly exceeds the rate of the enstrophy production $\eta \sim \sim k_0^2\varepsilon$. Thus, we conclude that most of E must be dissipated at the scales $\sim k_-$, i.e., that the energy cascade is *inverse* (with respect to its direction in 3D turbulence). Similarly, assuming *ad absurdum* that most of Ω is dissipated at $\sim k_-$ would also lead to a conclusion that the amount of E dissipated is much greater than the energy produced. Therefore, the cascade of Ω is direct; i.e., it is dissipated at $k \sim k_+ \gg k_0$.

Note that the quantities that determine the cascade directions are the k-space densities of the invariants or more precisely, the scaling of the ratios of these densities with k. In the CHM model, we have three invariants, and the cascade picture would necessarily be anisotropic (it is impossible to divide the 2D $\mathbf{k}$ space into three nonintersecting cascade regions in an isotropic way). Let us suppose that turbulence is produced near $\mathbf{k}_0 = (k_{0x}, k_{0y})$, and it can be dissipated only in regions which are separated in scales from the forcing scale, i.e., either at short scales, $k \gg k_0$, nearly zonal scales, $k_x \ll k_{0x}$, or at nearly meridional scales $k_y \ll k_{0y}$, see Fig. 1. Then, each of the invariants must cascade to the scales where its density is dominant over the densities of the other two invariants. The boundaries between the cascading ranges lie on the curves in the k space where the ratios of the different invariant densities, taken pairwise, remain constant (equal to the respective initial values). Note that $k^2 \le k_x{}^2 + 5k_y{}^2 \le 5k^2$, so we replace (6) with a simpler expression, $\tilde{\zeta}_{\mathbf{k}} \sim k_x{}^3/k^8$.

E/Ω boundary.—As for the 2D NS turbulence, the boundary separating the energy and the enstrophy cascades is defined by

$$k^2 \sim k_0^2, \tag{7}$$

(i.e., a circle in the 2D k space, $k_x^2 + k_y^2 = k_0^2$). This says that E must go to larger scales and Ω must go to smaller scales.

E/Z boundary.—Equating the ratio of the energy density $|\omega_k|$ to the zonostropy density $k_x{}^3/k^8$ to the initial value of this ratio, we get for this boundary

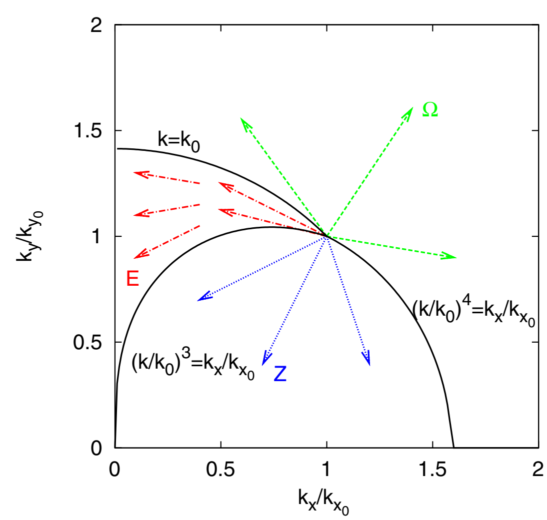

FIG. 1 (color online). Nonintersecting sectors for triple cascade as predicted by the generalized Fjørtoft argument.

$$k^3/k_x \sim k_0^3/k_{0x}. \tag{8}$$

This says that the E must go to the zonal scales, $k_y \gg k_x$. Moreover, this expression (8) also poses a particular restriction on the path of the energy to the zonal scales, e.g., for $k_y \gg k_x$, it should zonate at least as fast as $k_y = \text{const}\, k_x^{1/3}$, see Fig. 1.

Ω/Z boundary.—Equating the ratio of the enstrophy density k_x to the zonostropy density $k_x{}^3/k^8$ to the initial value of this ratio, we get for this boundary

$$k^4/k_x \sim k_0^4/k_{0x}. \tag{9}$$

This relation is also interesting. Because $k \geq k_x$, the curve (8) intersects the k_x axis at a finite distance, $k_x^* \sim k_0^{4/3}/k_{0x}^{1/3}$. We see that Z cannot cascade too far to large k unless one starts with nearly zonal scales, $k_{0y} \gg k_{0x}$. In particular, if $k_{0y} = k_{0x}$,we have $k_x^* = 2^{1/6}k_0$; i.e., the maximal allowed wave number for the Z cascade is practically the same as the initial scale. In other words, in this case, the zonostrophy can only cascade to the larger scales.

Numerical study.—A pseudospectral code has been used to solve the CHM equation. No dissipation is used, and the initial condition is given by

$$\hat{\psi}_{\mathbf{k}}|_{t=0} = Ae^{[(|\mathbf{k}-\mathbf{k}_0|^2/k_*{}^2)+i\phi_{\mathbf{k}}]} + \text{image}, \tag{10}$$

where $\mathbf{k_0} = (k_{0x}, k_{0y})$ and k_* are constants and $\phi_{\mathbf{k}}$ are random independent phases, and by "image" we mean the mirror-reflected spectrum with respect to the k_x axis. Since ψ is a real function, only the semiplane $k_x \geq 0$ was used in our computations because of the symmetry $\hat{\psi}_{-\mathbf{k}} = \hat{\psi}^*_{\mathbf{k}}$.

To quantify the cascades of the invariants in the time-evolving nondissipative turbulence, we introduce their *centroids* or " centers of mass" defined as follows:

$$\mathbf{k}_E(t) = \frac{1}{E}\int \mathbf{k}k^2|\hat{\psi}_k|^2 d\mathbf{k}, \tag{11}$$

$$\mathbf{k}_\Omega(t) = \frac{1}{\Omega}\int \mathbf{k}k^4|\hat{\psi}_k|^2 d\mathbf{k} \tag{12}$$

and

$$\mathbf{k}_Z(t) = \frac{1}{Z}\int \mathbf{k}\frac{k_x^4}{k^6}(k_x^2 + 5k_y^2)|\hat{\psi}_k|^2 d\mathbf{k}. \tag{13}$$

We have chosen two sets of parameters corresponding to weak and strong initial nonlinearities.

Weakly nonlinear case.—The initial spectrum and its width are $\mathbf{k_0} = (20, 20)$ and $k_* = 8$; the initial amplitude is $A = 10^{-6}$ with resolution 512^2. One can directly estimate the ratio of the linear and the nonlinear terms in Eqn. (1), leading to an estimate for the degree of nonlinearity σ to be

$$\sigma \sim \frac{2\sqrt{2\pi}k_0^3 k_* A}{\beta}, \tag{14}$$

which gives $\sigma \sim 0.09$. Numerically, we observe that all three invariants are well conserved, namely, the energy is conserved within 0.01%, the enstrophy within 0.15%, and the zonostrophy is conserved within 1%. For comparison, a nonconserved quantity $\int |\hat{\psi}_k|^2 d\mathbf{k}$ changes by over 200% over the same time interval.

The cascade directions for E, Ω, and Z are plotted in Fig. 2 in terms of the paths followed by the respective centroids. For convenience, we normalize the centroids to their initial values, k^c_{E0}, $k^c_{\Omega 0}$, and k^c_{Z0} so that the centroid paths start from the same point. We see that each invariant cascades well into its predicted sector. Interestingly, the enstrophy and the zonostrophy paths are well inside their respective cascade sectors, whereas the energy follows the boundary of its sector with the zonostrophy sector. One should remember, however, that the boundaries between the sectors are not sharp because the Fjørtoft argument operates with strong inequalities.

Strongly nonlinear case.—In this case, the resolution is 1024^2, the center of the initial spectrum is $\mathbf{k_0} = (40, 40)$, and its width is $k_* = 16$, and the initial amplitude is $A = 10^{-6}$. This time $\sigma \sim 0.7$ so that the initial turbulence is moderately strong. While the energy and enstrophy are still well conserved, within 0.2% and 1.2%, respectively, the zonostrophy is not conserved initially. It changes 65% over the first half of the run which is not surprising considering that the zonostrophy is only expected to be conserved if the nonlinearity is weak. What is more interesting, however, is that the zonostrophy growth saturates as time proceeds so that the zonostrophy is rather well conserved in this case for large times, within 5% over the second half of the run. This suggests that for large times, the scales that support

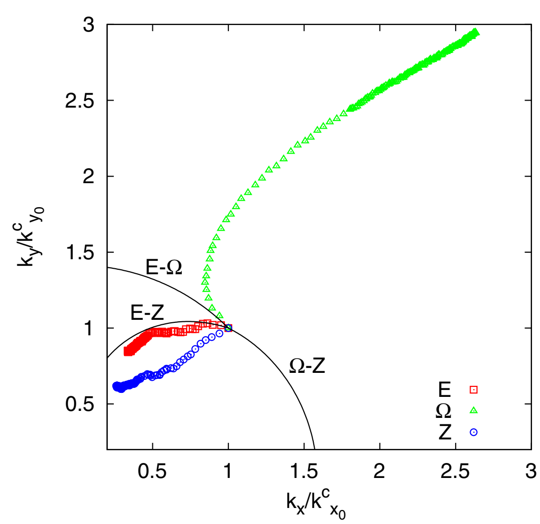

FIG. 2 (color online). The cascades of energy, enstrophy, and zonostrophy for the weakly nonlinear case, tracked by their centroids.

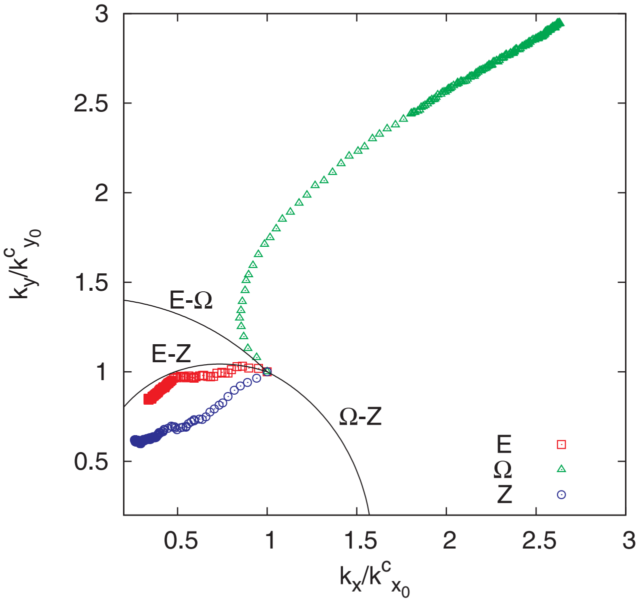

FIG. 3 (color online). The cascades of energy, enstrophy, and zonostrophy for the strongly nonlinear case, tracked by their centroids.

the zonostrophy invariant become weakly nonlinear, even though the energy scales probably remain moderately nonlinear, and the enstrophy scales are definitely strongly nonlinear.

The paths of the centroids are shown in Fig. 3. A picture similar to the weakly nonlinear case is observed; namely, the enstrophy and the zonostrophy cascades lie well inside their respective theoretically predicted sectors, and the energy cascade follows the boundary of its sector. This agreement could be explained by the fact that even in this strongly nonlinear regime, the zonostrophy invariant is conserved for large times and the triple cascade picture predicted using this invariant provides a reasonable description of the turbulence evolution and explanation of the zonal jet formation.

Summary.—In the present Letter, the generalized Fjørtoft argument was used to predict a triple cascade behavior of the CHM turbulence, in which the energy, the enstrophy, and the zonostrophy are cascading into their respective nonintersecting sectors in the scale space. These cascades are anisotropic, and the energy cascade is predicted to be directed to the zonal scales, which provides a physical explanation and the character of the formation of the zonal jets in such systems.

The zonostrophy conservation, as well as the triple cascade picture, were tested numerically for the cases of both weak and strong initial nonlinearities. The zonostrophy invariant was shown to be well conserved in the weakly nonlinear case. Moreover, the zonostrophy conservation was also observed for the case with strong initial nonlinearity after a transient nonconservative time interval. Presumably, this is because the zonostrophy moves in time to the scales that are weakly nonlinear even though the energy and the enstrophy remain in the strongly nonlinear parts of the Fourier space. Using the energy, the enstrophy and the zonostrophy centroids for tracking the transfers of these invariants in the Fourier space, we demonstrated that all three invariants cascade as prescribed by the triple cascade Fjørtoft argument in both the weakly nonlinear and in the strongly nonlinear cases. The energy appears to be somewhat special among the three invariants in that it tends to cascade along the edge of the sector allowed by the Fjørtoft argument, namely, along the curve $k \propto k_x^{1/3}$.

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